ebraic topology II - "Summary" of the material of the course.

1. H-spaces and operads

1.1 h-spaces

Definition 1.1: A pointed space M with base point 1 is called an h-space if there is a map $\mu: M \times M \longrightarrow M$ (written $\mu(x,y) = x \cdot y$) such that

1) $\mu \circ c_1 \simeq id_M$, where $c_1(x) = (x,1)$ ("$\mu(x,1) \simeq x$")
2) $\mu \circ c_2 \simeq id_M$, where $c_2(x) = (1,x)$ ("$\mu(1,x) \simeq x$")

Definition 1.2: M is called an h-monoid if μ is h-associative: $\mu \circ (id \times \mu) \simeq \mu \circ (\mu \times id)$ ("$(x \cdot y) \cdot z \simeq x \cdot (y \cdot z)$")

$$\begin{tikzcd}
 & M \times M \arrow[dr, "\mu"] \arrow[d, phantom, "\simeq"] & \\
M \times M \times M \arrow[ur, "id \times \mu"] \arrow[r, "\mu \times id"'] & M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

Definition 1.3: M is called h-commutative if $\mu \circ T \simeq \mu$, where $T(x,y) = (y,x)$ ("$x \cdot y \simeq y \cdot x$")

Definition 1.4: An h-monoid M is called an h-group if it has an h-inverse map $I: M \longrightarrow M$ satisfying

1) $I(1) = 1$
2) $\mu \circ (I \times id) \simeq c_1 \simeq \mu \circ (id \times I)$ ("$1 \simeq I(x) \cdot x \simeq x \cdot I(x)$")

$$\begin{tikzcd}
M \arrow[r, "I \times id"] \arrow[d, "id \times I"'] \arrow[dr, "c_1"] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

Remark: For all of these properties above, if we want to say that the equality up to homotopy is in fact strict, we mention that explicitly (e.g. 'strict associativity', 'strict neutral element', etc.)

Let's look at some examples of h-spaces:

Example 1.1: Topological groups are h-spaces with strict neutral element and inverse and are strictly associative.

Example 1.2: The closed loop space ΩX of a topological space X is an h-space with

$$(w_1 \circ w_2)(t) = \begin{cases} w_1(2t) & 0 \leq t \leq \frac{1}{2} \\ w_2(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

It is h-associative and has an h-inverse.

Example 1.3: The Moore-loop space $\Omega_{Moore}(X)$ is defined as the subset of $Map(\mathbb{R}_{\geq 0}, X) \times \mathbb{R}_{\geq 0}$ consisting of elements (w,a) such that $w(0) = x_0$ and $w(t) = x_0$ for $t \geq a$.

Then $\Omega_{Moore}(X, x_0)$ is an h-space via

$$(w_1, a_1) \circ (w_2, a_2) = (w_1 * w_2, a_1 + a_2),$$

where

$$w_1 * w_2(t) = \begin{cases} w_1(t) & 0 \leq t \leq a_1 \\ w_2(t-a_1) & a_1 \leq t \leq a_1 + a_2 \\ x_0 & t \geq a_1 + a_2 \end{cases}$$

In contrast to $\Omega(X, x_0)$, it is in fact strictly ~~commutative~~ associative.

We have the obvious inclusion

$$\begin{tikzcd} \Omega(X, x_0) \arrow[r, hook] & \Omega_{Moore}(X, x_0) \\ w \arrow[r, mapsto] & (w, 1) \end{tikzcd}$$

and it turns out this is a homotopy equivalence.

Example 1.4: Define the k-th ordered configuration space of X as

$$\widetilde{\mathrm{Conf}}^k(X) := \{(x_1, \dots, x_k) \in X^k \mid x_i \neq x_j \text{ for } i \neq j\}$$

Then the symmetric group $\mathfrak{S}_k$ acts freely (and properly discontinuous) on $\widetilde{\mathrm{Conf}}^k(X)$ by permuting the elements of the tuple.

The k-th unordered configuration space of X is then the quotient

$$\mathrm{Conf}^k(X) := \widetilde{\mathrm{Conf}}^k(X) / \mathfrak{S}_k$$

If X is a manifold, then $\widetilde{\mathrm{Conf}}^k(X)$ and $\mathrm{Conf}^k(X)$ are both manifolds as well.

Now we have another example of an h-space, by letting $X = \mathbb{R}^m$ and defining

$$C(\mathbb{R}^m) := \coprod_{k \geq 0} \mathrm{Conf}^k(\mathbb{R}^m)$$

Letting $\tau_1, \tau_2 : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be two disjoint embeddings, we get a multiplication $\mu : \mathrm{Conf}^k(\mathbb{R}^m) \times \mathrm{Conf}^\ell(\mathbb{R}^m) \longrightarrow \mathrm{Conf}^{k+\ell}(\mathbb{R}^m)$ by

$$\mu((x_1, \dots, x_k), (y_1, \dots, y_\ell)) := (\tau_1(x_1), \dots, \tau_1(x_k), \tau_2(y_1), \dots, \tau_2(y_\ell)),$$

which gives us a continuous map $\mu : C(\mathbb{R}^m) \times C(\mathbb{R}^m) \longrightarrow C(\mathbb{R}^m)$.

This h-space is h-associative. It is h-commutative if $m \geq 2$. It does not have an h-inverse.

Example 1.5: Let X be a connected space with basepoint $*$. We will construct an h-space $J(X)$ called the James reduced product of X, or the James construction.

Define

$$J_n(X) := X^n \Big/ (x_1, \dots, x_i, \dots, x_n) \sim (x_1, \dots, \hat{x}_i, \dots, x_n, x_i) \quad \text{if } x_i = *$$

(So if one of the x_i is the basepoint, we don't care about its position in the tuple.)

Using the natural inclusions $J_n(X) \hookrightarrow J_{n+1}(X) : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, *)$, we can define

$$J(X) := \lim_{n \to} J_n(X) \qquad \text{(with limit topology)}$$

A more intuitive way of viewing this space is by regarding its elements as words in the elements of X, where $*$ acts as a neutral element (e.g. $x_1 x_3 x_1 * x_2 x_3 = x_1 x_3 x_1 x_2 x_3$)

To make $J(X)$ into an h-space, we define the multiplication as 'concatination of words'. It is strict associative with a strict unit.

Proposition 1.1: $J(-)$ is a homotopy functor. That is, for a map $f : (X, x_0) \longrightarrow (Y, y_0)$ we get $J(f) : J(X) \longrightarrow J(Y)$ by sending the word $w_1 \cdots w_n$ to $f(w_1) \cdots f(w_n)$. If $f \simeq g$ rel x_0, then $J(f) \simeq J(g)$.

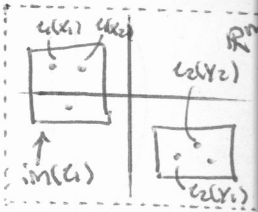
1.2 Operads

To motivate the general definition of an operad, let us take a closer look at the situation described in example 1.4, about the (un-) ordered configuration spaces of $\mathbb{R}^m$. For brevity, we will write $\tilde{C}^k$ for $\mathrm{Conf}^k(\mathbb{R}^m)$ here.

Given two disjoint embeddings $\iota_1, \iota_2 : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, we get a map

$$\tilde{C}^{k_1} \times \tilde{C}^{k_2} \longrightarrow \tilde{C}^{k_1+k_2}$$

$$((x_1, \dots, x_{k_1}), (y_1, \dots, y_{k_2})) \longmapsto (\iota_1(x_1), \dots, \iota_1(x_{k_1}), \iota_2(y_1), \dots, \iota_2(y_{k_2}))$$



Observe that we can get two such embeddings for each tuple (x,y) of two different points in $\mathbb{R}^m$ by drawing small boxes around these points of exactly the right size. Thus we obtain a continuous map

$$\gamma^2_{(k_1,k_2)} : \tilde{C}^2 \times (\tilde{C}^{k_1} \times \tilde{C}^{k_2}) \longrightarrow \tilde{C}^{k_1+k_2} \qquad \forall k_1, k_2 \in \mathbb{N}_{\geq 0}$$

In fact, there is nothing special about 2. Giving n different points in $\mathbb{R}^m$ will always allow us to combine n configurations into one:

$$\gamma^n_{(k_1,\dots,k_n)} : \tilde{C}^n \times (\tilde{C}^{k_1} \times \dots \times \tilde{C}^{k_n}) \longrightarrow \tilde{C}^{k_1+\dots+k_n}$$

This motivates the following definition:

Definition 1.5: An (topological) operad is a collection $(\vartheta_k)_{k\in\mathbb{N}}$ of sets (topological spaces) with a collection of (continuous) functions

$$\gamma^n_{(k_1,\dots,k_n)} : \vartheta_n \times (\vartheta_{k_1} \times \dots \times \vartheta_{k_n}) \longrightarrow \vartheta_{k_1+\dots+k_n},$$

called the structure maps that satisfy the following associativity relation: for integers $r, n_1, \dots, n_r, k_1^1, \dots, k_1^{n_1}, k_2^1, \dots, k_2^{n_2}, \dots, k_r^1, \dots, k_r^{n_r}$, the following diagram should commute:

$$\begin{tikzcd}
\vartheta_r \times (\vartheta_{n_1} \times (\vartheta_{k_1^1} \times \dots \times \vartheta_{k_1^{n_1}}) \times \dots \times \vartheta_{n_r} \times (\vartheta_{k_r^1} \times \dots \times \vartheta_{k_r^{n_r}})) \arrow[d, "\gamma^r_{(n_1,\dots,n_r)}"] \arrow[dr, "\gamma^{n_1}_{(k_1^1,\dots,k_1^{n_1})} \times \gamma^{n_2}_{(k_2^1,\dots,k_2^{n_2})} \times \dots \times \gamma^{n_r}_{(k_r^1,\dots,k_r^{n_r})}"] & \\
 & \vartheta_r \times (\vartheta_{\Sigma k_1^i} \times \vartheta_{\Sigma k_2^i} \times \dots \times \vartheta_{\Sigma k_r^i}) \arrow[d, "\gamma^r_{(\Sigma k_1^i,\dots,\Sigma k_r^i)}"] \\
\vartheta_{n_1+n_2+\dots+n_r} \times (\vartheta_{k_1^1} \times \dots \times \vartheta_{k_1^{n_1}} \times \dots \times \vartheta_{k_r^1} \times \dots \times k_r^{n_r}) \arrow[r, "\gamma^{n_1+\dots+n_r}_{(k_1^1,\dots,k_r^{n_r})}"'] & \vartheta_{\sum_{j=1}^{r} \sum_{i=1}^{n_j} k_j^i}
\end{tikzcd}$$

Example 1.6 (Little m-cubes operad)

The motivating example looks a lot like an operad. However, the structure maps satisfy the associativity relation only up to homotopy. We get a better operad if instead of choosing k distinct points in $\mathbb{R}^m$ we choose k disjoint affine embeddings:

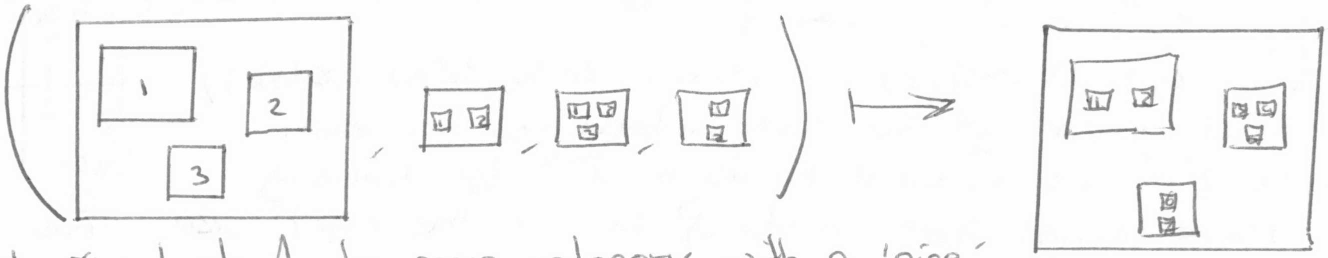
$$E_k^{(m)} := \{(\iota_1, \dots, \iota_k \mid \iota_i : \mathbb{I}^m \longrightarrow \mathbb{R}^m \text{ affine axis-parallel embeddings with disjoint images}\}$$

Identify $\mathbb{R}^m$ with the interior of $\mathbb{I}^m$.

Thus we get structure maps

$$\vartheta^n_{(k_1,\dots,k_n)}: E_n^{(m)} \times (E_{k_1}^{(m)} \times \dots \times E_{k_n}^{(m)}) \longrightarrow E^{(m)}_{k_1+\dots+k_n}$$

by composing the k_i embeddings of $E_{k_i}^{(m)}$ with the i-th embedding of $E_n^{(m)}$ and collecting them all together.



Example 1.7: Let A be some category with a 'nice' product (e.g. topological spaces or R-modules) and E an object in A. Let

$$\mathcal{O}_k := \mathrm{Hom}_A(E^k, E)$$

and

$$\vartheta^n_{(k_1,\dots,k_n)}: \mathcal{O}_n \times (\mathcal{O}_{k_1} \times \dots \times \mathcal{O}_{k_n}) \longrightarrow \mathcal{O}_{k_1+\dots+k_n}$$

$$(F, f_1(-,\dots,-), \dots, f_n(-,\dots,-)) \longmapsto F(f_1(-,\dots,-), f_2(-,\dots), f_n(-,\dots,-))$$

Example 1.8 $\mathrm{Ass}_k :=$ {rooted planar trees with k numbered leaves}

ϑ is given by attaching tree in Ass_{k_i} to the i-th leaf of the tree in Ass_n:

A topological operad actually gives us an h-space:

Proposition 1.2 If $((\mathcal{O}_k), (\vartheta^n_{k_1,\dots,k_n}))$ is a topological operad, then $M := \coprod_{k\in\mathbb{N}} \mathcal{O}_k$ is an h-associative h-space with h-neutral element, the operation given by fixing some $\xi \in \mathcal{O}_2$ and using the maps

$$\mu_{k\ell}(-,-) = \vartheta^2_{(k,\ell)}(\xi, -, -): \mathcal{O}_k \times \mathcal{O}_\ell \longrightarrow \mathcal{O}_{k+\ell}$$

1.3 Actions of h-spaces and operads on a topological space

Definition 1.6 For an h-space M with h-neutral element $1 \in M$ and a space X, we define an action of M on X to be a map $\rho: M \times X \longrightarrow X$ (denoted $\rho(m,x) = m \cdot x$) satisfying

(1) $E \simeq id_X: X \longrightarrow X$, where $E(x) = \rho(1,x)$ ("$1 \cdot x \simeq x$")

(2) $L \simeq R: M \times M \times X \longrightarrow X$, where ("$(m_1 \cdot m_2)\cdot x \simeq m_1 \cdot (m_2 \cdot x)$")

$$L(m_1, m_2, x) := \rho(m_1 \cdot m_2, x)$$
$$R(m_1, m_2, x) := \rho(m_1, \rho(m_2, x))$$

Example 1.9 (Master example)

Let $M = \Omega(B, b_0)$ be the closed loop space of a space B with base point b_0. This is an h-space. (Example 1.2)

Let $f : (E, e_0) \longrightarrow (B, b_0)$ be a pointed map and let

$$X := \mathrm{hfib}(f, b_0) = \{ (e, w) \in E \times B^I \mid w(0) = b_0,\ w(1) = f(e) \}$$

Then M acts on X by

$$\rho : \Omega(B, b_0) \times \mathrm{hfib}(f, b_0) \longrightarrow \mathrm{hfib}(f, b_0)$$

$$(\gamma, (e, w)) \longmapsto (e, \gamma * w),$$

i.e. first go through the loop from b_0 to b_0, then with w from b_0 to $f(e)$.

Definition 1.7 For an operad $\mathcal{O} = ((\mathcal{O}_k), (\gamma^n_{k_1 \dots k_n}))$ and a space X, we define an action of $\mathcal{O}$ on X as a collection of maps

$$\rho_n : \mathcal{O}_n \times X^n \longrightarrow X \qquad (\text{where } X^n = X \times \dots \times X)$$

such that for integers $n, k_1, \dots, k_n$ and $k := k_1 + \dots + k_n$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_n \times \mathcal{O}_{k_1} \times \dots \times \mathcal{O}_{k_n} \times X^{k_1} \times \dots \times X^{k_n} \arrow[r, "\rho_{k_1} \times \dots \times \rho_{k_n}"] \arrow[d, "\gamma^n_{(k_1 \dots k_n)}"] & \mathcal{O}_n \times X^n \arrow[d, "\rho_n"] \\
\mathcal{O}_k \times X^{k_1} \times \dots \times X^{k_n} \arrow[r, "\rho_k"] & X
\end{tikzcd}$$

Example 1.10 (Master example)

Consider the "little m-cubes operad" $E^{(m)}_*$ of example 1.6.

Let Y be a space with basepoint y_0 and let X be

$$X := \Omega^m(Y, y_0) = \mathrm{Map}(S^m, \infty ; Y, y_0)$$

$$\cong \mathrm{Map}(\mathbb{I}^m, \delta\mathbb{I}^m ; Y, y_0).$$

Define an action of $E^{(m)}_*$ on X by

$$\rho_n : E^{(m)}_n \times X^n \longrightarrow X$$

$$((c_1, \dots, c_n), (f_1, \dots, f_n)) \longmapsto F$$

n disjoint embeddings $c_i : \mathbb{I}^m \longrightarrow \mathbb{R}^m \subseteq \mathbb{I}^m$

n maps $f_i : \mathbb{I}^m \longrightarrow Y$ constant on $\delta\mathbb{I}^m$

where

$$F : (\mathbb{I}^m, \delta\mathbb{I}^m) \longrightarrow (Y, y_0) : x \longmapsto \begin{cases} f_i(z) & \text{if } x = c_i(z) \\ y_0 & \text{otherwise} \end{cases}$$

Essentially we construct $F : \mathbb{I}^m \longrightarrow Y$ by choosing n boxes in $\mathbb{I}^m$, using the map $f_i : \mathbb{I}^m \longrightarrow Y$ (after scaling) on the i-th box and letting F be constant outside of the n boxes.

We saw that $E_*^{(m)}$ can act on m-fold loop spaces. It turns out that this is actually the "only" type of space it can act on:

Theorem 1.1 (Recognition principle)

If a connected space X admits an action of the little m-cubes operad $E_*^{(m)}$, then it is homotopy equivalent to an m-fold loop space.

Remark: Note that it is essential that X must be connected. To see this, note that a topological operad $\mathcal{O}$ can always act on its associated h-space $M = \coprod_{k\in\mathbb{N}} \mathcal{O}_k$ by defining $\rho_n : \mathcal{O}_n \times M^n \longrightarrow M$ via

$$\gamma^n_{k_1 \ldots k_n} : \mathcal{O}_n \times (\mathcal{O}_{k_1} \times \ldots \times \mathcal{O}_{k_n}) \longrightarrow \mathcal{O}_{k_1+\ldots+k_n}.$$

In particular, $E_*^{(m)}$ acts on $M = \coprod_{k\in\mathbb{N}} E_k^{(m)}$, which is not connected, so the theorem doesn't apply. Indeed, one can prove that M is not homotopy equivalent to an m-fold loopspace.

Proof of recognition principle: Maybe later.

1.4 James reduced product $J(X)$

In example 1.5, we introduced the h-space $J(X)$ for a topological space X (connected, basepoint x_0).

We will give some facts about this space.

Theorem 1.2 (James)

If X is connected, there is a homotopy equivalence

$$\gamma : J(X) \xrightarrow{\simeq} \Omega(\Sigma X)$$

that sends the word $w = x_1 \cdot \ldots \cdot x_n$ to the loop in $\Sigma X = \frac{X \times I}{(x_0 \times I \cup X \times \partial I)}$ that in the i-th interval $[\frac{i}{n}, \frac{i+1}{n}] \subseteq [0,1]$ goes up in ΣX via $\{x_i\} \times I \subseteq \Sigma X$

Lemma 1.1: $$J_n(X)/J_{n-1}(X) \cong X^{(n)} := X \wedge X \wedge \ldots \wedge X \quad \text{(n-fold smas product)}$$

Theorem 1.3 (James)

If X is connected,

$$\Sigma J(X) \simeq \Sigma\Big(\bigvee_{n\geq 1} J_{(n)}(X)/J_{(n-1)}(X)\Big) \cong \Sigma\Big(\bigvee_{n\geq 1} X^{(n)}\Big)$$

Theorem 1.4 (Hilton-Milnor Theorem)

(The statement of the theorem in the lecture was not very precise, so I will leave it out.)

For a proof of theorem 1.3, see page 10.

2. Configuration spaces with labels

2.1 Definition and examples

Let M be an m-dimensional manifold. Recall the definition of the ordered configuration space of M:

$$\widetilde{\mathrm{Conf}}^k(M) := \{(\zeta_1, \dots, \zeta_k) \in M^k \mid \zeta_i \neq \zeta_j \text{ if } i \neq j\}.$$

Dividing out the free properly discontinuous action of the symmetric group $\mathfrak{S}_k$ gives us the unordered configuration space of M:

$$\mathrm{Conf}^k(M) := \widetilde{\mathrm{Conf}}^k(M) / \mathfrak{S}_k$$

The elements of $\mathrm{Conf}^k(M)$ can be regarded as configurations of k indistinguishable non-colliding particles in M.

Example 2.1:

$$\mathrm{Conf}^k(\mathbb{R}) \cong \{(\zeta_1, \dots, \zeta_k) \in \mathbb{R}^k \mid \zeta_1 < \dots < \zeta_k\} \simeq *$$

Example 2.2:

$$\widetilde{\mathrm{Conf}}^2(\mathbb{R}^m) \cong \mathbb{S}^{m-1} \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$$

$$(\zeta_1, \zeta_2) \longmapsto \left(\frac{\zeta_2 - \zeta_1}{\|\zeta_2 - \zeta_1\|}, \frac{\zeta_2 + \zeta_1}{2}, \|\zeta_2 - \zeta_1\|\right)$$

so

$$\mathrm{Conf}^2(\mathbb{R}^m) = \widetilde{\mathrm{Conf}}^2(\mathbb{R}^m) / \mathfrak{S}_2 \cong \mathbb{RP}^{m-1} \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$$

Example 2.3: (Braid groups)

The covering $\widetilde{\mathrm{Conf}}^k(\mathbb{R}^2) \longrightarrow \mathrm{Conf}^k(\mathbb{R}^2)$ gives us the important Braid group

$$\mathrm{Br}(k) \cong \pi_1(\mathrm{Conf}^k(\mathbb{R}^2))$$

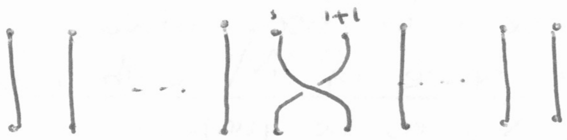
and its subgroup of pure braids

$$\mathrm{PBr}(k) \cong \pi_1(\widetilde{\mathrm{Conf}}^k(\mathbb{R}^2))$$

A presentation of $\mathrm{Br}(k)$ is given by

$$\mathrm{Br}(k) = \langle \beta_1, \dots, \beta_{k-1} \mid \beta_i\beta_j = \beta_j\beta_i \text{ if } |i-j| \geq 2,\ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} \rangle$$

where β_i should be visualised as



There is a surjection $\mathrm{Br}(k) \twoheadrightarrow \mathfrak{S}_k$ by sending β_i to the permutation $(i\ i{+}1)$ that switches i and $i+1$. We define the pure braids $\mathrm{PBr}(k) \subseteq \mathrm{Br}(k)$ to be the kernel of this map.

There is a sign map $\mathrm{Br}(k) \longrightarrow \mathbb{Z}$ by sending β_i to 1 for all i.

This gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{PBr}(k) \arrow[r, hook] \arrow[d, "\mathrm{sign}"] & \mathrm{Br}(k) \arrow[r, two heads] \arrow[d, "\mathrm{sign}"] & \mathfrak{S}_k \arrow[r] \arrow[d, "\mathrm{sign}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}_2 = \{\pm 1\} \arrow[r] & 0
\end{tikzcd}$$

Theorem 2.1 (Homotopy groups of configuration spaces of $\mathbb{R}^2$)

$$\pi_i(\widetilde{\mathrm{Conf}}^k(\mathbb{R}^2)) = \begin{cases} PBr(k) & i=1 \\ 0 & i\geq 2 \end{cases}$$

$$\pi_i(\mathrm{Conf}^k(\mathbb{R}^2)) = \begin{cases} Br(k) & i=1 \\ 0 & i\geq 2 \end{cases}$$

Theorem 2.2 (Homotopy groups of configuration spaces of $\mathbb{R}^\infty$)

$$\pi_i(\widetilde{\mathrm{Conf}}^k(\mathbb{R}^\infty)) = \begin{cases} 1 & i=1 \\ 0 & i\geq 2 \end{cases}$$

$$\pi_i(\mathrm{Conf}^k(\mathbb{R}^\infty)) = \begin{cases} \mathfrak{S}_k & i=1 \\ 0 & i\geq 2 \end{cases}$$

So $\widetilde{\mathrm{Conf}}^k(\mathbb{R}^\infty)$ is a contractible space on which $\mathfrak{S}_k$ acts freely.

The inclusion $\widetilde{\mathrm{Conf}}^k(\mathbb{R}^2) \hookrightarrow \widetilde{\mathrm{Conf}}^k(\mathbb{R}^\infty)$ turns out to give the map $Br(k) \longrightarrow \mathfrak{S}_k$ mentioned above on fundamental groups.

Theorem 2.2 can be proved using the Fadell-Neuwirth fibrations:

$$\begin{tikzcd} \widetilde{\mathrm{Conf}}^{k-1}(M\setminus\{p_1\}) \arrow[r, hook] & \widetilde{\mathrm{Conf}}^k(M) \arrow[r, "p"] & M \end{tikzcd}$$

$$(z_2,\dots,z_k) \longmapsto (p_1,z_2,\dots,z_k)$$

$$(z_1,\dots,z_k) \longmapsto z_1$$

A more general version is

$$\begin{tikzcd} \widetilde{\mathrm{Conf}}^{k-n}(M\setminus\{p_1,\dots,p_n\}) \arrow[r, hook] & \widetilde{\mathrm{Conf}}^k(M) \arrow[r, "p"] & \widetilde{\mathrm{Conf}}^n(M) \end{tikzcd}$$

$$(z_{n+1},\dots,z_k) \longmapsto (p_1,\dots,p_n,z_{n+1},\dots,z_k)$$

$$(z_1,\dots,z_n,z_{n+1},\dots,z_k) \longmapsto (z_1,\dots,z_n)$$

These are fibre bundles.

It is time to define configuration spaces with labels.

Definition 2.1 Let M be an m-dimensional manifold and $M_0 \subseteq M$ any subspace. Let X be a CW-complex with basepoint x_0.

Define the (unordered) configuration space of M with labels in X ~~as the quotient~~ denoted $C(M,M_0;X)$, as the quotient

$$\left(\coprod_{k\geq 1} \mathrm{Conf}^k(M) \times_{\mathfrak{S}_k} X^k\right)\Big/\sim$$

where the $\times_{\mathfrak{S}_k}$ notation means that for all $\sigma \in \mathfrak{S}_k$ we identify

$$(\xi_1,\dots,\xi_k : x_1,\dots,x_k) \quad \text{with} \quad (\xi_{\sigma(1)},\dots,\xi_{\sigma(k)} : x_{\sigma(1)},\dots,x_{\sigma(k)})$$

and $\sim$ is generated by the relations

① If $x_i = x_0$

$$(\xi_1,\dots,\xi_k : x_1,\dots,x_k) \sim (\xi_1,\dots,\hat{\xi}_i,\dots,\xi_k : x_1,\dots,\hat{x}_i,\dots,x_k)$$

② If $\xi_i \in M_0$

$$(\xi_1,\dots,\xi_k : x_1,\dots,x_k) \sim (\xi_1,\dots,\hat{\xi}_i,\dots,\xi_k : x_1,\dots,\hat{x}_i,\dots,x_k).$$

We can see elements of $C(M, M_0 ; X)$ as configurations [Seg
of indistinguishable non-colliding particles that all have a label in X and we can freely add or remove particles that either have label x_0 or are in M_0.

<u>Example 2.4</u> Let $M = \mathbb{R}$, $M_0 = \emptyset$, X any connected space.
There is a map

$$\Phi : C(\mathbb{R} ; X, x_0) \longrightarrow J(X, x_0)$$

$$[\xi_1, \dots, \xi_k ; x_1, \dots, x_k] \longmapsto w = x_1 \cdots x_k$$

This map turns out to be a homotopy equivalence.

Just as $J(X, x_0)$ has a filtration, $C(M, M_0 ; X, x_0)$ as well,

<u>Definition 2.2</u> Define

$$C_k(M, M_0 ; X, x_0) := \left(\coprod_{i=1}^{k} \widetilde{\mathrm{Conf}}^i(M) \times_{\mathfrak{S}_i} X^i \right) \Big/ \sim ① + ②$$

where ① and ② are the relations of before.
In other words we restrict to configurations of at most k particles.
Note that $C_1(M, M_0 ; X, x_0) = M \times X / (x_0 \times M \cup M_0 \times X) \cong (M / M_0) \wedge X$
Also note that the fibration strata give a $(X \setminus x_0)^k$-bundle:

$$\begin{tikzcd}
C_k \setminus C_{k-1} \arrow[d] \arrow[r, phantom, "\cong"] & \widetilde{\mathrm{Conf}}^k(M \setminus M_0) \times_{\mathfrak{S}_k} (X \setminus x_0)^k \arrow[d] \\
\mathrm{Conf}^k(M \setminus M_0) \arrow[r, phantom, "="] & \widetilde{\mathrm{Conf}}^k(M \setminus M_0) / \mathfrak{S}_k
\end{tikzcd}$$

For $X = S^q$, this is a vector bundle, since $(X \setminus x_0)^k \cong \mathbb{R}^{qk}$. Its Thom-space is C_k / C_{k-1}.

<u>2.2 Approximation Theorems</u>

In this subsection, we will describe several approximations of labeled configuration spaces.

<u>Theorem 2.3</u>: For every connected (well-pointed?) space X, there is a weak homotopy equivalence

$$\gamma : C(\mathbb{R}^m ; X, x_0) \xrightarrow{\simeq_w} \Omega^m \Sigma^m X$$

<u>Example 2.5</u> (Case $m = 1$)
For $m = 1$, the map γ is obtained by a process similar to theorem 1.2 for the map $J(X) \xrightarrow{\simeq} \Omega \Sigma X$. We can regard ΣX as $X \times I / (x_0 \times I \cup X \times \partial I)$. Given $\xi = [\xi_1, \dots, \xi_k ; x_1, \dots, x_k]$ in $C(\mathbb{R} ; X, x_0)$, we get a loop $\gamma(\xi)$ in ΣX by choosing k small intervals of length 2ε around the ξ_i, with ε small enough such that they don't intersect. Then the loop $\gamma(\xi)$ is defined on $\mathbb{R} \cup \{\infty\}$
the ith interval and be constant elsewhere

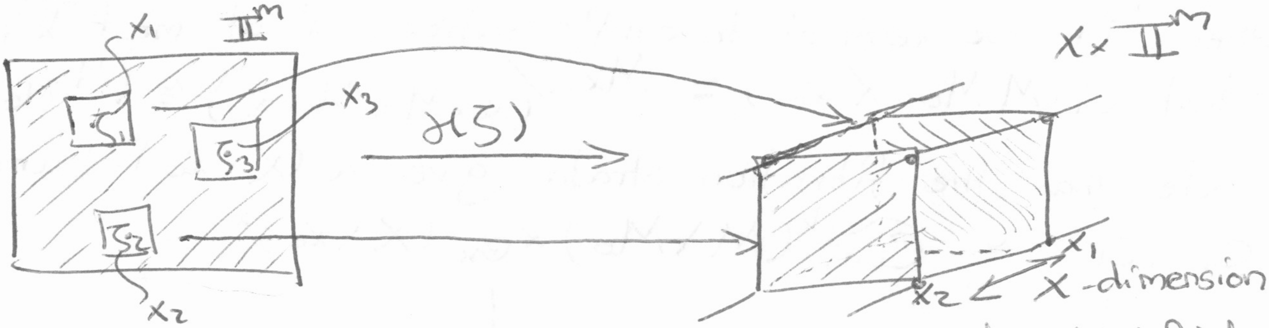
General description of γ for $m \geq 2$

For notational convenience we will define $\gamma: C(\mathring{\mathbb{I}}^m; X, x_0) \longrightarrow \Omega^m \Sigma^m X$ instead. We have

$$\Sigma^m X \cong X \times \mathbb{I}^m / (x_0 \times \mathbb{I}^m \cup X \times \partial \mathbb{I}^m)$$

and thus "loops" in $\Omega^m \Sigma^m X \cong \mathrm{Map}(\mathbb{I}^m, \partial \mathbb{I}^m; \Sigma^m X)$ can be represented by maps from $\mathbb{I}^m$ to $X \times \mathbb{I}^m / \ldots$ that are constant on the boundary.

Given $\zeta = [\zeta_1, \ldots, \zeta_k; x_1, \ldots, x_k] \in C(\mathring{\mathbb{I}}^m; X, x_0)$, we get such a map by again drawing k m-cubes around the k points $\zeta_i \in \mathring{\mathbb{I}}^m$ of length ε small enough. Then $\gamma(\zeta): \mathbb{I}^m \longrightarrow X \times \mathbb{I}^m /_\sim$ is defined on the i-th cube to map in an affine way onto $\{x_i\} \times \mathbb{I}^m \subseteq X \times \mathbb{I}^m /_\sim = \Sigma^m X$, and outside of these k cubes in $\mathbb{I}^m$, $\gamma(\zeta)$ just maps to the basepoint of $\Sigma^m X$.



(All "boundary" here gets identified to a single point.)

Note: we must choose $\varepsilon = \varepsilon(\zeta)$ in a continuous way for this to work.

There is a more general version of theorem 2.3, for which we need the idea of a "handle"

Definition 2.3 A handle of dimension m and index k is a pair (H, H_0) of an m-manifold with boundary and a submanifold $H_0 \subseteq H$ of dimension $m-1$ such that H is ~~iso~~ homeomorphic to an m-cube $\mathbb{I}^m = \mathbb{I}^k \times \mathbb{I}^{m-k}$ and H_0 corresponds under this homeomorphism to $\partial \mathbb{I}^k \times \mathbb{I}^{m-k} \subseteq \partial \mathbb{I}^m \cong \partial H$. $(0 \leq k \leq m)$

Theorem 2.4 Let (H, H_0) be a handle of index k $(0 \leq k \leq m)$. Then there is a weak homotopy equivalence

$$\gamma: C(H, H_0; X, x_0) \xrightarrow{\simeq_w} \Omega^{m-k} \Sigma^{m} X$$

Example 2.6: In particular:

- $k = m \Rightarrow C(\mathbb{D}^m, \partial \mathbb{D}^m; X, x_0) \simeq_w \Sigma^m X$
- $k = 0 \Rightarrow C(\mathbb{D}^m; X, x_0) \simeq_w \Omega^m \Sigma^m X$

So theorem 2.4 contains theorem 2.3 as a special case $k=0$.

Theorem 2.5 There is a weak homotopy equivalence

$$C(\mathbb{R}^\infty : X, x_0) \simeq_w \Omega^\infty \Sigma^\infty X$$

$$:= \lim_{\overrightarrow{k}} (\ldots \to \Omega^k \Sigma^k X \longrightarrow \Omega^{k+1} \Sigma^{k+1} X \to \ldots)$$

$$f \longmapsto \Sigma f$$

Proof: This follows directly from theorem 2.3 and the commutativity of the following square

$$\begin{tikzcd} C(\mathbb{R}^m : X) \arrow[r, "\gamma", "\simeq_w"'] \arrow[d] & \Omega^m \Sigma^m X \arrow[d, "\text{suspension}"] \\ C(\mathbb{R}^{m+1} : X) \arrow[r, "\gamma", "\simeq_w"'] & \Omega^{m+1} \Sigma^{m+1} X \end{tikzcd}$$

by taking limits on both sides.

We will now see a theorem that generalizes theorem 2.3 even more, allowing any pair (M, M_0). It has the form

$$\gamma : C(M, M_0 : X, x_0) \xrightarrow{\simeq_w} \mathrm{Sect}(W \setminus M_0, W \setminus M : E(W : X))$$

Let's first introduce the objects appearing above.

Preparation for theorem 2.6

Let M be an m-manifold with m-dimensional submanifold $M_0 \subseteq M$.
Let W be an m-manifold without boundary, containing M.
(If $\partial M = \emptyset$, we can choose $W = M$ for example. If $\partial M \neq \emptyset$, we can attach on open collar homeomorphic to $[0,1) \times \partial M$.)

Now consider the space

$$E(W : X) := V_m(TW) \times_{O(m)} \Sigma^m X.$$

Let's break this down. $V_m(TW)$ is the m-th Stiefel manifold of TW, ~~consisting of m-frames on W, i.e. a collection $(\alpha_1, \ldots, \alpha_m) \in (T_pW)^m$ of m vector fields on W.~~ (We probably implicitly want some Riemannian metric on W and let (α_i) be an orthonormal) basis of T_pW.

We get a fibre bundle $V_m(TW) \longrightarrow W$ with fibre equal to $V_m(\mathbb{R}^m) \cong O(m)$. For each point $p \in W$, the fibre consists of ortho(norm-)al bases $(\alpha_i)_{i=1}^m \in (T_pW)^m$ of T_pW.

Note that this fibre $O(m)$ acts on the space $\Sigma^m X \cong (S^m, \infty) \wedge (X, x_0)$ since it acts on $S^m = \mathbb{R}^m \cup \{\infty\}$ with ∞ as a fixed point.

So we can change the fibre bundle $V_m(TW) \longrightarrow W$ with fibre $O(m)$ to a bundle with fibre $\Sigma^m X$ by setting

$$E(W : X) := V_m(TW) \times_{O(m)} \Sigma^m X \text{ "}\cong\text{" } \bigsqcup V_m(T_pW) \times_{O(m)} \Sigma^m X.$$

So we have a fibre bundle

$$\Sigma^m X \longrightarrow E(W:X) \longrightarrow W$$

For each of the fibres we have the distinguished point $\infty \in \Sigma^m X$, since they are fixed points of the action of $O(m)$ on $\Sigma^m X$. Therefore we get two sections of this bundle:

~~$s_0 : W \longrightarrow E(W:X)$~~

$$s_\infty : W \longrightarrow E(W:X)$$

Finally, we need to define the specific type of sections of this bundle that we want to consider. If $B \subseteq A \subseteq W$ are submanifolds of dimension m, we can look at all sections of the restricted bundle $E(A:X) \longrightarrow A$ that are simply equal to s_∞ on B:

$$\mathrm{Sect}(A,B) := \{ s \in \mathrm{Sect}(\overset{\overset{E(W:X)}{\cup\!\mid}}{E(A:X)} \longrightarrow \overset{\overset{W}{\cup\!\mid}}{A}) \mid s|_B = s_\infty|_B \},$$

in other words: continuous maps $s : A \longrightarrow E(A:X)$ such that $s(a)$ lies in the fibre $V_m(T_aW) \times_{O(m)} \Sigma^m X \cong \Sigma^m X$ for all $a \in A$ and $s(b)$ is simply $\infty \in \Sigma^m X$ for $b \in B$.

The goal is now to find a weak homotopy equivalence

$$\gamma : C(M, M_0 : X, x_0) \xrightarrow{\simeq_w} \mathrm{Sect}(W \setminus M_0, W \setminus M : E(W:X))$$

that sends each configuration ζ to some section $\gamma(\zeta)$ of the bundle constructed above.

More precisely, $\gamma(\zeta) : W \setminus M_0 \longrightarrow E(W:X) = V_m(TW) \times_{O(m)} \Sigma^m X$ should assign to each point $w \in W \setminus M_0$ an element of $V_m(T_wW) \times_{O(m)} \Sigma^m X$, and if w lies outside of M we want it to be constant to $\infty \in \Sigma^m X$. (Since in the end we want it to be independent of the choice of W.)

Idea of the construction of γ

Let $\zeta = [z_1, \dots, z_n : x_1, \dots, x_n] \in C(M, M_0 : X, x_0)$ be a fixed configuration for now. We may assume $z_i \in M \setminus M_0 \subseteq W \setminus M_0$ and $x_i \neq x_0$ for all i.

The idea of defining $\gamma(\zeta) : W \setminus M_0 \longrightarrow E(W:X)$ is very similar to what we did in the case for the map

$$\gamma : C(\mathring{\mathbb{I}}^m ; X, x_0) \xrightarrow{\simeq} \Omega^m \Sigma^m X \simeq \mathrm{Map}(\mathbb{I}^m, \partial \mathbb{I}^m, \Sigma^m X)$$

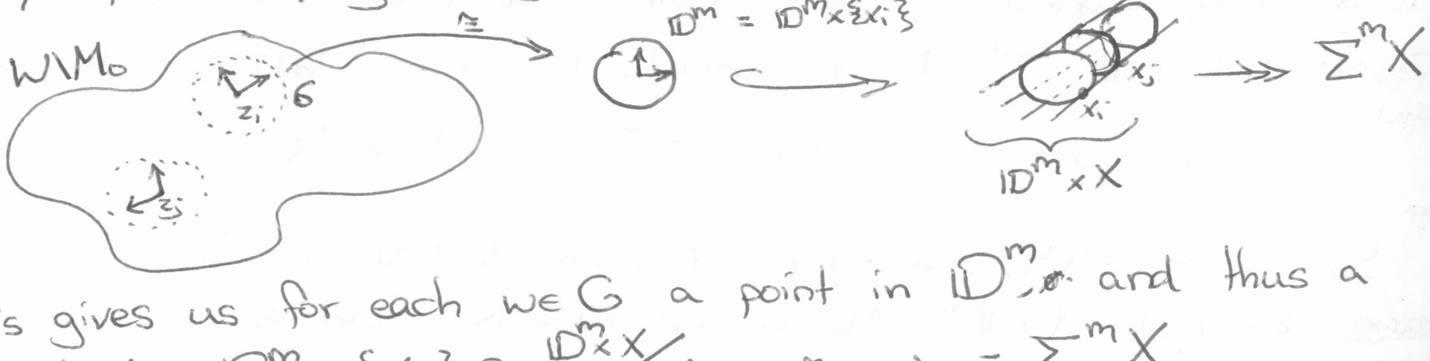
Given a configuration of n points in $\mathring{\mathbb{I}}^m$, we made a map from $\mathbb{I}^m$ to $\Sigma^m X$ by drawing n small boxes around the points on which we could define the map, and we extended it with the constant map to $\infty \in \Sigma^m X$ outside of these boxes. In particular, it was constant on $\delta \mathbb{I}^m$.

Now in this case, we are given n points $z_1, \dots, z_n$ in $W \setminus M_0$. Instead of boxes we want to draw geodesic discs around these points, small enough that they don't intersect. Again there will be a natural way of defining $\gamma(\xi)$ on these n geometric discs in $W \setminus M_0$ and we can extend it to all of $W \setminus M_0$ by letting it be constant ∞ outside the discs. In particular it will be constant on $W \setminus M$ if the discs are chosen small enough.

So given a geodesic disc G around $z_i \in W \setminus M_0$, what natural map $\gamma(\xi) : G \longrightarrow E(W; X) = V_m(TW) \times_{O(m)} \Sigma^m X$ do we have?

In the case of $M = \mathring{\mathbb{I}}^m$, we could just use some scaling to map a cube in $\mathring{\mathbb{I}}^m$ onto some cube $\{x_i\} \times \mathbb{I}^m \subseteq \Sigma^m X$, since the cube has a prefered orientation: a prefferred global frame. However, for a general manifold W there is not a canonical way to identify a geodesic disc to a disc $\mathbb{D}^m$: we have to make a <u>choice</u>. (However, we will see that this choice will not influence the final map $\gamma(\xi)$ due to ~~by~~ the way we divide out the action of $O(m)$.)

So how do we do it? Choose an orthonormal basis $(\alpha_1, \dots, \alpha_m)$ of tangent vectors in $T_{z_i} W$. This gives us a natural way of identifying the geodesic disc G around z_i with $\mathbb{D}^m$



This gives us for each $w \in G$ a point in $\mathbb{D}^m$, and thus a point in $\mathbb{D}^m \times \{x_i\} \subseteq \mathbb{D}^m \times X / (S^m \times X \cup \mathbb{D}^m \times \{x_0\}) = \Sigma^m X$.

Also, by parallel transporting the tangent vectors $(\alpha_1, \dots, \alpha_m) \in (T_{z_i} W)^m$ to an orthonormal basis $(\alpha_1(w), \dots, \alpha_m(w)) \in (T_w W)^m$ we get for each w an element in $V_m(T_w W)$.

Together this gives us a map

$$\gamma(\zeta) : G \longrightarrow V_m(TW) \times \Sigma^m X \twoheadrightarrow V_m(TW) \times_{O(m)} \Sigma^m X$$
$$= E(W:X)$$

which is indeed a section of $E(G:X) \to G$.

We now observe that a different choice of $(\alpha_1, \dots, \alpha_m)$ would have led to the same map, since $O(m)$ acts transitively on $V_m(T_{z_i} W)$ and the chosen points in $\Sigma^m X$ rotate accordingly if we rotate the basis $(\alpha_1, \dots, \alpha_m)$

We can do this for all i and have $\gamma(\zeta)$ defined on n geodesic discs in $W \setminus M_0$. Note that on the boundary of the geodesic discs we have just the section s_∞. This means we can extend $\gamma(\zeta)$ outside of these geodesic discs with s_∞ to finally obtain

$$\gamma(\zeta) : W \setminus M_0 \longrightarrow E(W:X)$$

i.e.

$$\gamma(\zeta) \in \mathrm{Sect}(W \setminus M_0, W \setminus M : E(W:X))$$

If we let the choice of geodesic discs around the z_i depend continuously on ζ, we thus have a map

$$\gamma : C(M, M_0 : X, x_0) \longrightarrow \mathrm{Sect}(W \setminus M_0, W \setminus M : E(W:X))$$

<u>Theorem 2.6</u>: The map γ described above is a weak homotopy equivalence, if M is compact

<u>Proposition 2.7</u>: For a connected based space (X, x_0), the inclusion

$$(\mathbb{D}^m, \partial\mathbb{D}^m) \wedge X \hookrightarrow C(\mathbb{D}^m, \partial\mathbb{D}^m : X)$$
$$z \wedge x \longmapsto [z : x]$$

is a strong deformation retract (with image $C_1(\mathbb{D}^m, \partial\mathbb{D}^m : X)$)

<u>Important remark</u>: If W is parallelizable, then $V_m(TW) \cong W \times O(m)$ and thus

$$E(W:X) = V_m(TW) \times_{O(m)} \Sigma^m X \cong W \times \Sigma^m X$$

So

$$\mathrm{Sect}(W \setminus M_0, W \setminus M; E(W:X)) \cong \mathrm{map}(W \setminus M_0, W \setminus M : \Sigma^m X)$$

<u>Example 2.7</u> Let $M = \mathbb{D}^m$, $M_0 = \emptyset$ and $W = \mathbb{R}^m$. Let X connected. We then see

$$C(\mathbb{R}^m; X) \cong C(\mathbb{D}^m : X) \simeq_w \mathrm{map}(\mathbb{R}^m, \mathbb{R}^m \setminus \mathbb{D}^m ; \Sigma^m X) \cong \Omega^m \Sigma^m X.$$

so we recover theorem 2.3.

<u>Example 2.8</u>. Let $M = G$ be a compact connected Lie-group, X connected. For $W = G$, we get

$$C(G; X) \simeq_w \mathrm{map}(G, \Sigma^m X)$$

In particular, we can use this for $M = S^1$. Let $M_0 = I \subseteq S^1$. Using $W = S^1$, we get

$$\begin{tikzcd}
C(I; X) \arrow[r, "j"] \arrow[d, "\simeq_w"'] & C(S^1; X) \arrow[r, "q"] \arrow[d, "\simeq_w"'] & C(S^1, I; X) \arrow[d, "\simeq_w"'] \\
\mathrm{map}(S^1, S^1 \setminus I; \Sigma X) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{map}(S^1, \emptyset; \Sigma X) \arrow[r] \arrow[d, phantom, "=" rotate=90] & \mathrm{map}(I, \emptyset; \Sigma X) \arrow[d, "\simeq"'] \\
\Omega\Sigma X \arrow[r] & \Lambda\Sigma X \arrow[r] & \Sigma X
\end{tikzcd}$$

<u>Example 2.9</u>: Let K be a finite complex, K_0 a subcomplex. Embed K in $\mathbb{R}^m$ and thicken it to an m-manifold M that deformation retracts onto K (and $M_0 \subseteq M$ def. retr. onto K_0). Then

$$C(M \setminus M_0, \partial M; X) \xrightarrow{\simeq_w} \mathrm{map}(K, K_0; \Sigma^m X)$$

Before we can prove the approximation theorem 2.6, we need the notion of a quasi-fibration.

<u>2.3 Quasifibrations</u>

<u>Definition 2.4</u>: A <u>quasifibration</u> is a ~~based~~ map $p: E \to B$ such that for all $b \in B$, $e \in F_b := p^{-1}(b)$ and $i \geq 0$, p induces an isomorphism

$$\pi_i(E, F_b, e) \xrightarrow[\cong]{p_*} \pi_i(B, b)$$

<u>Remark</u>: Note that this gives us the long exact sequence

$$\begin{tikzcd}
\arrow[r] & \pi_i(F_b, e) \arrow[r] & \pi_i(E, e) \arrow[r] \arrow[dr] & \pi_i(E, F_b, e) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_{i-1}(F_b, e) \arrow[r] & \pi_{i-1}(E, e) \arrow[r] & {} \\
& & & \pi_i(B, b) \arrow[ur] & & &
\end{tikzcd}$$

An equivalent definition is that $F_b \hookrightarrow \mathrm{hfib}(p, b)$ is a weak homotopy equivalence, which can be seen by comparing above sequence with

$$\to \pi_i(\mathrm{hfib}(p)) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(\mathrm{hfib}(p)) \to \pi_{i-1}(E) \to$$

<u>Note</u>: Pull-backs of quasi-fibrations are not necessarily quasi-fibrations. Fibrations (and in particular fibre bundles) are quasi-fibrations.

<u>Lemma 2.1</u> (Criteria for quasifibrations)
A map $p: E \to B$ of p.c. spaces is a q.f. if one of the following holds:

(I) $B = B_1 \cup B_2$, $B_0 := B_1 \cap B_2$, where each B_i is open and p.c. and $p_i : p^{-1}(B_i) \to B_i$ is a q.f. for $i = 0, 1, 2$.

(II) $B = \bigcup_{k \geq 0} B_k$, $B_0 \subseteq B_1 \subseteq \ldots \subseteq B$, $B = \varinjlim B_k$, where each B_k is path-connected and all $p_k : p^{-1}(B_k) \to B$ are q.f.

(III) E deformation retracts onto $E' \subseteq E$ and B onto B' s.t. $p(E') \subseteq B'$ and $p: E' \to B'$ is a q.f., and if $\hat{\Phi}_t : E \to E$ denotes the deformation retraction, the induced map on fibers

$$F_b = p^{-1}(b) \xrightarrow{\hat{\Phi}_1} p^{-1}(\Phi_1(b)) = F_{b'}$$

is a weak homotopy equivalence.

<u>Lemma 2.2</u> The map $p: E \to B$ is a quasi-fibration if

1) There is a family of spaces and maps

$$\begin{tikzcd}
F := E_0 \arrow[d, "p_0"] \arrow[r, phantom, "\subseteq"] & E_1 \arrow[d, "p_1"] \arrow[r, phantom, "\subseteq \cdots \subseteq"] & E_{k-1} \arrow[d, "p_{k-1}"] \arrow[r, phantom, "\subseteq"] & V_k \arrow[d, "\tilde{p}_k"] \arrow[r, phantom, "\subseteq"] & E_k \arrow[d, "p_k"] \arrow[r, phantom, "\subseteq \cdots \subseteq"] & E \arrow[d, "p"] \\
\{pt\} = B_0 \arrow[r, phantom, "\subseteq"] & B_1 \arrow[r, phantom, "\subseteq \cdots \subseteq"] & B_{k-1} \arrow[r, phantom, "\subseteq"] & U_k \arrow[r, phantom, "\subseteq"] & B_k \arrow[r, phantom, "\subseteq \cdots \subseteq"] & B
\end{tikzcd}$$

2) $U_k \subseteq B_k$ and $V_k \subseteq E_k$ are open and deformation retract onto B_{k-1}. resp. E_{k-1}, s.t. the induced map on fibers is a weak hpty equiv.

3) $B_{k-1} \subseteq U_k$ and $E_{k-1} \subseteq V_k$ are closed 5) Condition on fibers under def.

4) There is a homeomorphism $E_k \setminus E_{k-1} \cong (B_k \setminus B_{k-1}) \times F$.

<u>Proof</u>: Clearly p_0 is a q.f. Assuming p_{k-1} to be a q.f., we show that p_k is as well. By (III) of lemma 2.1, $\tilde{p}_k : V_k \longrightarrow U_k$ is a q.f. Then $B_k = U_k \cup (B_k \setminus B_{k-1})$ and ~~E_k~~ $E_k = V_k \cup E_k \setminus E_{k-1}$ are unions of two open sets. As $E_k \setminus E_{k-1} \cong (B_k \setminus B_{k-1}) \times F$, the restriction of p to (any subset of) $E_k \setminus E_{k-1}$ is a q.f., so by (I) of the lemma p_k is a q.f.

Then by (II), $p: E \longrightarrow B$ is a q.f.

<u>Proposition 2.2</u>: Let (M, M_0) be a manifold pair of dimension m and let $N \subseteq M$ have a good intersection with M. Let (X, x_0) based space. Then the sequence of embeddings

$$(N, N \cap M_0) \hookrightarrow (M, M_0) \longrightarrow (M, N \cup M_0)$$

induces a quasifibration

$$\underset{F}{\overset{\parallel}{C(N, N \cap M_0 ; X)}} \xrightarrow{j} \underset{E}{\overset{\parallel}{C(M, M_0 ; X)}} \xrightarrow{q} \underset{B}{\overset{\parallel}{C(M, N \cup M_0 ; X)}}$$

if X is connected or $(N, N \cap M_0)$ is connected.

<u>Proof</u>: 1) Use filtration $B_n := C_n(M, N \cup M_0 ; X)$

$E_n := q^{-1}(B_n)$

2) Note: - $q^{-1}(b) \cong F$ for all $b \in B$

- $E_n \setminus E_{n-1} \cong (B_n \setminus B_{n-1}) \times F$

3) Let U be open collar of N in M, and set

$U_n := \{ b \in B_{n+1} \mid$ at least one particle in $U \}$

$V_n := q^{-1}(U_n)$

Then U_n and V_n deformation retract onto B_n resp. E_n

4) For each $b \in U_n$, the induced monodrome on the fibre

$$F_b \longrightarrow F_{b'}$$

is a homotopy equivalence, since the particles of b in $U \setminus N$ pushed into N can be killed by moving them into M_0 or by moving their labels to x_0

Proof of the approximation theorem 2.6:

Let (M,M_0) be a compact manifold pair, X a based spaces.
Assume (M,M_0) is connected or X is path connected.
We want to show that the approximation map

$$\gamma : C(M,M_0 ; X) \longrightarrow \mathrm{Sect}(W\setminus M_0, W\setminus M ; E(W,X))$$

is a weak homotopy equivalence.

Case 1: (M,M_0) is a handle-pair of index $k \in \{0,\dots,m\}$ (definition 2.3, p.10)
The case $k=m$ is dealt with in proposition 2.1 (p14), which says

$$C(\mathbb{D}^m, \partial\mathbb{D}^m ; X) \simeq (\mathbb{D}^m, \partial\mathbb{D}^m) \wedge (X, x_0) \cong \Sigma^m X$$
$$\simeq \mathrm{map}(\mathbb{D}^m ; \Sigma^m X)$$
$$= \mathrm{map}(\mathbb{D}^m\setminus\partial\mathbb{D}^m, \mathbb{D}^m\setminus\mathbb{D}^m ; \Sigma^m X)$$

The case for general k goes by downwards induction, using a fibration of the form

$$(\mathbb{D}^m, \partial\mathbb{D}^k \times \mathbb{D}^{m-k}) \hookrightarrow (\mathbb{D}^m, \partial\mathbb{D}^k \times \mathbb{D}^{m-k} \cup \mathbb{D}^k \times \{1\} \times \mathbb{D}^{m-k-1})$$
$$\longrightarrow (\mathbb{D}^m, \partial\mathbb{D}^k \times \mathbb{D}^{m-k-1})$$

The middle pair is contractible and the third is a $(k+1)$-handle for which we know the result by induction. If we denote the three subspaces of $\mathbb{D}^m$ occurring before by M_1, M_2 and M_3 resp. then proposition 2.2 tells us that the upper row in

$$\begin{tikzcd}
C(\mathbb{D}^m, M_1) \arrow[r] \arrow[d, "\gamma_1"] & C(\mathbb{D}^m, M_2) \arrow[r] \arrow[d, "\gamma_2"] & C(\mathbb{D}^m ; M_3) \arrow[d, "\gamma_3"] \\
\mathrm{Sect}(\dots) \arrow[r] & \mathrm{Sect}(\dots) \arrow[r] & \mathrm{Sect}(\dots)
\end{tikzcd}$$

is a quasi-fibration. Also the bottom row is a fibration.
Since we know that γ_2 and γ_3 are weak homotopy equivalences applying the five-lemma to the long exact sequences of the (quasi-) fibrations shows that also γ_1 is a weak homotopy equiv.
This finishes case 1 by induction on k.

Case 2: For general (M,M_0), we use an important theorem saying that (M,M_0) can be built by finitely many handle attachments.
Thus we may inductively assume that $M = M' \cup_{H'} H$, where M' is a smaller manifold for which we know the result and (H,H') is some handle-pair dealt with in case 1.
We get by prop 2.2 a qf

$$\begin{tikzcd}
C(H,H' ; X) \arrow[r] \arrow[d, "\gamma_1"] & C(\overset{M}{\overset{\shortparallel}{M' \cup_{H'} H}}, M_0 ; X) \arrow[r] \arrow[d, "\gamma_2"] & C(M' \cup_{H'} H, H \cup M_0 ; X) \arrow[d, "\gamma_3"] \\
\mathrm{Sect}(W\setminus H' ; W\setminus H) \arrow[r] & \mathrm{Sect}(\dots) \arrow[r] & \mathrm{Sect}(\dots)
\end{tikzcd}$$

As $C(M' \cup_{H'} H, H \cup M_0) \cong C(M', M'_0 ; X)$, the induction hypothesis gives us that γ_3 is a weak homotopy equivalence and by step 1 we also know it for γ_1, so by the five-lemma applied to the long exact sequences also $\gamma_2 : C(M, M_0 ; X) \longrightarrow \mathrm{Sect}(\dots)$ is a weak homotopy equivalence. This finishes the proof

2.4 Snaith splitting for $C(M,M_0;X)$

Recall the splitting of $J(X)$ given in theorem 1.3:

$$\Sigma J(X) \xrightarrow{\simeq_w} \Sigma\Big(\bigvee_{n\geq 1} J_n(X)/J_{n-1}(X)\Big)$$

For the proof, write $V = \bigvee_{n\geq 1} J_n(X)/J_{n-1}(X)$. The goal is to find a map

$$s: J(X) \longrightarrow J(V),$$

since composing this with the weak hpty equiv $J(X) \xrightarrow{\simeq_w} \Omega\Sigma V$ gives $J(X) \longrightarrow \Omega\Sigma V$ and taking the adjoint gives us

$$s': \Sigma J(X) \longrightarrow \Sigma V$$

The idea of s is "take all subwords and organize them well"

For a word $w \in J(X)$, each subword of length n can be regarded as an element of $J_n(X)/J_{n-1}(X) \subseteq J\big(J_n(X)/J_{n-1}(X)\big) \subseteq J(V)$ so concatenating all subwords of length n gives a map

$$s_n: J(X) \longrightarrow J(V)$$

For each $w \in J(X)$, only finitely many of these are non-zero, so concatenating all of them gives a map

$$s: J(X) \longrightarrow J(V).$$

It is now possible to show (using an inductive argument) that the induced map $s': \Sigma J(X) \longrightarrow \Sigma V$ above is a weak homotopy equivalence. (See exercise 3.3)

We want to generalize this splitting to $C(M,M_0;X)$.

Theorem 2.7 (General Snaith Splitting)

Let (M,M_0) be a compact manifold pair and (X,x_0) a connected based space. Then there is a weak homotopy equivalence

$$\gamma: \Omega^\infty\Sigma^\infty C(M,M_0;X) \xrightarrow{\simeq_w} \Omega^\infty\Sigma^\infty\Big(\bigvee_{k\geq 1} C_k(M,M_0;X)/C_{k-1}(M,M_0;X)\Big)$$

Proof: Write $C := C(M,M_0;X)$ and $C_k := C_k(M,M_0;X)$. Also write $V := \bigvee_{k\geq 1} C_k/C_{k-1}$. The goal is to find a map

$$\Phi': C(\mathbb{R}^\infty, C) \xrightarrow{\simeq_w} C(\mathbb{R}^\infty, V)$$

since then by theorem 2.5

$$\Omega^\infty\Sigma^\infty C \simeq_w C(\mathbb{R}^\infty, C) \xrightarrow{\simeq_w} C(\mathbb{R}^\infty, V) \simeq_w \Omega^\infty\Sigma^\infty V.$$

We construct Φ' in two steps:

- first we define $C = C(M,M_0;X) \longrightarrow C(\mathbb{R}^\infty, V)$
- then we extend it to $C(\mathbb{R}^\infty, C)$

For the first step, note that each $\mathrm{Conf}^k(M)$ is a manifold and can thus be embedded in $\mathbb{R}^\infty$ via $e^k: \mathrm{Conf}^k(M) \hookrightarrow \mathbb{R}^\infty$.

We may assume these embeddings to be disjoint. What this means is that for a configuration $\zeta = [z_1, \dots, z_n : x_1, \dots, x_n]$ in $C = C_n(M, M_0 ; X)$, all possible subcollections of points $(z_{i_1}, \dots, z_{i_k}) \in \mathrm{Conf}^k(M)$ provide us a distinct element of $\mathbb{R}^\infty$.

In order to map to $C(\mathbb{R}^\infty, V)$, we have to give all these subcollections a label in V. The label assigned to $(z_{i_1}, \dots, z_{i_k})$ is

$$[z_{i_1}, \dots, z_{i_k} : x_{i_1}, \dots, x_{i_k}] \in C_k / C_{k-1} \subseteq \bigvee_{k \geq 1} C_k / C_{k-1} = V$$

So sending $\zeta \in C$ to all its subcollections and their labels gives a map

$$C \longrightarrow C(\mathbb{R}^\infty, V)$$

Note that the map is independent of representative $[z_1, \dots, z_n, x_1, \dots, x_n]$ of ζ, since:

- ▷ If one of the z_i is in M_0 or $x_i = x_0$, then the subcollections of ζ containing z_i will have trivial label in $C_k / C_{k-1} \subseteq V$ and thus won't contribute
- ▷ Since we take <u>all</u> subcollections, the order of the z_i doesn't matter.

To extend this map to $C(\mathbb{R}^\infty, C)$, we use

<u>Lemma 2.3</u>: Each map $f : Y \longrightarrow C(\mathbb{R}^\infty, Z)$ has an extension $f : C(\mathbb{R}^\infty, Y) \longrightarrow C(\mathbb{R}^\infty, Z)$.

<u>"Proof"</u>: To each configuration $\zeta \in C(\mathbb{R}^\infty, Y)$ we can use some "epsilon-map" to find a tuple of disjoint embeddings $\mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty$, one for each point in ζ. Applying these embeddings to the configuratio $f(y)$ given by the labels y, gives a new element of $C(\mathbb{R}^\infty, Z)$.

Thus we get a map $C(\mathbb{R}^\infty, C) \longrightarrow C(\mathbb{R}^\infty, V)$ and thus a map

$$\gamma : \Omega^\infty \Sigma^\infty C \longrightarrow \Omega^\infty \Sigma^\infty V$$

To show it's a weak homotopy equivalence, we show by induction that

$$\gamma_n : \Omega^\infty \Sigma^\infty C_n \longrightarrow \Omega^\infty \Sigma^\infty V_n \quad \left(= \Omega^\infty \Sigma^\infty \bigvee_{k=1}^{n} C_k / C_{k-1} \right)$$

is a weak hpty. equiv., the case $n = 1$ being clear.

We have

$$\begin{tikzcd}
\Omega^\infty \Sigma^\infty C_n / C_{n-1} \arrow[r, equal] & \Omega^\infty \Sigma^\infty C_n / C_{n-1} \\
\Omega^\infty \Sigma^\infty C_n \arrow[u] \arrow[r, "\gamma_n"] & \Omega^\infty \Sigma^\infty V_n \arrow[u] \\
\Omega^\infty \Sigma^\infty C_{n-1} \arrow[u] \arrow[r, "\gamma_{n-1}"] & \Omega^\infty \Sigma^\infty V_{n-1} \arrow[u]
\end{tikzcd}$$

The vertical maps are ~~co~~fibrations, so applying the five lemma to the long exact homotopy group sequences shows

is a weak homotopy equivalence, so is γ_n. □

3 Infinite symmetric product and Dold-Thom theorem

3.1 Definitions

Definition 3.1 : a) Let X be a (pointed) space. Define the n-th symmetric product of X as

$$SP_n(X) := X^n/\mathfrak{S}_n,$$

with elements denoted $[z_1,\dots,z_n]$ or $\sum_{j=1}^{n} z_j$ or $\sum_i k_i z_i$.

b) If X has base-point x_0 we have inclusions

$$SP_n(X) \hookrightarrow SP_{n+1}(X)$$

$$[z_1,\dots,z_n] \longmapsto [z_1,\dots,z_n,x_0]$$

c) Define the (infinite) symmetric product of X as

$$SP(X,x_0) := \operatorname*{colim}_n SP_n(X) \text{ “=” } \bigcup_{n\geq 0} SP_n(X)$$

Remark 3.1 : $SP(X,x_0)$ is the topological free abelian monoid generated by X with unit x_0.

If X doesn't have a base-point x_0, we still have

$$SP(X) := \coprod_{n\geq 0} SP_n(X),$$

which is the topological free abelian monoid generated by X.

Example 3.1 : Examples where symmetric products show up:

- Divisor of eigenvalues of a matrix
- Divisor of roots of a polynomial
- Divisor of zeroes (resp. of poles) of a meromorphic function.

Definition 3.2 : Let (X,x_0) be a pointed space and $\mathbb{A}$ a (Abelian) group/monoid. Define $SP\mathbb{A}(X,x_0)$ as

$$SP\mathbb{A}(X,x_0) = \Big(\coprod_{n\geq 0} X^n \times \mathbb{A}^n\Big)/\sim$$

where $\sim$ is generated by

1) $(x_1,\dots,x_n ; a_1,\dots,a_n) \sim (x_{\pi(1)},\dots,x_{\pi(n)} ; a_{\pi(1)},\dots,a_{\pi(n)})$ for $\pi\in\mathfrak{S}_n$

2) If $a_i=0$ or $x_i=x_0$

$$(x_1,\dots,x_n ; a_1,\dots,a_n) \sim (x_1,\dots,\hat{x}_i,\dots,x_n ; a_1,\dots,\hat{a}_i,\dots,a_n)$$

3) If $x_i=x_j$

$$(x_1,\dots,x_i,\dots,x_j,\dots,x_n ; a_0,\dots,a_i,\dots,a_j,\dots,a_n)$$
$$\sim (x_1,\dots,x_i,\dots,\hat{x}_j,\dots,x_n ; a_0,\dots,a_i+a_j,\dots,\hat{a}_j,\dots,a_n)$$

We'll write this simply as

$$SP\mathbb{A}(X,x_0) = \{\sum_{i=1}^{n} a_i x_i \mid a_i\in A,\ x_i\in X\}$$

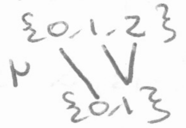
Remark 3.2 : Note that $SP\mathbb{N}(X,x_0)$ is just $SP(X,x_0)$.

We can even generalize this to Segel spaces:

Definition 3.3 : Let Γ, called "Segel's category" be the category with objects $\underline{n}=\{0,1,\dots,n\}$ for $n\geq 1$ and morphisms based maps $\varphi:\underline{n}\to\underline{m}$.

Example 3.2: Examples of morphisms in Γ are

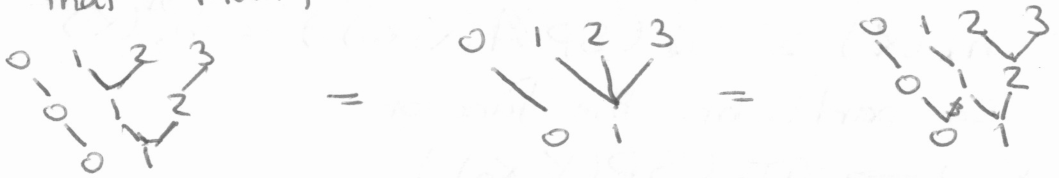
- All permutations $\sigma \circlearrowright \underline{n} = \{0,1,\dots,n\}$ fixing 0.
- The projection maps $d_i : \underline{n} \longrightarrow \underline{1}$ for $1 \le i \le n$ given by $d_i : \{0,1,\dots,n\} \longrightarrow \{0,1\} : j \longmapsto \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$
- The multiplication map $\mu : \underline{2} \longrightarrow \underline{1}$



Note that $\mu = \mu \circ \sigma_{12}$, where $\sigma_{12} : \underline{2} \longrightarrow \underline{2}$ permutes 1 and 2. We say that "multiplication is inherently commutative".

- In fact, any $\varphi : \underline{n} \longrightarrow \underline{n-1}$ that is injective except for $\varphi(i) = \varphi(j) \neq 0$ is a multiplication map, and we refer to it as "multiplication of the i-th and j-th factor".
- Multiplication maps that respect order.

We say that "multiplication is associative":



Definition 3.4: a) A Γ-functor is a functor $\mathbb{A} : \Gamma \longrightarrow Top_*$.
We have for $1 \le i \le n$ a map $\mathbb{A}(d_i) : \mathbb{A}(n) \longrightarrow \mathbb{A}(1)$ and we can combine them into a map

$$d : \mathbb{A}(n) \longrightarrow (\mathbb{A}(1))^n$$

b) A Segel space is a Γ-functor $\mathbb{A} : \Gamma \longrightarrow Top_*$ s.t.

1) $\mathbb{A}(0) \simeq *$
2) $d : \mathbb{A}(n) \xrightarrow{\simeq} (\mathbb{A}(1))^n$ is a homotopy equivalence.

c) For a general Γ-functor $\mathbb{A}$, the multiplication $m_{ij} : \underline{n} \longrightarrow \underline{n-1}$ multiplying factors i and j induces $\mathbb{A}(m_{ij}) : \mathbb{A}(n) \longrightarrow \mathbb{A}(n-1)$.

We can now define

$$SP\mathbb{A}(X, x_0) := \Big(\coprod_{n \ge 1} X^n \times A(n) \Big) / \sim$$

where $\sim$ is generated by

1) For $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ a permutation (and thus $\sigma : \underline{n} \to \underline{n}$)
$$(x_1,\dots,x_n ; a) \sim (x_{\sigma(1)},\dots,x_{\sigma(n)}, \mathbb{A}(\sigma)(a))$$

2) If $x_i = x_0$:
$$(x_1,\dots,x_i,\dots,x_n ; a) \sim (x_1,\dots,\hat{x}_i,\dots,x_n, \mathbb{A}(d_i)(a))$$

3) If $x_i = x_j$
$$(x_1,\dots,x_i,\dots,x_j,\dots,x_n ; a) \sim (x_1,\dots,x_i,\dots,\hat{x}_j,\dots,x_n ; \mathbb{A}(m_{ij})(a))$$

Example 3.3 Let $\mathbb{A}$ be an topological Abelian group/monoid. We get a Γ-functor by

$$\mathbb{A}^*: \quad \underline{n} \longmapsto \mathbb{A}^n$$

and sending the morphisms in Γ to the corresponding projections, multiplications and permutations of the coordinates.

The corresponding symmetric product space is just $SP\mathbb{A}(X,x_0)$ defined before.

It is clear by definition that $\mathbb{A}^*$ is a Segel-space.

Theorem 3.1: a) Let $\mathbb{A}$ be a Segel-space. Then the functors

$$(\text{CW-cplx})_0 \longrightarrow Ab: \qquad X \longmapsto h_q^{\mathbb{A}}(X) := \pi_q(SP\mathbb{A}(X,x_0))$$

give a homology theory when q ranges over $\mathbb{N}$.

b) If h_* is a connective homology theory: $CW_0 \longrightarrow Ab$, then it is equivalent to a functor of the above form, i.e. there is a Segel-space $\mathbb{A}$ such that

$$h_*(X) \cong \pi_*(SP\mathbb{A}(X,x_0)) = h_*^{\mathbb{A}}(X)$$

Corollary 3.1: In particular, the functor

$$X \longmapsto \pi_*(SP(X,x_0))$$

is a cohomology theory.

(We won't prove theorem 3.1, but we will later see a proof of corollary 3.1.

Example 3.4: Some Segel-spaces for which the homology theorems are known

1) For $SP(X) = SP\mathbb{N}(X)$, we get $H_*(X;\mathbb{Z})$

(Dold-Thom: we will see this result and prove it)

2) For an discrete Abelian group, $h_*^{\mathbb{A}}(X) \cong H_*(X;\mathbb{A})$

3) For a topological Abelian group, $h_*^{\mathbb{A}}(X) \cong \bigoplus_{i\geq 0} H_{*+i}(X;\pi_i(\mathbb{A}))$

Definition 3.5: An Eilenberg MacLane space (EML-space) is a connected based space with the homotopy type of a CW-complex with exactly one non-trivial homotopy group.

The space K is an EML-space of type (G,n) $(n\geq 1)$ if

$$\pi_i(K) = \begin{cases} G & i=n \\ 0 & i\neq n \end{cases}$$

A generic EML-space of type (G,n) will be denoted $K(G,n)$.

It is unique only up to homotopy type.

Example 3.5: Some $K(G,n)$-spaces:

1) S^1 is a $K(\mathbb{Z},1)$

2) $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2,1)$

3) $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$

By uniqueness of $K(G,n)$'s up to homotopy equivalence we have

Corollary 3.2: a) $K(G,n) \times K(H,n) \simeq K(G \times H,n)$

b) $K(G,n) \simeq \Omega K(G,n+1)$

Corollary 3.3: It follows from example 3.4 3) that for a topological Abelian group $\mathbb{A}$

$$SP\mathbb{A}(S^n) \simeq \prod_{i=0}^{n} K(\pi_i(\mathbb{A}), n-i)$$

3.2 The Dold-Thom theorem

Theorem 3.2 (Dold-Thom): There is a natural equivalence

$$\pi_*(SP(-)) \xrightarrow{\cong} H_*(-;\mathbb{Z})$$

of homology-functors $CW_0^{conn} \longrightarrow Ab$.

For the proof, we will need some propositions

Proposition 3.1: Let $A \xrightarrow{i} X \xrightarrow{q} X/A$ be a based cofibration, with X/A connected (i.e. (X,A) connected.)

Then we have a quasi-fibration $Q := Sp(q)$

$$\begin{tikzcd} SP(A,x_0) \arrow[r] \arrow[d, equal] & SP(X,x_0) \arrow[r, "Q"] \arrow[d, equal] & SP(X/A,\bar{x}_0) =: SP(X,A) \arrow[d, equal] \\ F & E & B \end{tikzcd}$$

Proof: We use lemma 2.2 (page 16). Filter B and E via

$B_n := SP_n(X,A)$ = at most n points outside A

$E_n := Q^{-1}(B_n)$.

1) Note that $E_n \setminus E_{n-1}$ = divisors with exactly n points outside A and some more in A

$\cong (B_n \setminus B_{n-1}) \times F$

2) There is an open neighborhood U of A in X with a strong deformation retract $\rho_t : (X,A,x_0) \longrightarrow (X,A,x_0)$ ($\rho_0 = id$, $\rho_1(U) \subseteq A$)

Let

$U_n = \{\xi \in B_{n+1} \mid$ at least one point of ξ is in $U\}$

$V_n = Q^{-1}(U_n)$.

Then U_n is an open nbhd of B_n in B_{n+1} that def. retr. onto B_n and similar for V_n.

3) The deformation retractions $R_t := SP(\rho_t)$ and $\tilde{R}_t$ induce homotopy equivalences on the fibers.

For fixed $b \in B_n$, some extra points previously not in A get pushed into A, thus adding points to each point in the fiber $F = SP(A,x_0)$. But for fixed b, these points are fixed and by homotoping them to the base-point x_0 (which is possible as (X,A) is connected) we can get rid of these points again.

Thus by lemma 2.2 Q is a quasifibration.

Proposition 3.2: We have a homology theory $h_*: CW_0^{conn} \longrightarrow Ab$ by

$$h_q(X) := \pi_q(SP(X, x_0))$$

Proof: Clearly a homotopy invariant functor satisfying excision. For the long exact sequence, we have by the previous proposition that the cofibration $(A, x_0) \xrightarrow{i} (X, x_0) \xrightarrow{q} (X, A)$ gives a quasi-fibration $SP(A, x_0) \longrightarrow SP(X, x_0) \longrightarrow SP(X, A) = SP(X/A)$, giving the l.e.s. □

We now want to find a natural transformation

$$\Theta: h_q(X) \longrightarrow H_q(X)$$

We need:

Lemma 3.1: $h_q(-)$ satisfies the Suspension axiom

Proof: Consider the cofibration $(X, x_0) \longrightarrow (X \times I, X \times \{1\} \cup x_0 \times I) \rightarrow (X \times I, X \times \partial I \cup x_0 \times I)$. This gives the quasi-fibration

$$\begin{tikzcd} SP(X, x_0) \arrow[r, "\simeq"] \arrow[d, equal] & SP(X \times I, X \times \{1\} \cup x_0 \times I) \arrow[r] \arrow[d, equal] & SP(X \times I, X \times \partial I \cup x_0 \times I) \arrow[d, equal] \\ SP(X, x_0) \arrow[r] & SP(CX, \bar{x}_0) \arrow[r] & SP(\Sigma X) \end{tikzcd}$$

with contractible total space, so from the l.e.s. we get

$$h_q(X) \cong \pi_q(SP(X, x_0)) \cong \pi_{q+1}(SP(\Sigma X)) = h_{q+1}(\Sigma X).$$

Corollary 3.4: $\tilde{h}_q(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$

Proof: Check this for $n=0$ and then use suspension isomorphism.

Corollary 3.5: $h_q(X_n, X_{n-1}) \cong h_q(\bigvee_{\substack{n\text{-cells} \\ \text{in } X}} \mathbb{S}^n) \cong \bigoplus_{\substack{n\text{-cells} \\ \text{in } X}} \mathbb{Z} \cong H_q(X_n, X_{n-1})$

Corollary 3.6: The above maps form a chain map between the cellular chain complexes $h_*(X_*, X_{*-1}) \xrightarrow{\Theta_*} H_*(X_*, X_{*-1})$, and thus a natural transformation on their homologies:

$$\Theta: h_*(X) \longrightarrow H_*(X),$$

which is in fact an isomorphism, as it is on the chain complex.

This proves the Dold-Thom-theorem.

Corollary 3.7: If $X = M(G, n)$ is a Moore-space of type (G, n), then $SP(X, x_0) = K(G, n)$ is an Eilenberg MacLane space of this type.

Corollary 3.8: For any connected CW-space, $SP(X, x_0)$ is an infinite loop-space.

Proof: As before (lemma 3.1) we have the quasi-fibration

$$SP(X, x_0) \longrightarrow SP(CX, \bar{x}_0) \longrightarrow SP(\Sigma X, \bar{x}_0),$$

so the fibre $SP(X, x_0)$ is $\overset{(\text{weakly})}{}$ $\overset{?}{}$ homotopy equivalent to the homotopy fibre of (a map homotopic to) the constant map, which is just the loop space $\Omega SP(\Sigma X)$. By induction $SP(X) \simeq_w \Omega^i SP(\Sigma^i X)$

Corollary 3.9: $SP(\mathbb{S}^n, \infty) \simeq \Omega^i SP(\mathbb{S}^{n+i}, \infty)$.

4. Brown Representability Theorem

For a contravariant functor $F: CW_0^{conn} \longrightarrow Sets_0$ consider the following three axioms:

BR1) Homotopy Invariance: If $f \simeq g$ rel x_0, then $f^* = g^*$

BR2) Mayer-Vietoris-Axiom: For a space $X = A \cup B$, we get the following pull-back-square of based sets:

$$\begin{tikzcd} F(A) & F(X) \arrow[l] \arrow[d] \\ F(A\cap B) \arrow[from=u] & F(B) \arrow[l] \end{tikzcd}$$

(i.e. if $\alpha \in F(A)$, $\beta \in F(B)$ have $\alpha|_{A\cap B} = \beta|_{A\cap B}$, there is a unique $\xi \in F(X)$ s.t. $\alpha = \xi|_A$ and $\beta = \xi|_B$)

BR3) Wedge-Axiom: The inclusions $i_i: X_i \longrightarrow \bigvee_{i\in I} X_i$ induce a bijection

$$F(\bigvee_i X_i) \longrightarrow \prod_i F(X_i)$$

Theorem 4.1 (Brown Representability theorem)

For a contravariant functor $F: CW_0^{conn} \longrightarrow Sets_0$ satisfying $Br_1 - Br_3$, there is a representing connected based CW-complex K with a universal element $\omega \in F(K)$, i.e. the natural transformation

$$\Theta_\omega^- : [-, K] \longrightarrow F(-)$$

given on $f: X \to K$ by

$$\Theta_\omega^X(f) := f^*(\omega) \in F(X)$$

is an equivalence of functors (in part. $[X, K] \cong F(X)$).

We won't repeat the whole proof, just the most important lemmas.

Call (K, ω) n-universal if $\Theta_\omega^{S^i}: [S^i, K] \longrightarrow F(S^i)$ is surjective for all $i \leq n$ and has trivial kernel for $i < n$.

Call (K, ω) ∞-universal if this holds for all i.

Lemma 4.1: For any connected Z and $\xi \in F(Z)$, there exists a ∞-universal pair (K, ω) with $Z \subseteq K$ and $\omega|_Z = \xi$.

Idea: Construct K inductively by attaching cells to Z, creating a sequence $Z = K_0 \subseteq K_1 \subseteq \dots \subseteq K_{n-1} \subseteq K_n \subseteq \dots \subseteq K = \mathrm{colim}_n K_n$.

Each K_n has $\omega_n \in F(K_n)$ s.t. $\omega_n|_{K_{n-1}} = \omega_{n-1}$, and (K_n, ω_n) is n-universal.

Then by Milnor's exact sequence we get $\omega \in F(K)$ ∞-universal.

Lemma 4.2: Let (K, ω) be ∞-universal and let (X, A, x_0) be any connected based CW-pair.

Then for each $\xi \in F(X)$ and $f: A \to K$ with $f^*(\omega) = \xi|_A$, there exists $g: X \to K$ with $g|_A \simeq f$ and $g^*(\omega) = \xi$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\simeq"] & K \\ X \arrow[ur, dashed, "\exists g"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} F(A) & F(K) \arrow[l] \\ F(X) \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} \xi|_A & \omega \arrow[l, mapsto] \\ \xi \arrow[u, mapsto] \arrow[ur, dashed, leftarrow] & \end{tikzcd}$$

Proof of BR-theorem: Lemma 4.1 tells us that a ~~ω~~ universal pair (K,ω) exists and it remains to show that is universal i.e. that $\Theta_\omega^X : [X,K] \longrightarrow F(X) : [f] \longmapsto f^*(\omega)$ is a bijection.

Surjectivity: Using lemma 4.2 with $A = \{x_0\}$ gives for every $\xi \in F(X)$ a map $g : X \longrightarrow K$ with $\xi = g^*(\omega)$. (The conditions become trivial.)

Injectivity: If $f_0, f_1 : X \longrightarrow K$ are two based maps with $\Theta_\omega^X([f_0]) = \Theta_\omega^X([f_1])$, define $\tilde{f} : X \times \{0,1\} \longrightarrow K$ by $f|_{X\times 0} = f_0$, $f|_{X\times 1} = f_1$.

This gives a based map $f : X \vee X \longrightarrow K$.

We want to apply lemma 4.2 to extend f to a homotopy

$g : X \wedge [0,1]_+ \longrightarrow K$, with $g|_{X\vee X} = f$, i.e. $f_0 \simeq f_1$.

For $p : X \wedge [0,1]_+ \longrightarrow X$ the projection, let $\xi = p^*(f_0^*(\omega)) = p^*(f_1^*(\omega)) \in F(X \wedge [0,1]_+)$. Then $\xi|_{X\vee X} = f^*(\omega)$ (which is not yet clear to me thh), so this g does indeed exist and we're done

Application: If we have a reduced cohomology theory $h^* = (h^n)_{n\in\mathbb{Z}}$ satisfying the Wedge axiom, we can apply BR and get representing spaces K_n and universal elements $\omega_n \in h^n(K_n)$ s.t.

$$[X, K_n] \xrightarrow{\cong} h^n(X) : [f] \longmapsto f^*(\omega_n).$$

By the Suspension isomorphism we thus have a natural equivalence of functors

$$[X, K_n] \cong h^n(X) \cong h^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}] \cong [X, \Omega K_{n+1}]$$

which implies by Yoneda-lemma that there is a weak homotopy equivalence $s_n : K_n \longrightarrow \Omega K_{n+1}$.

Thus $(K_n)_{n\in\mathbb{Z}}$ is an Ω-spectrum.

Definition 4.1: a) A spectrum E is a sequence $(E_n)_{n\in\mathbb{Z}}$ of spaces with structure maps $e_n : \Sigma E_n \longrightarrow E_{n+1}$

b) A spectrum E is called a suspension-spectrum (or Σ-spectrum) if all e_n are weak homotopy equivalences.

c) Dually, E is called an Ω-spectrum if all dual maps $e_n' : E_n \longrightarrow \Omega E_{n+1}$ are weak homotopy equivalences.

Example 4.1: For any space Z, we have the suspension spectrum of Z:

$$E_n := \Sigma^n Z, \quad e_n = id_{\Sigma^{n+1}Z}$$

Example 4.2: $E_n := K(\mathbb{Z}, n)$ gives an Ω-spectrum, with $E_{-k} = pt$

Remark: Each spectrum E gives a cohomology theory h^* via

$$h^n(X) = \lim_k \{[\Sigma^k X, E_{n+k}] \xrightarrow{\Sigma} [\Sigma^{k+1} X, \Sigma E_{n+k}] \xrightarrow{(e_{n+k})_*} [\Sigma^{k+1} X, E_{n+k+1}]\}$$

(For the MV-sequence use Blakers Massey and a limit argument.)

Remark: We can apply the construction of K in the proof of lemma 4.1 to $F(-) = H^1(-)$ and see explicitly that $K = \mathbb{S}^1$.

Observation: In general, we see

1) If $F(\mathbb{S}^i) = \{0\}$ for $i < n$, then one can assume K to be $(n-1)$ connected.

2) If $F(\mathbb{S}^i) = \{0\}$ for $i > n$, then $\pi_i(K) = 0$ for $i > n$.

3) If $F(\mathbb{S}^i) = \begin{cases} G & i = n \\ 0 & i \neq n, \end{cases}$ then $K = K(G,n)$ is an EML-space.

4) In particular $\tilde{H}^n(X;G) \cong [X, K(G,n)]$, i.e. the reduced singular homology theory $\tilde{H}^*(-;G)$ is represented by the Ω-spectrum of EML-spaces $\underline{K} = (K(G,n))_{n \geq 0}$.

Example 4.3: Let G_1, G_2 be abelian groups and $K_i = K(G_i, n)$, $i = 1,2$. Then

$$\begin{aligned} [K_1, K_2] &\cong \tilde{H}^n(K_1, G_2) \overset{uct}{\cong} \operatorname{Ext}(\hat{H}_{n-1}(K_1), G_2) \oplus \operatorname{Hom}(\hat{H}_n(K_1), G_2) \\ &\overset{hur}{\cong} 0 \oplus \operatorname{Hom}(\pi_n(K_1), G_2) \\ &\cong \operatorname{Hom}(G_1, G_2) \end{aligned}$$

given by

$$[f] \longmapsto \pi_n(f) : \pi_n(K_1) \longrightarrow \pi_n(K_2).$$

Example 4.4: Let $F : CW_0^{conn} \longrightarrow Ab$ be contravariant functor satisfying BR1) - BR3). This also gives $F^2 : CW_0^{conn} \longrightarrow Ab$ given by

$$F^2(X) = F(X)^2 = F(X) \times F(X).$$

Then $F(X) \cong [X, K_1]$, $F^2(X) \cong [X, K_2]$ for representing spaces K_1, K_2, and then

$$[X, K_2] \cong F^2(X) = F(X) \times F(X) \cong [X, K_1] \times [X, K_1] \cong [X, K_1 \times K_1]$$

shows that

$$K_2 \simeq_w K_1 \times K_1.$$

The natural transformation $F^2(-) \longrightarrow F(-)$ given by multiplication in the group $F(X)$ thus comes frome some map

$$\mu : K_1 \times K_1 \simeq_w K_2 \longrightarrow K_1,$$

making K_1 into an H-space.

In fact, we only need $F(X)$ to be a semi-group (multiplication).

- Each $F(X)$ has neutral element $\Longrightarrow$ K_1 has h-unit
- Each $F(X)$ is associative $\Longrightarrow$ K_1 is h-associative.
- ——— " ——— commutative $\Longrightarrow$ K_1 is h-commutative
- ——— " ——— has inverses $\Longrightarrow$ K_1 has an h-inverse.

Remark A multiplication in a cohomology theory h^* would be reflected

5. Fibre bundles 5.1 Definition and Classification Theorem

Throughout this chapter, G is a topological group and F a G-space, i.e. a space with a continuous G-action $\rho : G \longrightarrow \mathrm{Homeo}(F)$.

Definition 5.1: We say that a continuous surjective map $\xi : E \longrightarrow B$ with path-connected base B is a fibre bundle with fibre F and structure group G (short: (F,G)-bundle) if the following hold:

1) There is a trivializing open cover $\mathcal{U} = (U_i \mid i \in I)$ of B, i.e. there are homeomorphisms $h_i : E_i := \xi^{-1}(U_i) \xrightarrow{\cong} U_i \times F$ such that the following commutes

$$\begin{tikzcd} E_i \arrow[rr, "h_i", "\cong"'] \arrow[dr, "\xi|_{E_i}"'] & & U_i \times F \arrow[dl, "\mathrm{proj}"] \\ & U_i & \end{tikzcd}$$

2) Intersecting trivializations respect G-action, i.e. if $U_{ij} := U_i \cap U_j \neq \emptyset$, then $h_{ij} := (h_i|_{E_{ij}}) \circ (h_j|_{E_{ij}})^{-1} : U_{ij} \times F \xrightarrow{(h_j)^{-1}} E_{ij} := \xi^{-1}(U_{ij}) \xrightarrow{h_i|_{E_{ij}}} U_{ij} \times F$ is of the form $(b,x) \longmapsto (b, \rho_{ij}(b)(x))$, and we want that

$$\begin{tikzcd} \rho_{ij} : U_{ij} \arrow[rr] \arrow[dr] & & \mathrm{Homeo}(F) \\ & G \arrow[ur, "\rho"'] & \end{tikzcd}$$

: $b \longmapsto \rho_{ij}(b)$ factors over G

Example 5.1: Coverings (F discrete) and vector bundles ($F = \mathbb{R}^n$, $G = GL_n(\mathbb{R})$)

Definition 5.2: We call $\xi : E \longrightarrow B$ a principal G-bundle if $F = G$ and G acts on F by left multiplication.

Equivalently, $\xi : E \longrightarrow B$ is a principible G-bundle if there is a free action $\gamma : G \times E \longrightarrow E$ on E s.t. ξ is G-invariant ($\xi(g.e) = \xi(e)$) and

1) There exists a trivializing open cover $\mathcal{U} = (U_i \mid i \in I)$ of B with G-equivariant homeomorphisms $h_i : E_i \xrightarrow{\cong} U_i \times G$.
(i.e. $h_i(ge) = g h_i(e) = (\xi(e), g h_i'(e))$.

$$\begin{tikzcd} E_i \arrow[rr] \arrow[dr, "\xi"'] & & U_i \times G \arrow[dl, "\mathrm{proj}"] \\ & U_i & \end{tikzcd}$$

Definition 5.3: A morphism between (F,G)-bundles ξ and ξ' are maps $f, \hat{f}$ in

$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "\xi"] & E' \arrow[d, "\xi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

with some difficult condition concerning the G-action we will later specify.

For principal G-bundles, $\hat{f}$ is assumed to be G-equivariant.

If $f = id_B$ and $\hat{f}$ is a homeomorphism, then ξ and ξ' are called isomorphic (F,G)-bundles over B, and the set of isomorphism classes is denoted $\underline{\mathrm{Bun}_G^F(B)}$.

Similarly we have $\mathrm{Prin}_G(B)$ of principible G-bundles over B.

A trivial bundle over B is a bundle isomorphic to $F \times B \twoheadrightarrow B$.

Theorem: (Classification Theorem): There exists a space $K = BG$ and a universal (F,G)-bundle $\omega_G : EG \longrightarrow BG$ such that the map

$$[X, K] \xrightarrow{\cong} \mathrm{Bun}_G^F(X)$$

sending $[f : X \to K]$ to the pull-back $f^*(\omega_G)$ is a bijection.
In other words: every (F,G)-bundle $\xi : E \longrightarrow X$ is the pull-back of ω_G under some classifying map $f_\xi : X \longrightarrow BG$ and this map is unique up to homotopy.

The proof will just be applying Brown Representability, so let's prepare that.

Definition 5.4: The pull-back of a map $\xi : E \longrightarrow B$ along a map $f : B' \longrightarrow B$ is defined as $f^*(\xi) : B' \times_B E \longrightarrow B'$, where

$$B' \times_B E := \{ (b', e) \in B' \times E \mid f(b') = \xi(e) \}$$

$$\begin{tikzcd}
B' \times_B E \arrow[r, "pr_2"] \arrow[d, "f^*(\xi)"', "= pr_1"] & E \arrow[d, "\xi"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

Proposition 5.1: The pull-back of an (F,G)-bundle is again an (F,G)-bundle and the same holds for principle G-bundles.
Also pullbacks of equivalent bundles are again equivalent.

Proof: The "pull-back" $V_i := f^{-1}(U_i)$ of an atlas on B gives an atlas on B' (exercise)

This makes $\mathrm{Bun}_G^F(-) : CW_0^{conn} \longrightarrow Set_0$ into a contravariant functor, where $\mathrm{Bun}_G^F(X)$ has the trivial bundle as base point.

Definition 5.3: Let us now give the missing condition for $(f, \hat{f})$ to be a morphism between (F,G)-bundles ξ and ξ':

1)
$$\begin{tikzcd}
E \arrow[r, "\hat{f}"] \arrow[d, "\xi"] & E' \arrow[d, "\xi'"] \\
B \arrow[r, "f"] & B'
\end{tikzcd}$$
commutes

2) There are compatible trivializing atlases, i.e. an open cover $(U_i | i \in I)$ of B and $(V_j | j \in J)$ of B' with corresponding trivialization maps h_i, h_j, such that for all i $f(U_i) \subseteq V_{j(i)} \subseteq B'$ for some $j(i) \in J$ and in the diagram

$$\begin{tikzcd}
E_i \arrow[rr, "\tilde{f}"] \arrow[dr] \arrow[dd] & & E'_j \arrow[dr] \arrow[dd] & \\
& U_i \times F \arrow[rr, "(f, \varphi_{ij})"] \arrow[dl] & & V_j \times F \arrow[dl] \\
U_i \arrow[rr, "f"] & & V_j &
\end{tikzcd}$$

the induced map $\varphi_{ij}: U_i \times F \longrightarrow F$ is G-equivariant. *

This gives us the category of (F,G)-bundles: $\mathcal{B}un(F,G)$.

Lemma 5.1: The functor $Bun^F_G(-)$ satisfies the MV-Axiom and the Wedge-Axiom.

* often we actually need that $\varphi_{ij}(x): F \to F$ are homeomorphisms which come from $\rho: G \to Homeo(F)$ via
$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & Homeo(F) \\ & G \arrow[ur] & \end{tikzcd}$$

Proof: "Almost obvious." Exercise.

Homotopy invariance is more involved.

Lemma 5.2: Let $\zeta: E \to B \times I$ be a numerable (F,G)-bundle. Then there is a bundle map
$$\begin{tikzcd} E \arrow[r, "\hat{R}"] \arrow[d, "\zeta"'] & E \arrow[d, "\zeta"] \\ B \times I \arrow[r, "R"] & B \times I \end{tikzcd}$$
with
- ▷ $R(b,t) = (b,1)$ for all $b \in B$, $t \in I$
- ▷ The diagram is a pull-back
- ▷ $\hat{R}|_{B\times 1} = id : E|_{B\times 1} \longrightarrow E|_{B\times 1}$.

Proof: See notes. It uses a locally finite covering with partition of unity.

Corollary 5.1: $\zeta|_{B\times 0} \cong \zeta|_{B\times 1}$

Proof: Restrictions give
$$\begin{tikzcd} & E|_{B\times 0} \arrow[r, "\hat{R}"] \arrow[d, "\zeta|_{B\times 0}"] & E|_{B\times 1} \arrow[d, "\zeta|_{B\times 1}"] & \\ B\times 0 \cong & B \arrow[r, "R \cong id"] & B & \cong B\times 1 \end{tikzcd}$$
and bundle maps over the identity are homeomorphisms (exercise)

Corollary 5.2: If $f_0 \simeq f_1 : B' \longrightarrow B$ are homotopic and $\zeta: E \longrightarrow B$ is an (F,G)-bundle, then $f_0^*(\zeta) \cong f_1^*(\zeta)$ are isomorphic over B'.

Proof: Let $f: B' \times I \longrightarrow B$ be a homotopy. Applying corollary 5.1 to $f^*(\zeta): f^*(E) \longrightarrow B' \times I$ shows $f_0^*(\zeta) \cong f^*(\zeta)|_{B'\times 0} \cong f(\zeta)|_{B'\times 1} \cong f_1^*(\zeta)$.

Proof of Classification Theorem:

The functor $Bun^F_G(-): CW_0^{conn} \longrightarrow Set_0$ is homotopy invariant by corollary 5.2 and satisfies the MV-axiom and Wedge-axiom, so Brown Representability applies.

5.2 More bundle constructions and bundle theory

Corollary 5.3: By proposition 5.1, the restriction of an (F,G)-bundle (resp principle G-bundle) is an (F,G)-bundle (resp. principle G-bundle.)

Proposition 5.2: If $f: E \longrightarrow B$ is an (F,G)-bundle and $f': E' \longrightarrow E$ an (F',G)-bundle, and the spaces are 'nice enough', then the composition $f \circ f': E' \longrightarrow B$ is an $(F \times F', G)$-bundle.

Definition 5.5: Let $G' \leq G$ be a subgroup of the topological group G, F a G-space and $F' \subseteq F$ a G'-invariant subspace.
We say that $\xi' : E' \longrightarrow B$ is an (F', G')-subbundle of the (F, G)-bundle $\xi : E \longrightarrow B$ if there is a map $\varphi : E' \longrightarrow E$

$$\begin{tikzcd}
E' \arrow[rr, "\varphi"] \arrow[dr, "\xi'"'] & & E \arrow[dl, "\xi"] \\
& B &
\end{tikzcd}$$

such that there is a common atlas $(U_i \mid i \in I)$ ~~of~~ for ξ and ξ', such that in the diagram

$$\begin{tikzcd}
E_i' \arrow[rr, "\varphi"] \arrow[dr, "h_i'"] \arrow[dd, "\xi'"'] & & E_i \arrow[dr, "h_i"] \arrow[dd, "\xi"'] & \\
& U_i \times F' \arrow[rr, "(id, \varphi_i)"] \arrow[dl] & & U_i \times F \arrow[dl] \\
U_i \arrow[rr, "id"] & & U_i &
\end{tikzcd}$$

the induced map $\varphi_i : U_i \times F' \longrightarrow F$ gives for each $b \in U_i$ an injective G'-equivariant map $\varphi_i(b, -) : F' \longrightarrow F$.

Example: subvectorbundles, subcoverings.
This raises questions about decompositions of vector bundles into sums of line bundles and of decomposition of coverings.

Definition 5.6: If F is a G-space with a G-equivariant equivalence relation $\sim$, $\bar{F} := F/\sim$ is again a G-space and $F \twoheadrightarrow \bar{F}$ is G-equivariant.
Then for any (F, G)-bundle $\xi : E \longrightarrow B$, there is an induced G-equivariant equivalence relation $\sim$ on E via the local trivializations and $\bar{\xi} : \bar{E} = E/\sim \longrightarrow B$ is an $(\bar{F}, G)$-bundle, called a quotient bundle.

Definition 5.7: A section of an (F, G)-bundle $\xi : E \longrightarrow B$ is a map $s : B \longrightarrow E$ s.t. $\xi \circ s = id_B$.

Example: vector fields.

Lemma 5.3: A principle G-bundle is trivial iff it has a section.

Proof: Each fibre F has a free transitive G-action, so can be identified with G after choosing an element in F corresponding to $1 \in G$.

Lemma 5.4: If $(\hat{f}, id_B)$ is a morphism between (F, G)-bundles $\xi : E \to B$ and $\xi' : E' \longrightarrow B$, then $\hat{f}$ is a homeomorphism

$$\begin{tikzcd}
E \arrow[rr, "\hat{f}"] \arrow[dr, "\xi"'] & & E' \arrow[dl, "\xi'"] \\
& B &
\end{tikzcd}$$

Proof: See Husemöller: Fibre bundles for a proof for the case of

Equivalence of (F,G)-bundles and principle G-bundles.

We construct ~~functors~~ natural transformations

$$\Phi_{F,G} : \mathrm{Prin}_G(B) \longrightarrow \mathrm{Bun}^F_G(B)$$

and

$$\Psi_{F,G} : \mathrm{Bun}^F_G(B) \longrightarrow \mathrm{Prin}_G(B)$$

For a principle G-bundle $\zeta : P \longrightarrow B$ we have a G-action on P so we can define the (F,G)-bundle $\Phi_{F,G}(\zeta) : E \longrightarrow B$ as

$$E := P \times_G F \longrightarrow B,$$

for which we can use the same atlas as for ξ, since

$$(U_i \times G) \times_G F \cong U_i \times F.$$

It's easy to see that $\Phi_{F,G}$ preserves isomorphisms and is natural in B.

(Note that $\Phi_{F,G}$ is not necessarily injective if the G-action on F is not complicated enough.)

Conversely, for an (F,G)-bundle $\zeta : E \longrightarrow B$, consider the set $P = \mathrm{Hom}(t_F, \zeta)$ of morphisms of (F,G)-bundles from the trivial (F,G)-bundle $t_F : F \longrightarrow *$ to ζ.

So P consists of pairs $(\hat{f}, f)$ s.t.

$$\begin{tikzcd} F \arrow[r, "\tilde{f}"] \arrow[d, "t_F"'] & E \arrow[d, "\zeta"] \\ * \arrow[r, "f"] & B \end{tikzcd}$$

1) $\xi \circ \hat{f} = f \circ t_F$,

i.e. for $b := f(*)$, $\mathrm{im}(\hat{f}) \subseteq F_b = \zeta^{-1}(b) \subseteq E$.

2) $\hat{f} : F \xrightarrow{\cong} F_b$ must be a homeomorphism.

3) For each i with $b \in U_i$, we have a commutative diagram

$$\begin{tikzcd} F_b \arrow[r, hook] \arrow[d, "\cong"', "(\hat{f})^{-1}"] & \zeta^{-1}(U_i) \arrow[d, "\cong"', "h_i"] \\ F \cong b \times F \arrow[r, hook] & U_i \times F \end{tikzcd}$$

Define $\Psi_{F,G}(\zeta) : P \longrightarrow B$ by $(\hat{f}, f) \longmapsto f(*)$.

Since for given $b \in B$, the choice of $(\hat{f}, f) \in P$ with $f(*) = b$ comes down to choosing a homeomorphism $\hat{f} : F \longrightarrow F_b$ compatible with G, the fibre of $\Psi_{F,G}(\zeta)$ is homeomorphic to $\rho(G) \leq \mathrm{Homeo}(F)$, the image of the action map $\rho : G \to \mathrm{Homeo}(F)$.

We could give an atlas to show it is an $\rho(G)$-bundle, giving us $\Psi_{F,G} : \mathrm{Bun}^F_G(B) \longrightarrow \mathrm{Prin}_{\rho(G)}(B)$.

Theorem 5.2: If $\rho : G \longrightarrow \mathrm{Homeo}(F)$ is a faithful action (i.e. injective) then $\rho(G) \cong G$ and we have a natural equivalence

$$\mathrm{Bun}^F_G(-) \underset{}{\overset{\Psi_{F,G}}{\rightleftarrows}} \mathrm{Prin}_G(-)$$

5.3 Classifying space BG and the Milnor construction

In this section, we will give an explicit model/construction for the classifying space BG and universal bundle $\omega_G : EG \longrightarrow BG$.
Recall that this means that for each principal G-bundle $\xi : P \to X$, there is a classifying map $f : X \longrightarrow BG$ s.t. $\xi = f^*(\omega_G)$ is the pull-back of ω_G under f, and f is unique up to homotopy.

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, "\xi"'] & EG \arrow[d, "\omega_G"] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

Milnor construction:
Let $E_nG := G * \dots * G$ be the n-fold join of the topological group G, which has as points equivalence classes

$$[t_0 g_0 : t_1 g_1 : \dots : t_n g_n] \quad \text{with } \sum_{i=0}^{n} t_i = 1$$

where we identify such tuples only if $t_i = t_i'$ for all i, and $g_i = g_i'$ if $t_i = t_i' \neq 0$, but if $t_i = 0$, g_i is irrelevant!
Using the inclusion $E_nG \hookrightarrow E_{n+1}G$

$$[t_0 g_0 : \dots : t_n g_n] \mapsto [t_0 g_0 : \dots : t_n g_n : 0, 1]$$

we can define the Milnor total space as

$$EG := \underset{n}{\operatorname{colim}}\, E_nG,$$

for which we can write the points as

$$[t, g] = [t_0 g_0 : t_1 g_1 : t_2 g_2 : \dots]$$

with only finitely many nonzero t_i, and $\sum_{i=0}^{\infty} t_i = 1$.
For describing the topology on EG (the limit topology), we define the maps

$$\pi_i : EG \longrightarrow [0,1]$$
$$[t, g] \longmapsto t_i$$

and

$$\gamma_i : EG(i) := \pi_i^{-1}((0,1]) \longrightarrow G$$
$$[t, g] \longmapsto g_i$$

The topology on EG is the finest such that all π_i and γ_i are continuous maps.
We have a G-action on EG by

$$h[t, g] := [t, hg] = [t_0 hg_0 : t_1 hg_1 : \dots : t_n hg_n : \dots]$$

This is a free action and it leaves $EG(i)$ invariant.
Furthermore, π_i is G-invariant and γ_i is G-equivariant.

Define the Milnor universal bundle as the projection map

$$w_G : EG \twoheadrightarrow EG/G =: BG \qquad [t,g] \mapsto [[t,g]]$$

Theorem 5.3: $w_G : EG \longrightarrow BG$ is a principle G-bundle.

Proof: Take atlas $U_i = BG(i) := EG(i)/G$ with coordinates

$$\begin{tikzcd} EG(i) \arrow[rr, "h_i"] \arrow[dr, "w_G"'] & & BG(i) \times G \arrow[dl] \\ & BG(i) & \end{tikzcd} \qquad [t,g] \mapsto ([[t,g]], g_i)$$

The compositions $h_j \circ (h_i)^{-1}$ are G-equivariant, since the action is just by left translation.

Our next goal is to show that w_G is a universal G-bundle.

Lemma 5.5: The identity map $id_{EG} : EG \longrightarrow EG$ is homotopic to the map $j : EG \longrightarrow EG$ given by

$$j([t,g]) = [t_0, g_0 : 0, 1 : t_1, g_1 : 0, 1 : t_2, g_2 : \dots]$$

via a G-equivariant homotopy.

Proof: We construct the homotopy from j to id_{EG} by concatenating infinitely many simpler homotopies, the first of which is

$$[t,g] \longmapsto [t_0, g_0 : st_1, g_1, (1-s)t_1, g_1 : st_2, g_2 : (1-s)t_2, g_2 : \dots]$$

starting in j and ending in

$$[t,g] \mapsto [t_0, g_0 : t_1, g_1 : 0, 1 : t_2, g_2 : 0, 1 : t_3, g_3 : \dots]$$

and thus removing one of the zeroes.

Similarly we get a homotopy removing the n-th zero:

$$[t,g] \mapsto [t_0, g_0 : t_1, g_1, \dots : t_{n-1}, g_{n-1} : st_n, g_n : (1-s)t_n, g_n : st_{n+1}, g_{n+1}]$$

We get a homotopy from j to id_{EG} by applying the n-th homotopy in the interval $[\frac{n-1}{n}, \frac{n}{n+1}]$, which will give rise to a continuous function since for each $[t,g]$ only finitely many homotopies act non-trivially on it (as only finitely many t_i are non-zero.) □

Lemma 5.6: If $f_0, f_1 : X \longrightarrow BG$ are maps such that the pullbacks $f_0^*(w_G) \cong f_1^*(w_G)$ are isomorphic, then $f_0 \simeq f_1$ are homotopic.

Proof: Let $\hat{f}_0, \hat{f}_1 : \overset{f_0^*(EG) \cong f_1^*(EG)}{\overset{\cong}{P}} \longrightarrow EG$ be the corresponding morphisms of total spaces. By composing with j above, $\hat{f}_0$ is up to homotopy of the form

$$\hat{f}_0(y) = [t_0(y), g_0(y) : 0, 1 : t_1(y), g_1(y) : 0, 1 : t_2(y), g_2(y) \dots]$$

and similarly $\hat{f}_1$ is of the form (after applying G-equivariant homotopy)

$$\hat{f}_1(y) = [0, 1 : t_0'(y), g_0'(y) : 0, 1 : t_1'(y), g_1'(y) : 0, 1 : \dots].$$

Then $\hat{f}_0$ and $\hat{f}_1$ are homotopic along the G-equivariant homotopy

$$y \mapsto [st_0, g_0 : (1-s)t_0', g_0' : st_1, g_1 : (1-s)t_1', g_1' : st_2, g_2 : \dots]$$

This homotopy descends to a homotopy from f_0 to f_1. □

Note that this proves uniqueness of a classifying map up to homotopy

<u>Theorem 5.4</u>: $\omega_G : EG \longrightarrow BG$ is a <u>universal</u> bundle

<u>Proof</u> Let $\zeta : P \longrightarrow X$ be a principle G-bundle. We want to find $f : X \longrightarrow BG$ st. $\zeta \cong f^*(\omega_G)$, since by lemma 5.6 we already have uniqueness up to homotopy.

Let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be an atlas of X with a partition of unity $\{t_i : X \longrightarrow [0,1]\}$ and trivializations $h_i : \zeta|_{U_i} \xrightarrow{\cong} U_i \times G$.

Let γ_i be the composition

$$\gamma_i : P_i := \zeta^{-1}(U_i) \xrightarrow[\cong]{h_i} U_i \times G \longrightarrow\!\!\!\!\rightarrow G$$

Define $\hat{f} : P \longrightarrow EG$ by

$$\hat{f}(y) = [t_0(\zeta(y)), \gamma_0(y) : t_1(\zeta(y)), \gamma_1(y) : \dots],$$

which is well-defined, since

- ▷ For all y, only fin. many $t_i(\zeta(y)) \neq 0$ and $\sum_{i=0}^{\infty} t_i(\zeta(y)) = 1$
- ▷ If $\gamma_i(y)$ is not defined, then $y \notin \zeta^{-1}(U_i)$, so $\zeta(y) \notin U_i$ and $t_i(\zeta(y)) = 0$.

The γ_i are G-equivariant, so $\hat{f}$ is and thus induces a map $f : X \longrightarrow BG$. Then $\hat{f}$ factors over $f^*(EG)$:

$$\begin{tikzcd}
P \arrow[rr, "\hat{f}"] \arrow[dr] \arrow[dd, "\zeta"'] & & EG \arrow[dd, "\omega_G"] \\
& f^*(EG) \arrow[ur] \arrow[dl, "f^*(\omega_G)"] & \\
X \arrow[rr, "f"] & & BG
\end{tikzcd}$$

So by lemma 5.4 (page 531), $\zeta \cong f^*(\omega_G)$.

So $\omega_G : EG \longrightarrow BG$ is a universal principle G-bundle. □

<u>Theorem 5.5</u>: EG is contractible.

This follows easily from lemma 5.5, since $id_{EG} \simeq j$, and $j \simeq *$ via the null homotopy

$$[st_0, g_0 : 1-s, 1 : st_1, g_1 : 0, 1 : st_2, g_2 : 0, 1 : \dots]$$

<u>Lemma 5.7</u>: (Conjugation with $\sigma \in G$ on EG induces identity on BG.

I don't get the statement of this lemma sorry.

Lemma 5.8: There is a map $\varphi : G \longrightarrow \Omega BG$ sitting inside a map of fibre bundles

$$\begin{tikzcd}
 & G \arrow[r, "\varphi"] \arrow[d] & \Omega BG \arrow[d] & \\
* \simeq & EG \arrow[r, "\tilde{\varphi}"] \arrow[d] & PBG \arrow[d] & \simeq * \\
 & BG \arrow[r, equal] & BG &
\end{tikzcd}$$

Proof: For $[t,g] \in EG$, define the path $\tilde{\varphi}([t,g])$ in BG by the composition of two paths from

$$b_0 = [[1,1 : 0,1 : 0,1 : \dots]] = [[1,1 : 0,g_0 : 0,g_1 : 0,g_2 : \dots]]$$

to

$$[[0,1 : t_0,g_0 : t_1 g_1 : t_2,g_2 : \dots]]$$

and after that a path similar to the one in lemma 5.5 to

$$[[t_0 g_0 : t_1 g_1 : t_2,g_2 : \dots]] = [[t,g]]$$

In particular if $[t,g] = [1,g_0 : 0,1 : 0,1 : \dots]$ lies in the fibre of ω_G over the basepoint b_0 in BG, this gives a closed loop in BG.

Corollary 5.4: $\varphi : G \longrightarrow \Omega BG$ is a homotopy equivalence~~s~~.

Proof: Clear from long exact sequences, since id_{BG} and $\tilde{\varphi}$ are.

Corollary 5.5: $\pi_n(BG) \cong \pi_{n-1}(G)$ for all n

Corollary 5.6: If G is discrete, BG is a $K(G,1)$.

Theorem 5.6: A principal G-bundle $\zeta : P \longrightarrow B$ is universal if and only if P is contractible.

Proof: The proof was omitted in the lecture. I will provide a proof for the "only-if" direction. I haven't found any proof for the "if" direction and wonder whether it's actually true.

If $\zeta : P \to B$ is universal, then universality of ζ gives us a map $\varphi : EG \longrightarrow P$ and universality of ω_G gives us a map $\psi : P \longrightarrow EG$, and both compositions are homotopic to the identity by universality, so $P \simeq EG$; and EG is contractible by theorem 5.5.

5.4 Classifying space of a category

Definition 5.8: For a category C we define a simplicial set $\mathbb{B}_\bullet C$ as follows:

$$\mathbb{B}_q C := \{ (X_0, \dots, X_q ; \varphi_0, \dots, \varphi_{q-1}) \mid X_0 \xrightarrow{\varphi_0} X_1 \to \dots \xrightarrow{\varphi_{q-1}} X_q \}$$

We have maps $d_i : \mathbb{B}_q C \to \mathbb{B}_{q-1} C$ for $0 \le i \le q$ that "forget about X_i":

$$d_i (X_0, \dots, X_q ; \varphi_0, \dots, \varphi_{q-1}) = \begin{cases} (X_1, \dots, X_q ; \varphi_1, \dots, \varphi_{q-1}) & i=0 \\ (X_0, \dots, \hat{X_i}, \dots, X_q ; \varphi_0, \dots, \varphi_{i+1}\varphi_i, \dots, \varphi_{q-1}) \\ (X_0, \dots, X_{q-1} ; \varphi_0, \dots, \varphi_{q-2}) & i=q \end{cases}$$

and similarly maps $s_i : \mathbb{B}_{q-1} C \to \mathbb{B}_q C$ that "sneak in id_{X_i}":

$$s_i (X_0, \dots, X_{q-1}, \varphi_0, \dots, \varphi_{q-2}) = (X_0, \dots, X_{i-1}, X_i, X_i, X_{i+1}, \dots, X_{q-1} ; \varphi_0, \dots, \varphi_{i-1}, id_{X_i}, \varphi_i, \dots, \varphi_{q-2})$$

One can check that these satisfy the simplicial relations.

We now define the <u>classifying space BC</u> of C as the geometric realization of $\mathbb{B}_\bullet C$:

$$BC := |\mathbb{B}_\bullet C| = \left(\coprod_{q \ge 0} \mathbb{B}_q C \times \Delta^q \right) / \sim$$

(If C has a topology on the objects/on the morphisms we could give each $\mathbb{B}_q C$ a topology by regarding it as a subset of $Ob(C)^q \times Mor(C)^{q-1}$.)

Example 5.2: For a group G we have the two categories $\widetilde{C}_G$ and C_G:

- $Ob(\widetilde{C}_G) = G$

  $Mor(g, g') = \{ (g, g') \}$

$$\widetilde{C}_G \qquad \begin{tikzcd} g \arrow[r, "{(g,g')}"] & g' \\ 1 \arrow[u, "{(1,g)}"] \arrow[ur, "{(1,g')}"'] & \end{tikzcd}$$

- $Ob(C_G) = \{ \mathbb{1} \}$

  $Mor(\mathbb{1}, \mathbb{1}) = G$

$$C_G \qquad \begin{tikzcd} \bullet \arrow[loop above, "g'"] \arrow[loop right, "g"] \end{tikzcd}$$

One can check that:

$$B\widetilde{C}_G \simeq EG, \quad BC_G \simeq BG$$

Observations:

- Let $\Phi : C \to C'$ be a <u>functor</u>. This induces a simplicial map $\mathbb{B}_\bullet \Phi : \mathbb{B}_\bullet C \to \mathbb{B}_\bullet C'$ via

  $$(X_0, \dots, X_q ; \varphi_0, \dots, \varphi_{q-1}) \mapsto (\Phi(X_0), \dots, \Phi(X_q), \Phi(\varphi_0), \dots, \Phi(\varphi_{q-1}))$$

  and thus a map $B\Phi : BC \to BC'$ between spaces.

- For the trivial category $C_{triv} = (p, id_p)$, we have $BC_{triv} \simeq *$

$$C_{triv}: \begin{tikzcd} p \arrow[loop right, "id_p"] \end{tikzcd}$$

- For the "interval category" $\mathcal{I} = (\{0, 1\}, \{id_0, id_1, t: 0 \to 1\})$ we have $B\mathcal{I} \simeq [0,1]$

$$\mathcal{I}: \begin{tikzcd} 0 \arrow[loop left, "id_0"] \arrow[r, "t"] & 1 \arrow[loop right, "id_1"] \end{tikzcd}$$

▷ A natural transformation $\eta : \Phi \longrightarrow \Psi$ between functors induces a homotopy $B\eta$ from map $B\Phi$ to $B\Psi$.

For this first observe that a natural transformation η between $\Phi, \Psi : \mathcal{C} \longrightarrow \mathcal{C}'$, is equivalent to a functor

$$\eta' : \mathcal{C} \times \mathcal{I} \longrightarrow \mathcal{C}'$$

$$\text{on objects } \left\{ \begin{array}{l} (x, 0) \longmapsto \Phi(x) \\ (x, 1) \longmapsto \Psi(x) \end{array} \right.$$

$$\text{on morphisms } \left\{ \begin{array}{l} (\varphi, id_0) \longmapsto \Phi(\varphi) \\ (\varphi, id_1) \longmapsto \Psi(\varphi) \\ (id_x, t) \longmapsto \eta_x : \Phi(x) \longrightarrow \Psi(x) \end{array} \right.$$

So this functor η' gives a map

$$\begin{tikzcd}
B\eta' : & B(\mathcal{C} \times \mathcal{I}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & B\mathcal{C}' \\
& B\mathcal{C} \times B\mathcal{I} \arrow[d, phantom, sloped, "\cong"] & \\
& B\mathcal{C} \times I \arrow[uur] &
\end{tikzcd}$$

that on $B\mathcal{C} \times 0$ is $B\Phi$ and on $B\mathcal{C} \times 1$ is $B\Psi$.

▷ We have a functor $v : \widehat{\mathcal{C}_G} \longrightarrow \mathcal{C}_G$ sending all objects g to $\mathbb{1}$ and the morphism $(g, g') : g \longrightarrow g'$ to $g' g^{-1} : \mathbb{1} \to \mathbb{1}$.

~~B~~Then $Bv : B\widehat{\mathcal{C}_G} \longrightarrow B\mathcal{C}_G$ corresponds to $\omega_G : EG \longrightarrow BG$.

▷ $\widehat{\mathcal{C}_G}$ is equivalent to the trivial category $\mathcal{C}_{triv} = (p, id_p)$.

$$\begin{array}{rl} T : \widehat{\mathcal{C}_G} & \longrightarrow \mathcal{C}_{triv} \\ g & \longmapsto p \\ (g g') & \longmapsto id_p \end{array}$$

$$\begin{array}{rl} S : \mathcal{C}_{triv} & \longrightarrow \widehat{\mathcal{C}_G} \\ p & \longmapsto 1 \in G = ob(\widehat{\mathcal{C}_G}) \\ id_p & \longmapsto (1, 1) = id_1 . \end{array}$$

Clearly $T \circ S = Id_{\mathcal{C}_{triv}}$.

Conversely define $\eta : S \circ T \longrightarrow Id_{\widehat{\mathcal{C}_G}}$ by setting for all $g \in G$

$$\eta_g = (1, g) : \underset{\overset{\shortparallel}{S \circ T(g)}}{1} \longrightarrow \underset{\overset{\shortparallel}{Id(g)}}{g}$$

$$\cancel{S \circ T(gg')}$$

$$\begin{tikzcd}
S \circ T(g) \arrow[d, phantom, sloped, "="] & S \circ T(g') \arrow[d, phantom, sloped, "="] \\
1 \arrow[r, "{(1,1)}"] \arrow[d, "\eta_g"', "{(1,g)}"] \arrow[dr, dashed, "{(1,g')}"] & 1 \arrow[d, "\eta_{g'}"] \\
id(g) = g \arrow[r, "{(g g')}", "id(gg')"'] &
\end{tikzcd}$$

So $\widehat{\mathcal{C}_G} \cong \mathcal{C}_{triv}$.

This gives a different proof of contractibility of $EG \simeq B\widehat{\mathcal{C}_G} \simeq B\mathcal{C}_{triv} \simeq *$. ▷

Chapter 6: Characteristic Classes

Let G be a topological group and F a faithful G-space.

Definition 6.1: A characteristic class is an assignment

$$c_X : \mathrm{Bun}_G^F(X) \longrightarrow H^k(X;R) \qquad \text{for all } X$$

such that $c_{B'}(f^*(\xi)) = f^*(c_B(\xi))$ for all (F,G)-bundles $\xi : E \to B$ and maps $f : B' \longrightarrow B$, where $f^*(\xi) : E' \longrightarrow B'$ is the pull-back bundle and $f^* : H^k(B^*;R) \longrightarrow H^k(B';R)$

In other words, it's a natural transformation

$$c : \mathrm{Bun}_G^F(-) \longrightarrow H^k(-;R)$$

of functors for some k.

The collection $\mathrm{CharCl}(G,R)$ of such characteristic classes forms an R-algebra, by using the R-algebra structure of $H^*(X;R)$ pointwise

Theorem 6.1: There is an isomorphism

$$\mathrm{CharCl}(G,R) \cong H^*(BG;R)$$

Proof: $\mathrm{Bun}_G^F(-) \cong [-,BG]$ is representable, so by Yoneda

$$\begin{aligned}\mathrm{CharCl}(G,R) &= \mathrm{NatTrans}(\mathrm{Bun}_G^F(-),\ H^*(-;R))\\ &\cong \mathrm{NatTrans}([-,BG],\ H^*(-;R))\\ &\cong H^*(BG;R).\end{aligned}$$

Example 6.1 (Stiefel-Whitney classes)

For $G = O(n)$, $F = \mathbb{R}^n$, Bun_G^F = n-dimensional real vector bundles.

With $R = \mathbb{Z}/2$, we see

$$\mathrm{CharCl}_*(O(n);\mathbb{Z}/2) \cong H^*(BO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[W_1,\dots,W_n].$$

The $W_1,\dots,W_n$ are called the universal Stiefel-Whitney classes.

Similarly for $G = U(n)$, $F = \mathbb{C}^n$, we get complex vector bundles and

$$\mathrm{CharCl}_*(U(n);\mathbb{Z}) \cong \mathbb{Z}[Z_1,\dots,Z_n] \qquad (|Z_i| = 2)$$

The $Z_1,\dots,Z_n$ are called universal Chern classes.

Remark: We have a ring map $cl : H^*(BG;R) \longrightarrow \mathrm{CharCl}_*(G,R)$ by sending $\alpha \in H^k(BG;R)$ to the characteristic class $cl(\alpha) : \mathrm{Prin}_G(X) \longrightarrow H^k(X,R)$ via the composition

$$\begin{tikzcd}
\mathrm{Prin}_G(X) \arrow[r, "\cong"] & {[X,BG]} \arrow[r] & H^k(X,R) \\
{[\xi]} \arrow[r, mapsto] & {[f_\xi]} \arrow[r, mapsto] & f_\xi^*(\alpha).
\end{tikzcd}$$

Theorem 6.1 says that cl is an isomorphism of rings.

Definition 6.2: (Stiefel-Whitney classes)

Define the i-th Stiefel-Whitney class w_i as the characteristic class $w_i := cl(W_i)$ associated to $W_i \in H^i(BO(n);R) \cong \mathbb{Z}/2[W_1,\dots,W_n]$

$(X) = \{n\text{dim vec bun. over } X\} \longrightarrow H^i(X;\mathbb{Z}/2)$

takes an n-dimensional real vector bundle $\xi : E \longrightarrow X$ and associates to it $w_i(\xi) := f_\xi^*(W_i) \in H^i(X; \mathbb{Z}/2)$, where $f_\xi : X \longrightarrow BO(n)$ is the classifying map for ξ.

Also set $w_0(\xi) = 1$, $w_i(\xi) = 0$ if $i > n$, for all ξ.

<u>Lemma 6.1 (Cartan formula)</u>

For $w(\xi) := \sum_{i=0}^{\infty} w_i(\xi)$ (note: finite sum), we have

$$w(\xi_1 \oplus \xi_2) = w(\xi_1) \cup w(\xi_2).$$

<u>Lemma 6.2</u>: For a real vector bundle ξ

$$w_1(\xi) = 0 \iff \xi \text{ is orientable}$$

<u>Proof</u>: Exercise

<u>Lemma 6.3</u>: If $\mathbb{R}P^n$ is parallellizable, then $n = 2^k - 1$ for some k.

<u>Proof</u>: Let $\tau : T\mathbb{R}P^n \longrightarrow \mathbb{R}P^n$ be the tangent bundle. If τ is trivial, then $w_i(\tau) = 0$ for all $i > 0$.

If $\alpha \in H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/\alpha^{n+1}$ is the generator, we apparently have in general that

$$w(\tau) = (1 + \alpha)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} \alpha^i = \sum_{i=0}^{n} \binom{n+1}{i} \alpha^i$$

Thus $w_i(\tau) = 0$ implies $\binom{n+1}{i} \equiv 0 \pmod 2$ for all $1 \le i \le n$, which is only the case if $n+1$ is a power of 2.