Algebraic Topology II - Notes - Prof. C.F. Bödigheimer

Lecture 1 - 09-04-18

"And if I had said everthing mod 2... that would have been the truth."

Overview of the lecture course:

1. Fibre bundles

- Principal G-bundles
- Fibre bundle of type (F,G)
- Classification: classifying space $EG \to BG$

$$\mathrm{Isom}^F_G(X) \cong [X, BG]$$

- Coverings, vector bundles

2. Eilenberg-MacLane spaces

- $K(G,n)$ is a space with only nontrivial homotopy group π_n equal to G.
- Construction of a $K(G,n)$.
  1) Brute force (attaching higher and higher cells)
  2) Infinite symmetric product $SP_\infty(X) := \lim\limits_{n} X^n/\mathfrak{S}_n$

     Dold-Thom theorem: $\pi_i(SP_\infty(X)) \cong H_i(X,\mathbb{Z})$

     So $K(G,n) \cong SP_\infty(M(G,n))$
- <u>Theorem</u>:

$$[X, K(G,n)] \cong H^n(X;G)$$

- Product on the left is induced by $G \times G \to G$ (G abelian) giving a product

$$K(G,n) \times K(G,n) \cong K(G\times G, n) \longrightarrow K(G,n)$$

- We have $\Omega K(G,n) \simeq K(G,n-1)$, since

$$\begin{aligned}[X, K(G,n-1)] \cong H^{n-1}(X;G) &\cong H^n(\Sigma X; G)\\ &\cong [\Sigma X, K(G,n)]\\ &\cong [X, \Omega K(G,n)].\end{aligned}$$

So $K(G,n)$ contains more information than $K(G,n-1)$

- Let $E_0, E_1, \ldots$ be a sequence of spaces with maps $\sigma_n : \Sigma E_n \longrightarrow E_{n+1}$ (called a "spectrum")

Define

$$h^n(X) := \lim \left\{ \begin{tikzcd}[\Sigma^k X, E_{n+k}] \arrow[dr] & & [\Sigma^{k+1} X, E_{n+1+k}] \\ & [\Sigma^{k+1} X, \Sigma E_{n+k+1}] \arrow[ur, "\sigma^*_{n+k}"] & \end{tikzcd} \right.$$

This is a ~~R~~cohomology theory.

3. Spectral homology and cohomology theories.

- ▷ A family of functors

4. Spectral sequences

- ▷ $E^{r+1}_{p,q} = H_*(E^r_{p,q}, d^r)$
- ▷ Application: fibre bundle $F \to E \to B$
  Express $H_*(E)$ using $H_*(B)$ and $H_*(F)$
  (Serre spectral sequence)

5. Homology of fibre spaces

6. Theorems of Serre

Recommended literature for this course:
Hatcher, Bredon, Spanier, Weibel, tom Dieck.

So, let's start ^^

0. Groups and co-groups up to homotopy

Definition 0.1 A pointed space M with basepoint $x_0 = 1$ is called a homotopy monoid if there is a map $\mu : M \times M \to M$, written $\mu(x,y) = x \cdot y$, such that

1) $\mu \circ \iota_1 \simeq id$ where $\iota_1(x) = (x,1)$ "$\mu(x,1) \simeq x$"
2) $\mu \circ \iota_2 \simeq id$ where $\iota_2(x) = (1,x)$ "$\mu(1,x) \simeq x$"

$$\begin{tikzcd} & M\times M \arrow[dr, "\mu"] \arrow[d, phantom, "\simeq"] & \\ M \arrow[ur, "\iota_1"] \arrow[rr, "id"] \arrow[dr, "\iota_2"'] & {} & M \\ & M\times M \arrow[ur, "\mu"'] \arrow[u, phantom, "\simeq"] & \end{tikzcd}$$

Remark: a) 1 is a neutral element up to homotopy
b) These spaces are called H-spaces.
c) We don't insist on connectivity

Corollary 0.1: $\pi_0(M)$ is a discrete monoid, with the path component $W(1)$ of 1 as (strict) neutral element.

Corollary 0.2: $\pi_1(M,1)$ is abelian (exercise)

Definition 0.2: M is called h-associative if $\mu \circ (id \times \mu) \simeq \mu \circ (\mu \times id)$

$$\begin{tikzcd} & M\times M \arrow[dr, "\mu"] & \\ M\times M\times M \arrow[ur, "id\times\mu"] \arrow[dr, "\mu\times id"'] & & M \\ & M\times M \arrow[ur, "\mu"'] & \end{tikzcd}$$

M is called <u>h-commutative</u> if $\mu \circ T \simeq \mu$.   "$\mu(x,y) \simeq \mu(y,x)$"
where $T(x,y) = (y,x)$

M has an <u>h-inverse</u> if there is a map $I : M \to M$ s.t.

- $I(1) = 1$
- $\mu \circ (I \times id) \simeq c_1 \simeq \mu(id \times I)$
  "$1 \simeq \mu(I(x),x) \simeq \mu(x,I(x))$"

$$\begin{tikzcd} M \arrow[r, "I\times id"] \arrow[d, "id\times I"'] \arrow[dr, "1" description, "\simeq", "\simeq"'] & M\times M \arrow[d, "\mu"] \\ M\times M \arrow[r, "\mu"'] & M \end{tikzcd}$$

<u>Examples</u>:

1) Topological groups.
   in particular Lie Groups
   $S^1, S^1 \times S^1, \ldots, T^k \cong (S^1)^k$
   $(\mathbb{R},+), (\mathbb{C},+), (\mathbb{R}^*,\cdot), (\mathbb{C}^*,\cdot)$
   $SO(n), O(n), SU(n)$
   For M a manifold:
   $\mathrm{Diff}(M) \supseteq \mathrm{Diff}^+(M)$ (i.e. oriented differentiable maps)
   $\supseteq \mathrm{Diff}_{\simeq id}(M)$
   For a space X:
   $\mathrm{Homeo}(X) \supseteq \mathrm{Homeo}_{\simeq id}(X)$

2) $M = \Omega X$, the closed loop space of a ~~loop~~ space X.

$$(\omega_1 \cdot \omega_2)(t) = \begin{cases} \omega_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \omega_2(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

1 is constant path at x_0

This is <u>not</u> strictly h-associative and 1 is <u>not</u> strictly a unit. (But it <u>is</u> h-associative and has an h- inverse)

3) <u>Moore loop space</u>

$M(X, x_0) :=$ ~~$\{(w,a) \in \mathrm{Map}([0,a],X) \times \mathbb{R}$~~
$= \{(\omega, a) \mid w : [0,a] \to X,\ w(0) = w(a) = x_0\}$

$(\omega_1, a_1) \cdot (\omega_2, a_2) = (\omega_3 * a_1 + a_2)$

4) $C := \coprod_{k \geq 0} C^k(\mathbb{R}^m)$, where $C^k(X)$ is the unordered configuration space

$C^0(\mathbb{R}^m) = \{\emptyset\}$ (empty configuration)
$C^1(\mathbb{R}^m) = \mathbb{R}^m$
$C^2(\mathbb{R}^m) \cong \mathbb{R}^m \times \mathbb{RP}^{m-1} \times \mathbb{R}_{>0}$

Define $C^k(\mathbb{R}^m) \times C^\ell(\mathbb{R}^m) \longrightarrow C^{k+\ell}(\mathbb{R}^m)$ by

This does not have an h-inverse.

It is h-commutative, if $m \geq 2$

Lecture 2 - 11-04-18

Last time we saw the h-space $M(X, x_0)$ the Moore loop space.
It contains $\Omega(X, x_0)$ via the inclusion

$$\begin{aligned} \Omega(X, x_0) &\longrightarrow M(X, x_0) \\ \omega &\longmapsto (\omega, 1) \end{aligned}$$

It turns out that this inclusion is a homotopy equivalence.

Given a group G we have a classifying space BG (Milner construction.)
Do we have a similar classifying space BM for a topological monoid M up to homotopy?

4) Another example of a homotopy monoid was

$$C(\mathbb{R}^m) := \coprod_{k\geq 0} C^k(\mathbb{R}^m)$$

where

$$C^k(\mathbb{R}^m) := \{(x_1, \dots x_k) \in (\mathbb{R}^m)^k \mid x_i \neq x_j\} / \mathfrak{S}_k$$

is the unordered configuration space.
We get an h-multiplication $\mu: C^k(\mathbb{R}^m) \times C^\ell(\mathbb{R}^m) \longrightarrow C^{k+\ell}(\mathbb{R}^m)$ via

Example 4) leads to the idea of an "operad"

$\mathcal{O} = \{\mathcal{O}_k\}_{k\in\mathbb{N}}$. ($\mathcal{O}_k$ is a connected space
$\mathcal{O}_0$ is often just a base point)

Instead of putting the two boxes at fixed positions in above example, we could add anoter parameter that tells us where to put them. For this we need 2 different points in the space $\mathbb{R}^m$ around which we put the boxes, so: an element of $\tilde{C}^2(\mathbb{R}^m)$

We see that for every $\beta \in \tilde{C}^{2}(\mathbb{R}^m)$, we have a map

$$\mu_\beta : \tilde{C}^{k}(\mathbb{R}^m) \times \tilde{C}^{\ell}(\mathbb{R}^m) \longrightarrow \tilde{C}^{k+\ell}(\mathbb{R}^m).$$

In other words we have a map

$$\tilde{C}^{2}(\mathbb{R}^m) \times \tilde{C}^{k}(\mathbb{R}^m) \times \tilde{C}^{\ell}(\mathbb{R}^m) \longrightarrow \tilde{C}^{k+\ell}(\mathbb{R}^m)$$

$$(\beta, \xi_1, \xi_2) \longmapsto \mu_\beta(\xi_1, \xi_2).$$

This idea can be generalized. Let $\mathcal{O} = \{\mathcal{O}_k\}_{k \in \mathbb{N}}$ be a set of spaces as before. We can consider maps of the form

$$\vartheta^{n}_{(k_1, \dots, k_n)} : \mathcal{O}_n \times \mathcal{O}_{k_1} \times \dots \times \mathcal{O}_{k_n} \longrightarrow \mathcal{O}_{k_1 + k_2 + \dots + k_n},$$

so in some sense putting n "configurations" together, using a parameter living in $\mathcal{O}_n$.

Now consider r of these maps $\vartheta^1, \dots, \vartheta^r$:

$$\vartheta^i : \mathcal{O}_{n_i} \times \mathcal{O}_{k_{1,i}} \times \dots \times \mathcal{O}_{k_{n_i,i}} \longrightarrow \mathcal{O}_{k_{1,i} + \dots + k_{n_i,i}}$$

We can now make a map from

$$\mathcal{O}_r \times (\mathcal{O}_{n_1} \times \mathcal{O}_{k_{1,1}} \times \dots \times \mathcal{O}_{k_{n_1,1}}) \times \dots \times (\mathcal{O}_{n_r} \times \mathcal{O}_{k_{r,r}} \times \dots \times \mathcal{O}_{k_{n_r,r}})$$

to

$$\mathcal{O}_{(k_{1,1} + k_{2,1} + \dots + k_{n_1,1}) + \dots + (k_{r,1} + k_{r,2} + \dots + k_{n_r,n_r})}$$

in two ways:

- Either by first going to $\mathcal{O}_r \times \mathcal{O}_{k_{1,1} + \dots + k_{n,1}} \times \dots \times \mathcal{O}_{k_{r,1} + \dots + k_{r,n_r}}$ using the ϑ^i, and then use another ϑ.
- Or by first collecting the $\mathcal{O}_r \times (\mathcal{O}_{n_1} \times \dots \times \mathcal{O}_{n_r})$ to the front and use some ϑ^* to map this to $\mathcal{O}_{n_1 + \dots + n_r}$, and then use a map

$$\mathcal{O}_{n_1 + \dots + n_r} \times (\mathcal{O}_{k_{1,1}} \times \dots \times \mathcal{O}_{k_{n_1,1}}) \times (\dots) \times \dots \times (\mathcal{O}_{k_{r,r}} \times \dots \times \mathcal{O}_{k_{n_r,r}})$$

$$\longrightarrow \mathcal{O}_{(k_{1,1} + k_{2,1} + \dots + k_{n_2,1}) + \dots + (k_{r,1} + \dots + k_{n_r,r})}$$

Actions of h-spaces on spaces

Let M be an h-space with unit $1 \in M$, and X be a space.

We define an action of M on X to be a map $\rho : M \times X \longrightarrow X$ s.t.

(1) $\rho(m_1 \cdot m_2, x) \simeq \rho(m_1, \rho(m_2 x))$ (i.e. $L \simeq R$)

$\shortparallel$ $L(m_1 m_2, x)$ $\shortparallel$ $R(m_1 m_2 x)$

(2) $\underset{E}{\underset{\shortparallel}{\rho(1,x)}} \simeq \underset{id}{\underset{\shortparallel}{x}}$

$$\begin{tikzcd} X \arrow[r, bend left, "E"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\simeq"] & X \end{tikzcd}$$

$$\begin{tikzcd} M \times M \times X \arrow[r, bend left, "L"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\simeq"] & X \end{tikzcd}$$

Example (master example)

Let B be a space with basepoint $b_0 \in B$.

Let $M = \Omega(B, b_0)$, this is an h-space.

Let $f : E \longrightarrow B$ be a map, $e_0 \in E$ baspoint, $f(e_0) = b_0$.

Let

$$X := hFib(f) = \{(e,\omega) \in E \times W(B, b_0) \mid \omega(0) = b_0, \ \omega(1) = f(e)\}$$

Action of M on X

Let $\gamma \in \Omega(B, b_0) = M$ act on $(e,w) \in X$ by

$$\rho(\gamma, (e,w)) = (e, \gamma * w)$$

So first go through the loop γ, then through w.

This is an action of M on X: $\rho : M \times X \longrightarrow X$.

Action of an operad on a space X

Let $\Theta = \{\Theta_k\}$ be an operad, X a topological space. Then an action of Θ on X is a ~~map~~ collection of maps

$$\rho_n : \Theta_n \times X^n \longrightarrow X \qquad (\text{where } X^n = X \times \ldots \times X)$$

such that for $k_1, \ldots, k_n \in \mathbb{N}$ and $k := k_1 + k_2 + \ldots + k_n$ we have a commutative diagram

$$\begin{tikzcd} \Theta_n \times \Theta_{k_1} \times \ldots \times \Theta_{k_1} \times X^{k_1} \times \ldots \times X^{k_n} \arrow[r, "\rho_{k_1} \times \ldots \times \rho_{k_n}"] \arrow[d, "\gamma^n_{k_1,\ldots,k_n}"] & \Theta_n \times X^n \arrow[d, "\rho_n"] \\ \Theta_k \times X^{k_1} \times \ldots \times X^{k_n} \arrow[r, "\rho_k"] & X \end{tikzcd}$$

Note: It was not really clearly stated in the lecture but it seems an operad is a pair $(\{\Theta_k\}, \{\gamma^n_{(k_1,\ldots,k_n)}\})$ of a collection of spaces and a collection of maps, making the right diagram commute.

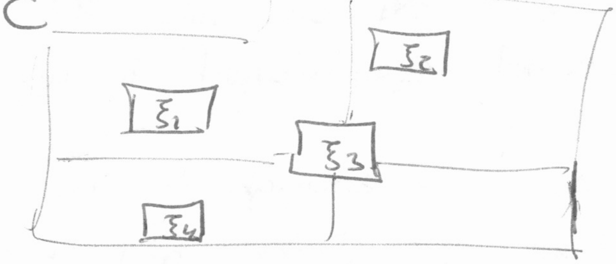
Example of operad (master example)

$\vartheta_k := \tilde{C}^k(\mathbb{R}^m)$ = "little m-cube operad"

$\vartheta : \tilde{C}^n \times \tilde{C}^{k_1} \times \dots \times \tilde{C}^{k_n} \longrightarrow \tilde{C}^{k_1 + \dots + k_n}$

$(\xi, \xi_1, \dots, \xi_n) \longmapsto$

position of boxes, box 1, box n



Operation of $\tilde{C}^k$-operad

Let Y be a space, $y_0 \in Y$ a basepoint.

Let $X = \Omega^m(Y, y_0) = \mathrm{Map}_0(\mathbb{S}^m, \infty; Y, y_0)$

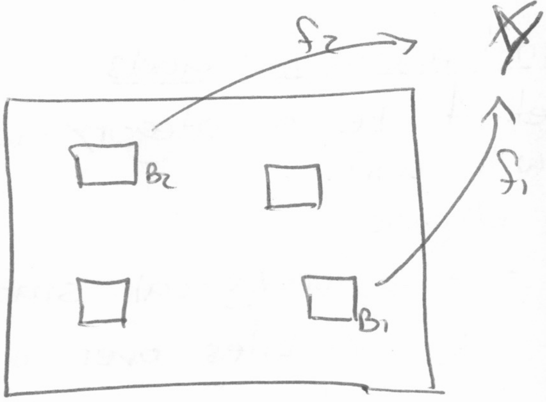
$\hat{=} \ \mathrm{map}(I^m, \partial I^m; Y, y_0)$

The action of $(\{\tilde{C}^k\}, \{\vartheta\})$ on X is given by

$\wp_n : \tilde{C}^n \times X^n \longrightarrow X$

n distinct points
$\Downarrow$
n disjoint cubes of side lengths $\frac{1}{3}$ · min dist between two points

maps $f_1, \dots, f_n$ defined on cubes.



So for $\wp_n(\xi, f_1, \dots, f_n)$, we select n boxes in I^n using the n points of ξ. On those boxes, we use $f_1, \dots, f_n$ to map to Y. On the rest of the cube, we choose constant y_0.

The unit is the constant map $I^n \to Y$; $x \mapsto y_0$.

Note that for all m, $\Omega^m(Y, y_0)$ is an h-monoid, just as $\Omega(Y, y_0)$.

For $\Omega(Y, y_0)$, we just had "two" multiplications: $\alpha * \beta$ or $\beta * \alpha$.

But now on $\Omega^m(Y, y_0)$, we have a multiplication for each $\xi \in \tilde{C}^2(\mathbb{R}^m)$

Very often, $\Omega^m(Y, y_0)$ is not connected.

$(\pi_0(\Omega^m(Y, y_0)) = \pi_m(Y, y_0))$.

But often, the connected components are homotopy equivalent.

Lecture 3 - 16-04-18

Theorem (Recognition Principle)
A connected space X on which the operad $\mathcal{C}(m) = (\widetilde{\mathrm{Conf}_k(\mathbb{R}^m)})_{k\in\mathbb{N}}$ acts is homotopy equivalent to an m-fold loop space.

"Proof"
By induction, it is enough to show that there is some space BX such that $X \simeq \Omega BX$ and BX has an action of $\mathcal{C}(m-1)$.
BX is constructed via the Milnor construction/generalized bar construction.

$$\begin{tikzcd}
\Omega X \arrow[r, "\simeq"] \arrow[d] & \Omega X \arrow[d] \\
PX \arrow[r] \arrow[d] & EX \simeq * \arrow[d] \\
X \arrow[r] & \Omega BX
\end{tikzcd}$$

Non-topological operads

Let A be a category with a monoidal symmetric product with unit.
For example:
- A = topological spaces with Cartesian product and $\mathbb{1} = \{*\}$
- A_R = modules over a commutative ring R, with the tensor product $A \otimes_R B$ and $\mathbb{1} = R$.

~~An operad in this category is a collection $\{\mathcal{O}_k\}_{k\in\mathbb{N}}$ of objects in A together with a collection of morphism~~

This gives an example of a non-topological operad:

$$\mathcal{O}_k := \mathrm{Hom}_A(E^{\otimes k}, E) \quad \text{for fixed object } E \text{ in } A$$

$$\vartheta : \mathcal{O}_k \times \mathcal{O}_{n_1} \times \dots \times \mathcal{O}_{n_k} \longrightarrow \mathcal{O}_{n_1+\dots+n_k}$$

$$(F, f_1, \dots, f_k) \longmapsto F(\underbrace{f_1(-,\dots,-)}_{n_1}, \underbrace{f_2(-,\dots,-)}_{n_2}, \dots, \underbrace{f_k(-,\dots,-)}_{n_k})$$

Example 2: ASS

$$ASS_k = \{\text{rooted planar trees with } k \text{ numbered leaves}\}$$

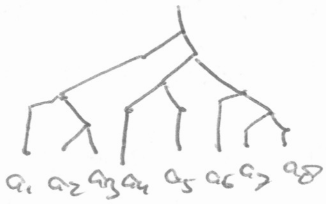
$$\vartheta : ASS_k \times ASS_{n_1} \times \dots \times ASS_{n_k} \longrightarrow ASS_{n_1+\dots+n_k}$$

is given by connecting last k trees to the first

Let A be the category of algebras over R. We have an action of Ass_k on A by

$$ASS_k \times A^{\otimes k} \longrightarrow A$$

$$(T; a_1, \dots, a_k) \longmapsto$$



<u>Example 3</u>: Comm

<u>Example 4</u>: LIE

<u>Proposition</u>: If $\mathcal{O} = (\mathcal{O}_k \mid k \in \mathbb{N})$ is an operad in the category of spaces then

$$M := \coprod_{k \in \mathbb{N}} \mathcal{O}_k$$

is a topological h-space. Moreover

- It is h-associative
- It has an h-neutral element.
- If $\mathcal{O}_2 / \Sigma_2$ is path connected, it is h-commutative.

<u>Proof</u>: We need $\mu : M \times M \longrightarrow M$, so we simply define μ by picking an $\xi \in \mathcal{O}_2$ and setting

$$\mu(x,y) := \gamma^2_{k,\ell}(\xi, x, y) \in \mathcal{O}_{k+\ell}. \quad \mu : \mathcal{O}_k \times \mathcal{O}_\ell \longrightarrow \mathcal{O}_{k+\ell}$$

where $\gamma^2_{k\ell} : \mathcal{O}_2 \times \mathcal{O}_k \times \mathcal{O}_\ell \longrightarrow \mathcal{O}_{k+\ell}$.

By the properties of an operad this is h-associative and has an h-neutral element.

<u>Remark</u>: For $\mathcal{O} = \tilde{C}(m)$, the little cube operad, we see that $M = \coprod_k \widehat{Conf_k}(\mathbb{R}^m)$ is <u>not</u> connected, so we can <u>not</u> use the recognition principle to conclude that it is an m-fold loop space. (Indeed it turns out it isn't.)

Note that $C(m)$ acts on M

$$C_k \times M^k \xrightarrow{\rho} M$$

by

$$\rho : C_k \times C_{n_1} \times \dots \times C_{n_k} \xrightarrow{\gamma^k_{n_1 \dots n_k}} C_{n_1 + \dots + n_k}$$

If it <u>were</u> an Ω^m-fold loop space, all path components would be h-equivalent, but $\widehat{Conf_1}(\mathbb{R}^m) \cong \mathbb{R}^m \simeq *$, $\widehat{Conf_2}(\mathbb{R}^m) \simeq S^{m-1}$.

More examples of h-spaces

Let X be a (connected) space; $x_0 \in X$ a basepoint.

(1) James Construction $J(X)$ = the free (non-abelian) monoid generated by X, with x_0 as neutral element.

It consists of words in the elements of X ($x_1 x_3 x_2 x_2 x_0 x_1$) with concatination as associative multiplication and x_0 acting as a unit. (Note: it doesn't have inverses)

Note that this multiplication is not commutative.

We want a topology on $J(X)$ such that

- the inclusion $X \hookrightarrow J(X)$ is continuous
- multiplication $J(X) \times J(X) \longrightarrow J(X)$ is continuous.

The easiest way to do this is to first define

$$J_n(X) = X^n / (A x_0 B) \sim (A B x_0)$$

and then using the inclusions $J_n(X) \longrightarrow J_{n+1}(X) : A \mapsto (A, x_0)$ define

$$J(X) := \varinjlim_n J_n(X) \qquad \text{(with the limit topology)}$$

The topology could also be defined directly on $J(X)$ by taking as the open neighborhoods of a word $x_1 x_2 \ldots x_n$ the set of all words obtained by letting x_i vary in some open neighborhood U_i of x_i for all i.

Proposition: 1) $J(X)$ is an h-space with strict associativity and unit

2) $J(-)$ is a homotopy functor: if $f: (X, x_0) \longrightarrow (Y, y_0)$ is a map of pointed spaces we get $J(f): J(X) \to J(Y)$ by

$$f(w_1 \ldots w_n) \longmapsto f(w_1) \ldots f(w_n)$$

If $f \simeq g$ rel x_0 then $J(f) \simeq J(g)$.

Corollary: If $X \simeq *$, then $J(X) \simeq *$.

Theorem (James): If X is connected, then there is a homotopy equivalence

$$J(X) \xrightarrow{\simeq} \Omega(\Sigma X)$$

$w = x_1 x_2 \ldots x_n \longmapsto$ the loop in ΣX that starts in the basepoint $[x_0 \times I]$, goes up n times going through $x_i \times I$ in the ith iteration

Also note:

Lemma: $J_n(X)/J_{n-1}(X) \cong \underbrace{X \wedge X \wedge \dots \wedge X}_{n\text{-fold}} =: X^{(n)}$ (n-fold smash product)

Theorem (James) For a connected space X

$$\Sigma J(X) \simeq \Sigma \left(\bigvee_{n \geq 1} X^{(n)} \right)$$

Corollary: Certain "attaching maps" are null homotopic.

Remark: Note that the map $J(X) \longrightarrow \Omega\Sigma X$ we gave before factors through the Moore loop space $M(\Sigma X, \overline{x}_0)$ via a multiplicative map $\gamma : J(X) \longrightarrow M(\Sigma X, \overline{x}_0)$ $(\simeq \Omega \Sigma X)$

We want to generalize this to higher dimensions.

We will do this next Wednesday.

Lecture 4 - 18-04-18

We have seen that for the James construction $J(X)$

1.) $J(X) \xrightarrow[\simeq]{\gamma} \Omega\Sigma X$

An example of how to use this is for $X = \mathbb{S}^n$:

$$\begin{tikzcd} & \pi_i(\Omega\Sigma S^n) \cong \pi_{i+1}(\mathbb{S}^{n+1}) \arrow[d, "hur"] \\ H_i(J(\mathbb{S}^n)) \arrow[r, "\gamma_*", "\cong"'] & H_i(\Omega\Sigma\mathbb{S}^n) \end{tikzcd}$$

from which it follows that $J(\mathbb{S}^n)$ is $(n-1)$-connected.

Furthermore we saw the splitting

2.) $\Sigma J(X) \xrightarrow{\simeq} \Sigma\Big(\bigvee_{n\geq 1} \underbrace{J_n(X)/J_{n-1}(X)}_{X^{(n)} = X \wedge \dots \wedge X}\Big)$

$$\begin{tikzcd} & \pi_i(\Sigma\Omega\Sigma X) \\ \pi_{i-1}(J(X)) \arrow[r, "\gamma_*", "\cong"'] \arrow[d] & \pi_{i-1}(\Omega\Sigma X) \arrow[u, "\cong \text{ in stable range}"'] \\ H_{i-1}(J(X)) & \end{tikzcd}$$

For proving 2.), we will eventually use the following idea:

We will find a map

$$S : J(X) \longrightarrow \Omega\Sigma V,$$

where $V = \bigvee_{n \geq 1} J_n(X)/J_{n-1}(X)$

It turns out that we get a commutative diagram

$$\begin{tikzcd} J(X) \arrow[r, "S"] \arrow[d] & \Omega\Sigma V \arrow[d, "\simeq"] \\ JJ(X) \arrow[r] & J(V) \end{tikzcd}$$

We will show that the bottom map is a homotopy equivalence.

so then $\Omega\Sigma J(X) \simeq JJ(X) \simeq J(V) \simeq \Omega\Sigma V$.

so $\Sigma J(X) \simeq \Sigma V$.

3.) The Hilton-Milnor Theorem

$$\pi_i(S^n \vee S^m) \cong \bigoplus_k \pi_{j_k}(S^{l_k})$$

where the sum is taken over infinitely many k and j_k and l_k depend on i, n and m.

~~More general, there is a theorem that~~

We also have

$$J(X \vee Y) \simeq \prod_k^{w} J(Z_k)$$

(infinitely many k. Z_k depends on X and Y.)

~~Note~~

We have

$$\Sigma J(S^n) \simeq \bigvee_k \Sigma S^{k \cdot n}$$

Chapter II: Configuration spaces with labels

Let M be an m-dimensional manifold.

Let (X, x_0) be a well-based CW-complex.

The ordered configuration space is

$$\widetilde{\mathrm{Conf}}^k(M) := \{(\xi_1, \ldots, \xi_k) \in M^k \mid \xi_i \neq \xi_j \text{ for } i \neq j\}$$

There is a clear action of $\mathfrak{S}_k$ and the unordered configuration space is the quotient

$$\mathrm{Conf}^k(M) := \widetilde{\mathrm{Conf}}^k(M) / \mathfrak{S}_k .$$

Do you mean $\mathfrak{S}_k$?

We can regard each element of $\mathrm{Conf}^k(M)$ as a collection of k indistinguishable particles in M, that are not allowed to collide.

Example 1: We can write

$$\widetilde{\mathrm{Conf}}^k(\mathbb{R}) \cong \coprod_{\sigma \in \mathfrak{S}_k} K_\sigma \simeq \coprod_{\sigma \in \mathfrak{S}_k} *_\sigma \cong \coprod_{k!} *$$

where all K_σ are homeomorphic and

$$\mathrm{Conf}^k(\mathbb{R}) \cong K_1 = \{(\xi_1, \ldots, \xi_k) \mid \xi_1 < \xi_2 < \ldots < \xi_k\} \simeq *$$

Example 2:

$$\widetilde{\mathrm{Conf}}^2(\mathbb{R}^m) \cong \mathbb{S}^{m-1} \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$$

$$(\xi_1, \xi_2) \longmapsto \left(\frac{\xi_2 - \xi_1}{\|\xi_2 - \xi_1\|}, \frac{\xi_2 + \xi_1}{2}, \|\xi_2 - \xi_1\| \right)$$

So

$$\mathrm{Conf}^2(\mathbb{R}^m) \cong \widetilde{\mathrm{Conf}}^2(\mathbb{R}^m) / \mathfrak{S}_2 \cong \mathbb{RP}^{m-1} \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$$

Example 3: The covering $\widetilde{\mathrm{Conf}}^k(\mathbb{R}^2) \longrightarrow \mathrm{Conf}^k(\mathbb{R}^2)$ gives us the important Braid group

$$Br_k := \pi_1(\mathrm{Conf}^k(\mathbb{R}^2))$$

and its subgroup of pure braids:

$$PBr_k := \pi_1(\widetilde{\mathrm{Conf}}^k(\mathbb{R}^2))$$

The configuration space of M with labels in X is

$$C(M; X) := \left(\coprod_{n \geq 1} \widetilde{\mathrm{Conf}}^k(M) \times_{\mathfrak{S}_k} X^k \right) / \sim$$

The symbol $\times_{\mathfrak{S}_k}$ means that we identify for every $\sigma \in \mathfrak{S}_k$

The ζ_i have to be seen as particles and x_i is the "label" of ζ_i. The relation $\sim$ will allow to add or remove particles with label x_0:

② $$(\overbrace{\zeta_1, \ldots, \zeta_k}^{\text{particles}} : \overbrace{x_1, \ldots, x_k}^{\text{labels}}) \sim (\zeta_1, \ldots, \hat{\zeta_i}, \ldots, \zeta_k : x_1, \ldots, \hat{x_i}, \ldots, x_k)$$

<u>if</u> $x_i = x_0$

Given a subspace $M_0 \subseteq M$, we can also define

$$C(M, M_0 : X) := C(M, \cancel{M_0} : X)/_\sim$$

where

③ $$(\zeta_1, \ldots, \zeta_k : x_1, \ldots, x_k) \sim (\zeta_1, \ldots, \hat{\zeta_i}, \ldots, \zeta_k : x_1, \ldots, \hat{x_i}, \ldots, x_k)$$

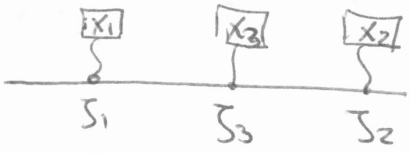
<u>if</u> $\zeta_i \in M_0$.

In other words: particles in M_0 can freely be added or removed.

<u>Example 1</u>: $M = \mathbb{R}$, $M_0 = \emptyset$, X arbitrary connected space.

We have a map

$$\Phi : C(\mathbb{R} : X, x_0) \xrightarrow{\simeq} J(X, x_0) \; (\simeq \Omega\Sigma X)$$



$$\longmapsto \quad w = x_1 x_3 x_2$$

This turns out to be a homotopy equivalence.

An inverse is based on the following map:

$$w = x_1 x_2 x_3 \cdots x_n \longmapsto (1, 2, 3, \ldots, n, x_1, x_2, \ldots, x_n)$$

However, this does not quite work yet, as it is not continuous. The problem lies in the fact that in w we can add or remove the letter x_0. This means that if one of the x_i's is very close to x_0, the image of w in $C(\mathbb{R} : X; x_0)$ must also be very close to...

Okay, I didn't get this.

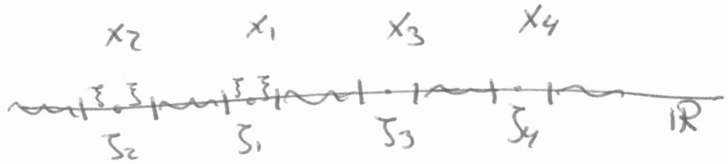
Theorem: There is a $\overset{\text{weak}}{\text{a}}$ homotopy equivalence

$$\gamma : C(\mathbb{R}^m; X, x_0) \xrightarrow{\simeq_w} \Omega^m \Sigma^m X$$

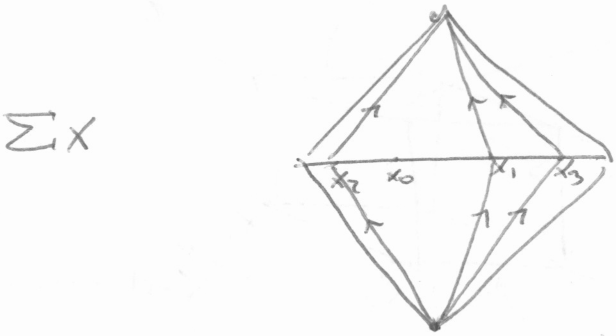
for every path-connected space X.

Example: For $m=1$, $C(\mathbb{R}, X)$ is another model for $\Omega\Sigma X$, like $J(X)$.

The map $C(\mathbb{R}, X) \xrightarrow{\simeq} \Omega\Sigma X$ can be described as follows: given a labeled configuration $(\zeta_1, \dots, \zeta_n, x_1, \dots, x_n)$



Define ~~ξ~~ $\varepsilon = \frac{1}{3} \min\{|\zeta_i - \zeta_j| : i \neq j\}$. Then we can draw small intervals of length 2ξ around the ζ_i that don't intersect. We can now define a loop in ΣX by staying constant outside of the intervals, and in the interval of ζ_i, go from the South pole in ΣX to the North pole in ΣX along the line $\{x_i\} \times I$



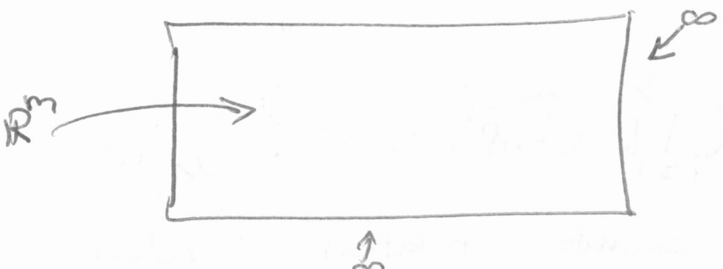
By scaling in the right way, this gives a loop in $\Omega\Sigma X$, thus giving a map $\gamma : C(\mathbb{R}, X, x_0) \longrightarrow \Omega\Sigma X$.

Situation for higher dimensions:

Let $m \geq 2$. In order to draw $\Sigma^m X$, we use that

$$\Sigma^m X \cong X \times (\mathbb{R}^m \cup \infty) \Big/ \left(x_0 \times (\mathbb{R}^m \cup \infty) \cup X \times \{\infty\}\right)$$

and we draw $X \times (\mathbb{R}^m \cup \infty)$ as

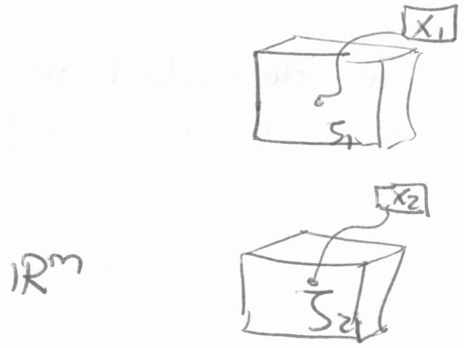


(~~X is t~~
(the X-dimension goes out of the paper)

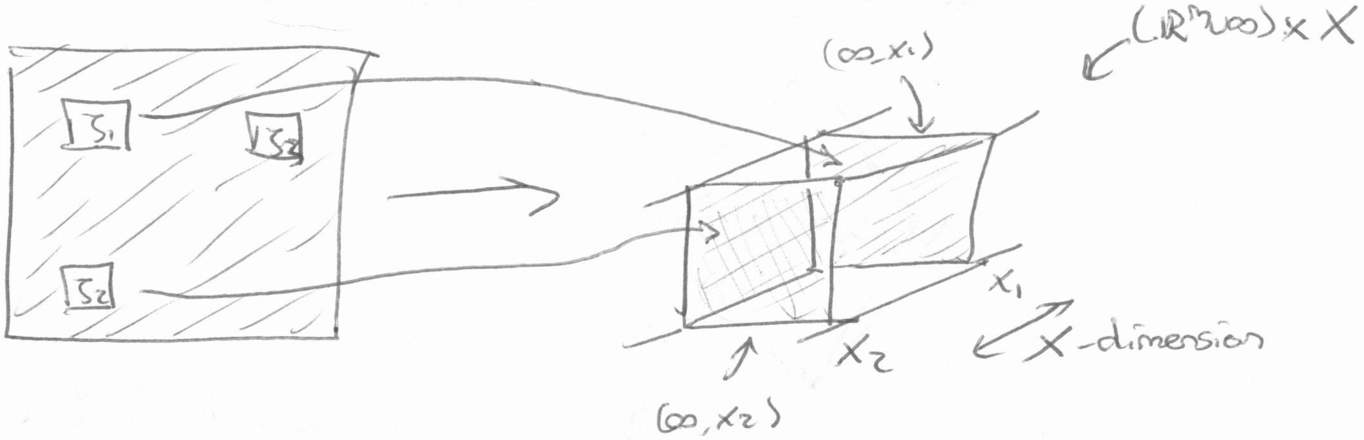
An element of $\Omega^m \Sigma^m X$ can thus be given by a map

$$\mathbb{R}^m \cup \infty \longrightarrow (\mathbb{R}^m \cup \infty) \times X.$$

How we give this map if we are given a configuration $(\xi_1, \dots, \xi_k; x_1, \dots, x_k)$ is as follows:



Again draw small cubes around the ξ_i of size ε (small enough). Outside of these cubes we choose the map to $((\mathbb{R}^m \cup \infty) \times X)/\sim$ to be constant to the base point. Inside the boxes we send it to the sheet $(\mathbb{R}^m \cup \infty) \times \{x_i\}$, using an appropriate scaling but keeping x_i fixed.



This way every configuration $(\xi_1, \dots, \xi_k, x_1, \dots, x_k)$ gives an element of $\Omega^m \Sigma^m X$.

Notice that we have now seen a higher-dimensional analog of

$$J(X) \simeq \Omega \Sigma X.$$

Now for $J(X)$ we had a filtration. Do we also have that for $C(M, M_0; X, x_0)$?

Define

$$C_k(M, M_0; X, x_0) := \Big(\coprod_{i=1}^{k} \widetilde{\mathrm{Conf}}^i(M) \overset{\sim \text{①}}{\times_{G_i}} X^i \Big) \Big/_{\text{②}\,\text{③}}.$$

where ①, ② and ③ are the equivalence relation of before.

We have

$$C_1 \subseteq C_2 \subseteq \dots \subseteq C_{k-1} \subseteq C_k \subseteq \sim \subseteq C := C(M, M_0 ; X, x_0)$$

$$\begin{tikzcd} C_1 \arrow[d, phantom, "\shortparallel"] \\ M \times X / (x_0 \times M) \cup (M_0 \times X) \arrow[d, phantom, "\shortparallel"] \\ (M / M_0) \wedge X \end{tikzcd}$$

The fibration strata are

$$\begin{tikzcd} C_k \setminus C_{k-1} \cong \widetilde{Conf}^k(M \setminus M_0) \times_{\mathfrak{S}_k} (X \setminus x_0)^k \arrow[d] \\ \widetilde{Conf}^k(M \setminus M_0) / \mathfrak{S}_k = Conf^k(M \setminus M_0) \end{tikzcd}$$

For $M = \mathbb{R}^m$, $M_0 = \emptyset$, $X = \mathbb{S}^n$, $x_0 = \infty$, we get $X \setminus x_0 \cong \mathbb{R}^m$, so we have

$$\begin{tikzcd} C_k \setminus C_{k-1} \cong \widetilde{Conf}^k(\mathbb{R}^m) \times_{\mathfrak{S}_k} (\mathbb{R}^m)^k \arrow[d] \\ Conf^k(\mathbb{R}^m) \end{tikzcd}$$

Lecture 5 - 23-04-18

Fadell-Neuwirth fibrations:

$$\begin{tikzcd} \tilde{C}^{n-1}(M \setminus p_1) \arrow[r] & \tilde{C}^n(M) \arrow[r, "p"] & M \\ (z_2, \dots, z_n) \arrow[r, mapsto] & (p_1, z_2, \dots, z_n) & \\ & (z_1, z_2, \dots, z_n) \arrow[r, mapsto] & z_1 \end{tikzcd}$$

This is a fibre bundle, as $p^{-1}(p_1) \cong \tilde{C}^{n-1}(M \setminus p_1)$

Remark: Configuration spaces of graphs are very useful in robotics. For example

$Y =$ 

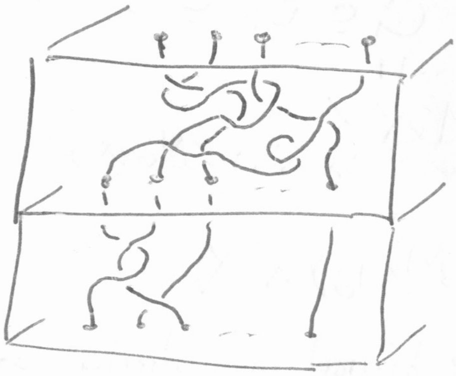
$$\begin{tikzcd} \tilde{C}^2(Y) \cong \mathbb{S}^1 \arrow[d, "(2)"] \\ C^2(Y) \cong \mathbb{S}^1 \end{tikzcd}$$

More generally:

$$\begin{tikzcd} \tilde{C}^{n-k}(M \setminus \{p_1, \dots, p_k\}) \arrow[r] & \tilde{C}^n(M) \arrow[r, "p"] & \tilde{C}^k(M) \\ (z_{k+1}, \dots, z_n) \arrow[r, mapsto] & (p_1, \dots, p_k, z_{k+1}, \dots, z_n) & \end{tikzcd}$$

Remember the Braid group

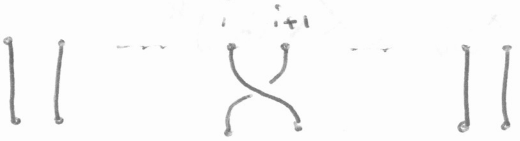
$$Br(n) := \pi_1(C^n(\mathbb{R}^2))$$
$$PBr(n) := \pi_1(\tilde{C}^n(\mathbb{R}^2))$$



A ~~re~~presentation of $Br(n)$ is

$$\left\langle \beta_1, \dots, \beta_{n-1} \;\middle|\; \begin{array}{ll} ① \ \beta_i\beta_j = \beta_j\beta_i & |i-j| \geq 2 \\ ② \ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} & \end{array} \right\rangle$$

where β_i represents



We have a surjection $Br(n) \longrightarrow \mathfrak{S}_n : \beta_i \longrightarrow (i\ i{+}1)$. The kernel of this map is by definition $PBr(n)$.

$$\begin{tikzcd}
 & PBr(n) \arrow[d] \arrow[r] & \mathbb{Z} \arrow[d, "\cdot 2"] \\
\boxed{?} \arrow[r] \arrow[d, "\times" marking] & Br(n) \arrow[d, two heads] \arrow[r, "\text{``sign''}"] & \mathbb{Z} \arrow[d] \\
A_n \arrow[r] & \mathfrak{S}_n \arrow[r, "\text{sign}"] & \mathbb{Z}_2 = \{\pm 1\}
\end{tikzcd}$$

$\text{sign}(\beta_i) = +1$

$\text{sign}(\beta_i^{-1}) = -1$

Theorem 1 $(M = \mathbb{R}^2)$

(i) $\pi_1(C^n(\mathbb{R}^2)) = Br(n)$

$\pi_1(\tilde{C}^n(\mathbb{R}^2)) = PBr(n)$

And the upper diagram of group extensions commutes

(ii) $\pi_i(\tilde{C}^n(\mathbb{R}^2)) = 0$ for $i \geq 2$

$\pi_i(C^n(\mathbb{R}^2)) = 0$ for $i \geq 2$

Theorem 2 $(M = \mathbb{R}^\infty)$

(i) $\pi_1(\tilde{C}^n(\mathbb{R}^\infty)) = 1$

$\pi_1(C^n(\mathbb{R}^\infty)) = \mathfrak{S}_n$.

(ii) $\pi_i(\tilde{C}^n(\mathbb{R}^\infty)) = 0$ for $i \geq 2$

$\pi_i(C^n(\mathbb{R}^\infty)) = 0$

($\tilde{C}^n(\mathbb{R}^\infty)$ is a contractible space on which $\mathfrak{S}_n$ acts freely.)

The inclusion $\tilde{C}^n(\mathbb{R}^2) \hookrightarrow \tilde{C}^n(\mathbb{R}^\infty)$ turns out to give the map $Br(n) \twoheadrightarrow \mathfrak{S}_n : \beta_i \mapsto (i\ i{+}1)$ on fundamental groups.

Exercise:

1) $Br(1) = 1$ 2) $Br(2) \cong \mathbb{Z}$ 3) $Br(3) \twoheadrightarrow \mathbb{Z}_6$ — $Br(3) \to SL_2(\mathbb{Z}) \to 0$

Remember the filtration $C_1 \subseteq C_2 \subseteq \dots \subseteq C = C(M, M_0 ; X)$, where $C_i(M, M_0 ; X) =: C_i$.

We have filtration strata

$$\begin{tikzcd}
C_n \setminus C_{n-1} \cong \tilde{C}^n(M \setminus M_0) \times_{\mathfrak{S}_n} (X \setminus x_0)^n \arrow[d] \\
\tilde{C}^n(M \setminus M_0)/_{\mathfrak{S}_n} = C^n(M \setminus M_0)
\end{tikzcd}$$

If $M_0 = \emptyset$, $X = S^q$, $x_0 = \infty$, we get

$$\begin{tikzcd}
(\mathbb{R}^q)^n \arrow[r] & \tilde{C}^n(M) \times_{\mathfrak{S}_n} (\mathbb{R}^q)^n \arrow[r, "\xi^n(M, \mathbb{R}^q)"'] & C^n(M),
\end{tikzcd}$$

This is a vector bundle.

We have the quotients

$$C_n/_{C_{n-1}} = \mathrm{Thom}(\xi^n(M; \mathbb{R}^q))$$

Approximation maps

$$\gamma : C(\mathbb{R}^m ; X) \longrightarrow \Omega^m \Sigma^m X$$

Goal: this is a (weak) homotopy equivalence.

Theorem I:

$$\gamma : C(\mathbb{R}^m ; X) \xrightarrow{\simeq_w} \Omega^m \Sigma^m X$$

Theorem II: Let (M, M_0) be a handle of index k $(0 \leq k \leq m)$. Then there is a weak homotopy equivalence

$$\gamma : C(M, M_0 ; X) \xrightarrow{\simeq_w} \Omega^{m-k} \Sigma^m X.$$

Special cases:

$k=m$ $\quad C(D^m, \partial D^m ; X) \simeq_w \Sigma^m X$

$$C_1 = D^m \times X$$

$$C_0 = (\partial D^m \times X) \cup (D^m \times \{x_0\})$$

$$C_1/_{C_0} \cong S^m \wedge X$$

$k=0$ $\quad C(D^m ; X) \simeq_w \Omega^m \Sigma^m X.$

So theorem II implies theorem I.

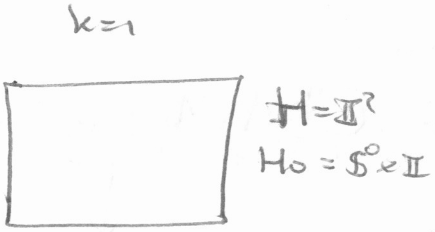
Handles

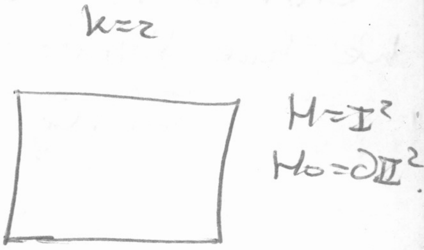
What is a handle pair of dimension m and index k? $(0 \leq k \leq m)$

It is a pair (H, H_0) of a m-dimensional manifold with boundary, and $H_0 \subseteq H$ is a submanifold, such that H is homeomorphic to $I^m = I^k \times I^{m-k}$ and H_0 corresponds under this homeomorphism

$\partial I^k \times I^{m-k} \subseteq \partial I^m$

Example: $m=2$





Theorem III $(M=\mathbb{R}^{\infty})$

$$C(\mathbb{R}^{\infty}:X) \simeq_w \Omega^{\infty}\Sigma^{\infty}X =$$

$$:= \varinjlim_k \left(\cdots \longrightarrow \Omega^k\Sigma^k X \longrightarrow \Omega^{k+1}\Sigma^{k+1}X \longrightarrow \cdots \right)$$

$$f \longmapsto \Sigma f$$

$$(f: S^k \longrightarrow \Sigma^k X) \quad (\Sigma f: S^{k+1} \longrightarrow \Sigma^{k+1} X)$$

(Pay attention to the connection with Freudenthal suspension!)

$$\pi_k^{stab}(X) = \pi_k(\Omega^{\infty}\Sigma^{\infty}X) \quad \text{(homology theory!)}$$

By commutativity of

$$\begin{tikzcd} C(\mathbb{R}^m:X) \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft\!\!\simeq"] & \Omega^m\Sigma^m X \arrow[d, "\text{suspension}"] \\ C(\mathbb{R}^{m+1}:X) \arrow[r, "\gamma"] & \Omega^{m+1}\Sigma^m X \end{tikzcd}$$

theorem III follows from theorem I, by taking limits.

Theorem IV If X connected or (M,M_0) is connected.

$$C(M,M_0:X) \xrightarrow{\simeq_w} \mathrm{Sect}(W\setminus M_0, W\setminus M : E_W(X))$$

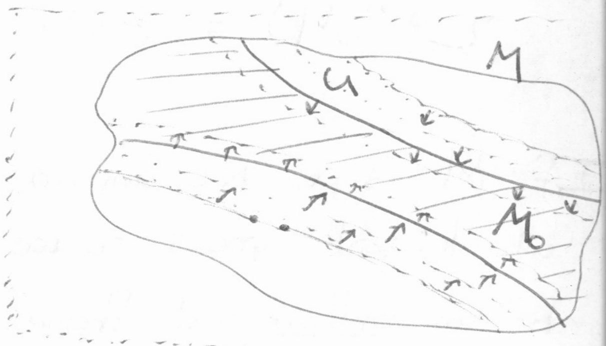
where W is an m-manifold without boundary containing M.

$$\begin{tikzcd} E_W(X) := V_m(TW) \times_{O(m)} (S^m_+ \wedge X) \arrow[d] \\ M_0 \subseteq M \subseteq W \end{tikzcd}$$

Lecture 6 - 25-04-18

Approximation Theorems

Let M be an m-manifold. $M_0 \subseteq M$ a submanifold of dimension m, closed as a subspace. Assume that M and M_0 are smooth (we want a tubular neighborhood of M_0 (for example $M_0 \cup$ (a tubular nbhd of ∂M_0))). Also assume M_0 and M to be compact.



Let X be a CW-complex with well-based basepoint $x_0 \in X$.

We now first want to define $\widetilde{Cub}^n(M)$, consisting of configurations of n disjoint boxes in M.

Let's first look at an example: $M = \mathbb{R}^m$

$$\widetilde{Cub}^n(\mathbb{R}^m) := \{ (c_1, \dots, c_n) \in Map(\mathbb{I}^m, \mathbb{R}^m)^n \mid ① \text{ and } ② \}$$

① We have affine embeddings (in each coordinate), axis parallel. (preferably with same slope, so we really get <u>cubes</u>)

② The images are disjoint.

We have a map

$$\widetilde{Cub}^n(\mathbb{R}^m) \xrightarrow[F\,.\,.\,\varepsilon\,.\,.]{\simeq} \tilde{C}^n(\mathbb{R}^m)$$

$$(c_1, \dots, c_n) \longmapsto (z_1, \dots, z_n)$$

z_i = barycenter of image of c_i

This has fibre homeomorphic with $\mathbb{R}_{\geq 0}$, and a homotopy inverse is given by the "ε-map", for example

$$\varepsilon = \tfrac{1}{3} (\text{min. distance between } z_i \text{ and } z_j)$$

and let the boxes around the z_i all have length ε.

Then $\widetilde{Cub}^n(\mathbb{R}^m)$ is an operad, called the 'little m-cubes operad'.

Now, since in a general manifold M, there is not such a concept of "affineness", we cannot make such a neat definition for a general M.

We can try some things:

$$\widetilde{Cub}^n(M) := \{ (c_1, \dots, c_n) \in Map(\mathbb{I}^m, M)^n \mid ① \text{ and } ② \}$$

① n embeddings into some local chart of $z_i \in M$ with $c_i(b) = z_i$, where $b = (\frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{I}^m$, such that all c_i are affine and parallel. (So in at least one coord. chart)

② The images are disjoint.

However, this space is huge, since for many "disformed" cubes, we would still be able to find some coordinate chart in which it is an affine cube.

A more 'rigid' construction is given by choosing only m directions in each z_i in which we want to extend the boxes, i.e.

$$\widetilde{\mathcal{F}}^n(M) := \left\{ (v_1, \dots, v_n) \mid \begin{array}{l} v_i = (v_i^1, \dots, v_i^m) \in (T_{z_i}(M))^m \\ \text{is a frame at } z_i \in M,\ z_i \neq z_j \end{array} \right\}$$

If M is a Riemannian manifold, we have an inner product on all tangent spaces, so we could choose each $v_i = (v_i^1, \dots, v_i^m)$ to be an <u>orthonormal</u> frame.

In that case, we also have a map $\widetilde{\mathcal{F}}^n(M) \longrightarrow \widetilde{\mathrm{Cub}}^n(M)$ by extending the orthonormal frame $v_i \in (T_{z_i}(M))^m$ to a cube in $T_{z_i}(M)$ and use the exponential map to go to M.

<u>Definition of γ</u>:

The goal is now to give a more rigorous definition of the map

$$\gamma : C(M, M_0; X) \longrightarrow \mathrm{Sect}(W \setminus M_0, W \setminus M; E(W; X))$$

that we mentioned.

Choose a manifold W containing M, $\dim W = m$, $\partial W = \emptyset$.

(W may or may not be compact / connected.)

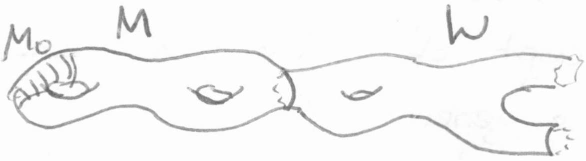
<u>ex.</u> $W = M$ if $\partial M = \emptyset$

$W = M \cup (\text{open collar})$ if $\partial M \neq \emptyset$.

(W need not be homeomorphic or even homotopy equiv. to M.)

Define.

$$E(W; X) = V_m(TW) \times_{O(m)} \Sigma^m X.$$



Note that $\Sigma^m X$ is the reduced m-fold suspension of X, which is equal to

$$(S^m, \infty) \wedge (X, x_0)$$

Then $O(m)$ acts on S^m, as it is the one-point compactification of $\mathbb{R}^m$ and $O(m)$ acts on $\mathbb{R}^m$.

This action on $\mathbb{S}^m$ always has two fixed points: $0 \in \mathbb{R}^m$ and $\infty \in \mathbb{S}^m$.

Note that $O(m)$ also acts on $V_m(TW)$, as ~~choosing~~ we can act on a global orthonormal m-frame in each tangent space by rotation.

Note that we have the following principle $O(m)$-bundle:

$$\begin{tikzcd} & V_m(\mathbb{R}^m) \circlearrowleft O(m) \arrow[dl] \\ V_m(TW) \arrow[d] & \\ W \cong V_m(TW)/O(m) & \end{tikzcd}$$

↖(Stiefel manifold of $\mathbb{R}^m$)

↖(Stiefel manifold of TW)

So we get

$$\begin{tikzcd} \Sigma^m X \arrow[d] \\ E(W;X) := V_m(TW) \times_{\leftarrow O(m) \rightarrow} \Sigma^m X \arrow[d] \\ V_m(TW)/O(m) \cong W \end{tikzcd}$$

Now, the two fixed points ~~to~~ 0 and ∞ give us two fixed points $(0,x)$ and (∞,x) in each fibre. So we get two sections

$$s_0 : W \longrightarrow E(W;X)$$
$$s_\infty : W \longrightarrow E(W;X)$$

For $B \subseteq A \subseteq W$ any subsets, we define

$\mathrm{Sect}(A,B) := \{$ ~~Sbdd~~ $s \in \mathrm{Sect}\Big(\overset{\overset{E(W;X)}{\cup}}{E(A;X)} \underset{s}{\leftrightarrows} \overset{\overset{W}{\cup}}{A}\Big) \mid s|_B = s_\infty|_B \}$

So $\mathrm{Sect}(W \setminus M_0, W \setminus M ; E(W;X))$ is the space of continuous sections of $E(W;X) \longrightarrow W$, only defined on $W \setminus M_0$, that on $W \setminus M$ is equal to the section s_∞.

The map $\gamma : C(M, M_0 ; X) \longrightarrow \mathrm{Sect}(W \setminus M_0, W \setminus M ; E(W, X))$ is called a "scanning map". To define it, fix a configuration ζ with labels

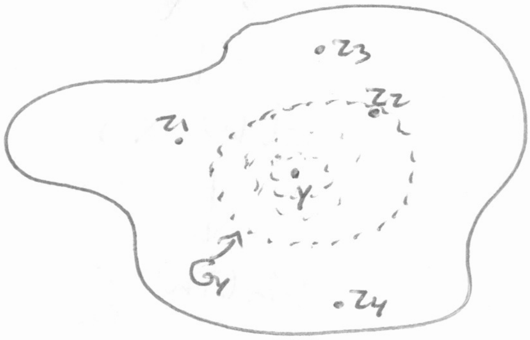
$$\zeta = [z_1, \dots, z_n ; x_1, \dots, x_n] \in C(M, M_0 ; X)$$

We may assume ~~$z_i \neq$~~ $z_i \in M \setminus M_0$, $x_i \neq x_0$.

We now want a section $\gamma(\zeta) : W \setminus M_0 \longrightarrow E(W \setminus M_0 ; X)$

Let $y \in W \setminus M_0$ and choose a geodesic disc $G_y \subseteq W \setminus M_0$ around y that is small enough s.t. it contains at most one of the z_i. (Better said: exactly one point, otherwise we could make it arbitrarily small and there is no 'natural stopping time'. Well, and there are some other considerations. If there are two or more z_i that have the same minimal distance to ~~z~~ y, we allow all of these.)

We can now regard ζ as an element of $C(M, M_0 \cup (M \setminus \overset{\circ}{G}_y) ; X)$



We'll continue on Monday

Lecture 7 - 30-04-18

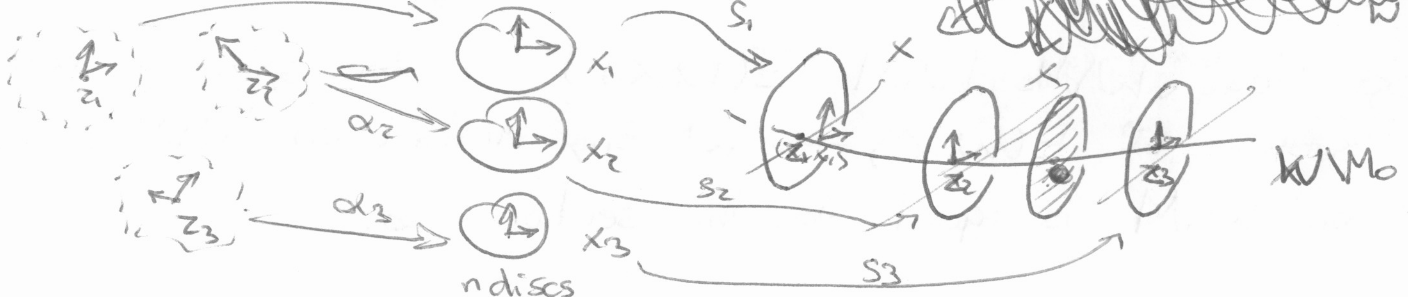
Recall that we constructed a fibre bundle

$$\Sigma^m X \longrightarrow E(W, X) \longrightarrow W$$

and two sections $s_0, s_\infty : W \longrightarrow E(W, X)$ given by $0, \infty \in \Sigma^m X$.

For $\zeta = [z_1, \dots, z_n ; x_1, \dots, x_n] \in C(M, M_0 ; X)$, we were constructing a section $\gamma(\zeta) : W \setminus M_0 \longrightarrow E(W \setminus M_0 ; X)$ of this bundle that is ~~oo~~ equal to s_∞ on $W \setminus M$.

Choose geodesic discs around the z_i



So we get a map from n geodesic discs in $W \setminus M_0$ into

$$(W \setminus M_0) \times D^m \times X$$

by mapping the geodesic disc around $z_i \in M \setminus M_0 \subseteq W \setminus M_0$ to

$$\{z_i\} \times D^m \times \{x_i\}.$$

We can now divide out $(W \setminus M) \times D^m \times X$, and $(W \setminus M_0) \times \partial D^m \times X$ and $(M \setminus M_0) \times D^m \times \{x_0\}$, and call the point obtained after quotienting out the union of these three subspaces ∞_W.

This quotient is essentially

$$(W \setminus M_0, W \setminus M) \wedge \Sigma^m X$$

Notice that the boundaries of the geodesic discs in $W \setminus M_0$ that we chose now get mapped to ∞_W in this quotient.

So we can ~~define~~ extend the map from these n discs in $W \setminus M_0$ to all of $W \setminus M_0$ by just sending all other points to ∞_W.

<u>Notice</u> that in order to identify the n geometric discs in $W \setminus M_0$ to the discs $\{z_i\} \times D^m \times \{x_i\}$, we must choose an orthonormal frame $(\alpha_1, \dots, \alpha_m)$ of W, i.e. an element of $V_m(TW)$. We thus have a map

$$\gamma(\xi) : W \setminus M_0 \longrightarrow V_m(TW) \times_{O(m)} \Sigma^m X = E(W;X)$$

by choosing $(\alpha_1, \dots, \alpha_m)$ in the first coordinate and the previously constructed map for the second component.

The map is <u>independent</u> of the choice of $(\alpha_1, \dots, \alpha_m)$ as we divide out the action of $O(m)$ and the map into $\Sigma^m X$ would have changed in the right way if we had chosen different $(\alpha_1, \dots, \alpha_m)$.

A more systematic definition of $\gamma(\zeta)$

Let $\zeta \in C(M, M_0 : X)$ be given and $w \in W \setminus M_0$.

We need a geometric disc $G(w)$ around w. $\{$ This comes from a choice of a Riemannian metric and the construction of the exponential map. $\}$ We now have a natural choice $G(w)$ for each w.

$$\begin{tikzcd}
T(W\setminus M_0) \arrow[r, phantom, "\supseteq"] & D(W\setminus M_0) \arrow[r, "\exp"] & W\setminus M_0 \arrow[d, phantom, "\cup\!|"] \\
T_w(W\setminus M_0) \arrow[u, hook] \arrow[r, phantom, "\supseteq"] & D_w(W\setminus M_0) \arrow[u, hook] \arrow[r, two heads] & G(w)
\end{tikzcd}$$

- $G(w)$ moves continuously in the space of embeddings of a standard m-disc $\mathbb{D}^m$ into ~~$M\setminus M_0$~~ $W\setminus M_0$
- $$\begin{tikzcd}
C(M, M_0 : X) \arrow[r] & C(M, M_0 \cup (M \setminus \mathring{G}(w)) : X) \\
\zeta \arrow[r, mapsto] & \zeta_w
\end{tikzcd}$$

  by "ignoring" all particles outside of $\mathring{G}(w)$.

  By a sort of "excision", we have

  $$C(M, M_0 \cup (M \setminus \mathring{G}(w)) : X) \cong C(G(w), \partial G(w) : X)$$
- Now we use the following proposition:

Proposition Let X be a connected space with basepoint x_0. Then the inclusion

$$\begin{tikzcd}
(\mathbb{D}^m, \partial \mathbb{D}^m) \wedge X \arrow[r, hook] & C(\mathbb{D}^m, \partial \mathbb{D}^m : X) \\
z \wedge x \arrow[r, mapsto] & {[z : x]}
\end{tikzcd}$$

$(\cong C(\mathbb{R}^m, \mathbb{R}^m \setminus \mathbb{D}^m : X))$

(with image $C_1(\mathbb{D}^m, \partial \mathbb{D}^m : X)$) is a strong deformation retract

(i.e. $R_t : C \longrightarrow C$ s.t. $R_0 = id_C$, $R_1(C) \subseteq C_1$ and $R_t(C_1) \subseteq C_1$ pointwise fixed.

Proof: Later. (Hmmm, okay. I am not sure if we should trust him on that...)

Anyway, back to our problem. We see that for each $w \in W \setminus M_0$ we get in a continuous way an element

$$\zeta_w \in C(G(w), \partial G(w) : X) \simeq (\mathbb{D}^m, \partial \mathbb{D}^m) \wedge X$$
$$\cong \{w\} \times (\mathbb{D}^m, \partial \mathbb{D}^m) \wedge X$$
$$\subseteq (D(W\setminus M_0) / S(W \setminus M_0)) \wedge X$$

So for each $w \in W\setminus M_0$ we get an element in

$$\mathbb{D}(W\setminus M_0)/_{S(W\setminus M_0)} \wedge X \cong E(W\setminus M_0, X)$$

disc bundle over $W\setminus M_0$

sphere bundle

(The bundle $E(W\setminus M_0; X) \longrightarrow W$ has fibres of the form $(\mathbb{D}^m, \mathbb{S}^m) \wedge X$.)

So each $\xi \in C(M, M_0; X)$ gives a section

$$\gamma(\xi) : W\setminus M_0 \longrightarrow E(W\setminus M_0, X).$$

(We used a retraction

$$C(G(w), \partial G(w); X) \longrightarrow (G(w), \partial G(w)) \wedge X$$

for all w at the same time. This can be done because the retraction of the proposition is "so beautiful".)

So we get a map

$$\gamma : C(M, M_0, X) \times (W\setminus M_0) \longrightarrow E(W; X)$$
$$(\xi, w) \longmapsto \gamma(\xi)(w)$$

This is continuous in ξ, so we can take the adjoint

$$\gamma : C(M M_0, X) \longrightarrow \mathrm{Sect}(W\setminus M_0, E(W; X)).$$

We can check that on $W\setminus M$ this is just s_∞, so γ maps in fact to $\mathrm{Sect}(W\setminus M_0, W\setminus M; E(W; X))$.

Proof of proposition

① Consider the $(\mathbb{R}^m, \mathbb{R}^m\setminus \mathring{\mathbb{D}}^m)$ and the dilatation

$$\psi_t : (\mathbb{R}^m, \mathbb{R}^m\setminus\mathring{\mathbb{D}}^m) \longrightarrow (\mathbb{R}^m, \mathbb{R}^m\setminus\mathring{\mathbb{D}}^m)$$
$$y \longmapsto (1+t)y.$$

for all $t \geq 0$.

Then
- $\psi_0 = id$
- ψ_t keeps $\mathbb{R}^m\setminus\mathring{\mathbb{D}}^m$ invariant (.

② There is an induced map

$$\Psi_t : C(\mathbb{R}^m, \mathbb{R}^m\setminus\mathring{\mathbb{D}}^m) \longrightarrow C(\mathbb{R}^m, \mathbb{R}^m\setminus\mathring{\mathbb{D}}^m)$$

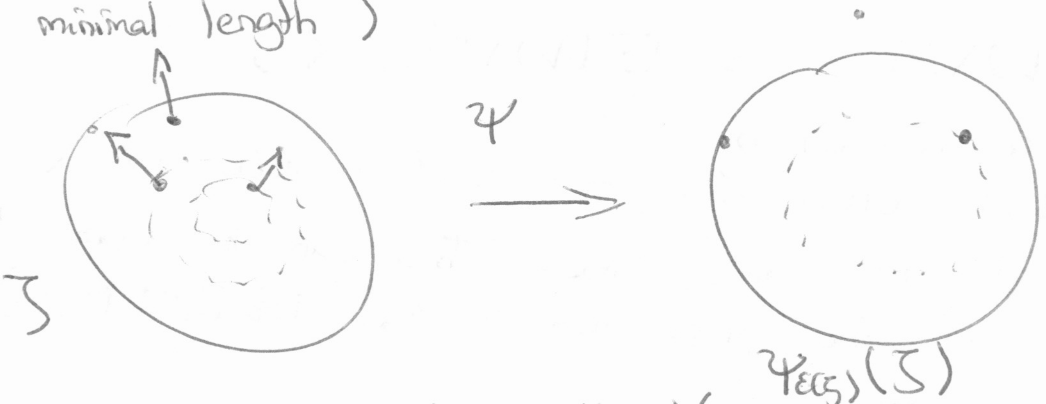
$$\Psi_0 = id.$$

③ Define for each ζ the stopping time $\varepsilon(\zeta)$ as

$$\varepsilon(\zeta) = \frac{1-d(\zeta)}{d(\zeta)}$$

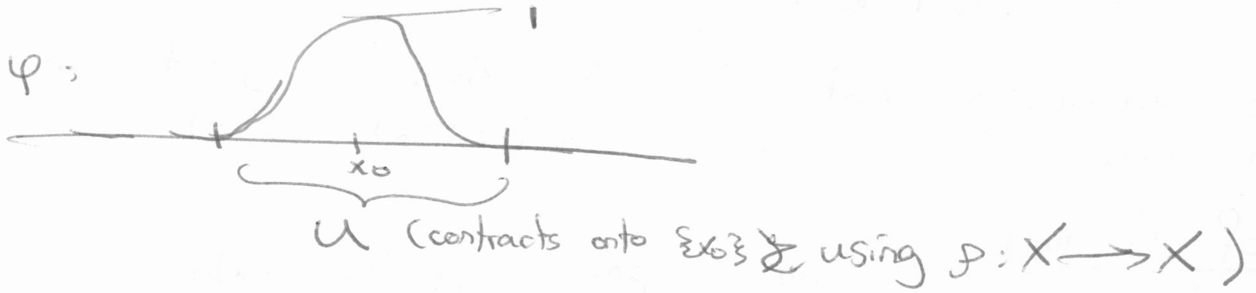
where $d(\zeta)$ is the second smallest $|z_i|$, or the smallest if there is more than one smallest $|z_i| = |z_j|$.

The idea will be to "strech out" the space untill all particles but one are gone (or several, if they have the same minimal length)



④ Now we have to deal with X and x_0. We can add a point with label x_0 very close to the center, and it will be the smallest and we have to strech suddenly very far. This would no longer be continuous.

Therefore we need some weighting function φ



We weigh the x's with $1-\varphi(x)$.

(Boedigheimer didn't make this more precise.)

⑤ We can now set

$$R_t : C(M, M_0 ; X) \longrightarrow C(M, M_0 ; X)$$

$$\zeta = [z_1, \dots, z_n, x_1, \dots, x_n] \longmapsto [\Psi_{t \cdot \varepsilon(\zeta)}, \rho(x_1), \dots, \rho(x_n)]$$

This is a deformation retraction into $C_1(M, M_0 ; X)$.

Examples

① $M = \mathbb{D}^m$, $M_0 = \emptyset$, $W = \mathbb{R}^m$

$$\begin{tikzcd} \gamma : C(\mathbb{D}^m ; X) \arrow[r, "\simeq_w"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{maps}(W, W \setminus \mathbb{D}^{\circ}, \Sigma^m X) \\ C(\mathring{\mathbb{D}}^m ; X) \arrow[d, phantom, "\cong" rotate=90] & \\ C(\mathbb{R}^m ; X) & \end{tikzcd}$$

Notice that $TW \cong \mathbb{R}^m \times \mathbb{R}^m$, so

$$\begin{tikzcd} E(\mathbb{R}^m ; X) \arrow[d] & = & \mathbb{R}^m \times (\mathbb{S}^m \wedge X) \arrow[d] \\ \mathbb{R}^m & & \mathbb{R}^m \end{tikzcd}$$

(When W is parallizable, $E(W, X) \cong \Sigma^m X$.)

So we get

$$\begin{aligned} \gamma : C(\mathbb{R}^m ; X) \longrightarrow & \ \mathrm{maps}(W, W \setminus \mathring{\mathbb{D}}^m ; \Sigma^m X) \\ & = \mathrm{maps}(\mathbb{R}^m, \mathbb{R}^m \setminus \mathring{\mathbb{D}}^m ; \Sigma^m X) \\ & \cong \mathrm{maps}(\mathbb{D}^m, \partial \mathbb{D}^m ; \Sigma^m X) \\ & \cong \Omega^m \Sigma^m X . \end{aligned}$$

For $m=1$ we just get

$$\begin{tikzcd} C(\mathbb{R}^1 ; X) \arrow[rr, "\simeq_w"] \arrow[dr, "\simeq"'] & & \Omega \Sigma X \\ & J(X) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

② $M = \mathbb{S}^1$, $M_0 = \mathbb{I}$, $W = \mathbb{S}^1$

$$\begin{tikzcd} C(M_0 ; X) \arrow[d] \arrow[rr] & & \Omega \Sigma X \arrow[d] & \\ C(\mathbb{S}^1, X) \arrow[r, "\simeq_w"] \arrow[d] & \mathrm{map}(\mathbb{S}^1 ; \Sigma X) \simeq & \Lambda \Sigma X \arrow[d, "\mathrm{eval}"] & \text{free loop space} \\ C(\mathbb{S}^1, M_0 ; X) \arrow[r, "\simeq_w"] & \mathrm{map}(\mathbb{S}^1 \setminus M_0 ; \Sigma X) \simeq & \Sigma X & \end{tikzcd}$$

Lecture 8: 02-05-18

Remember the approximation map

$$\gamma^W_{(M,M_0;X)} = \gamma : C(M, M_0; X, x_0) \longrightarrow \mathrm{Sect}(W \setminus M_0, W \setminus M; E(W;X))$$

- This is natural w.r.t. embeddings $\underset{\dim = m}{(M, M_0)} \hookrightarrow \underset{\dim = m' \geq m}{(M', M_0')}$ up to homotopy
- This is natural w.r.t. maps $(X, x_0) \longrightarrow (X', x_0')$

Note that

- $C(\mathrm{int}(M); X) \xhookrightarrow{\simeq} C(M; X)$ is a homotopy equivalence

Recall the following

Proposition: The inclusion

$$\begin{aligned} (\mathbb{D}^m, \partial\mathbb{D}^m) \wedge X &\xhookrightarrow{\simeq} C(\mathbb{D}^m, \partial\mathbb{D}^m; X) \cong C(\mathbb{D}_2^m, \mathbb{D}_2^m \setminus \mathring{\mathbb{D}}_1^m; X) \\ z \wedge x &\longmapsto [z; x] \end{aligned}$$

is a strong deformation retract, i.e. there is a retraction $R_t : C(--) \longrightarrow C(-)$ s.t. $R_0 = id$, $R_1(C(-)) \subseteq (\mathbb{D}^m, \partial\mathbb{D}^m) \wedge X$

Quasifibrations

Definition: A map $p : (E, e) \longrightarrow (B, b)$ with fibre $F_b := p^{-1}(b)$ is called a quasifibration if p induces isomorphisms

$$\pi_i(E, F_b, e) \xrightarrow[\cong]{p_*} \pi_i(B, b)$$

for all $b \in B$, $e \in F_b$ and all $i \geq 0$.

Note that this gives us the long exact sequence

$$\begin{tikzcd}
{} \arrow[r] & \pi_i(F_b, e) \arrow[r] & \pi_i(E, e) \arrow[r] \arrow[dr] & \pi_i(E, F_b, e) \arrow[r] \arrow[d, "\cong"] & \pi_{i-1}(F_b, e) \arrow[r] & \pi_i(E, e) \arrow[r] & {} \\
& & & \pi_i(B, b) \arrow[ur] & & &
\end{tikzcd}$$

It also follows that the inclusion

$$\begin{aligned} F_b &\hookrightarrow \mathrm{hfib}(p, b) \\ e &\longmapsto (e, c_b) \end{aligned}$$

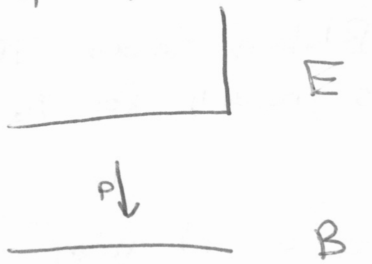
is a weak homotopy equivalence, ~~by~~ by comparing above sequence with

$$\to \pi_i(\mathrm{hfib}(p)) \longrightarrow \pi_i(E, e) \longrightarrow \pi_i(B, b) \longrightarrow \pi_{i-1}(\mathrm{hfib}(p)) \longrightarrow \pi_{i-1}(E, e) \longrightarrow$$

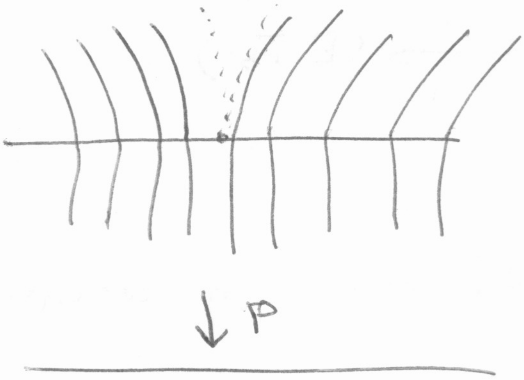
Drawback: Pull-backs of quasi-fibrations are not necessarily again quasi-fibrations.

Both fibre bundles and fibrations are preserved under taking pull-back (in particular under restricting.)

Example: (a) An example for a quasi-fibration that is not a fibration is



(b)



$E = \mathbb{R}^2 - (i\,\mathbb{R}_{>0})$

$B = \mathbb{R}$

Note that fibrations (and in particular fibre bundles) are quasi-fibrations.

Lemma (Criteria for quasifibrations)

A map $p : E \longrightarrow B$ of path-connected spaces is a quasifibration if any of the following criteria hold:

(I) $B = B_1 \cup B_2$, $B_0 := B_1 \cap B_2$, where B_i is open and path-connected and $E_i := p^{-1}(B_i)$ and each $p_i = p|_{E_i} : E_i \longrightarrow B_i$ is a q.f.

(II) $B = \bigcup_{k \geq 0} B_k$, $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots \subseteq B$, $B = \varinjlim B_k$ where each B_k is path connected and each restriction

$$p_k : E_k := p^{-1}(B_k) \longrightarrow B_k$$

is a q.f.

(III) There are subspaces $E' \subseteq E$, $B' \subseteq B$ with a deformation

$$\begin{tikzcd}
E' \arrow[rrr, "\widetilde{\Phi}_t"] \arrow[dd, "p'"'] \arrow[dr, hook, "\iota"'] & & & E' \arrow[dr] \arrow[dd] & \\
 & E \arrow[rrr] \arrow[dd, "p"] & & & E \arrow[dd, "p"] \\
B' \arrow[dr, hook] & & & B' \arrow[dr, hook] & \\
 & B \arrow[rrr, "\Phi_t"] & & & B
\end{tikzcd}$$

s.t.

1) $p \circ \widehat{\Phi}_t = \Phi_t \circ p$

2) $\widetilde{\Phi}_0 = id_E$, Φ_0

3) $p'(E') \subseteq B'$, $p' := p|_{E'}$

4) $\widetilde{\Phi}_t(E') \subseteq \cancel{\Phi}\ E'$

$\Phi_t(B') \subseteq B'$

5) $\widehat{\Phi}_1(E) \subseteq E'$, $\widetilde{\Phi}_1(B) \subseteq B'$

6) The map $F_b = p^{-1}(b) \xrightarrow{\widetilde{\Phi}_1} p^{-1}(\Phi_1(b)) = F_{b'}$ is a weak homotopy equivalence

and p' is a quasi-fibration.

Proof

(I) Exercise (Clever application of Blakers-Massey Theorem (i.e. Mayer-Vietoris property for π_*-groups.))

(II) Clear from a limit argument

(III) Let $b' := \Phi_1(b)$. We have a diagram

$$\begin{tikzcd}
(E, F_b) \arrow[r, "\tilde{\Phi}_1"] \arrow[d, "p"'] & (E', F_{b'}) \arrow[r, hook, "\iota"] \arrow[d, "p'"] & (E, F_{b'}) \\
(B, b) \arrow[r, "\Phi_1"] & (B', b') &
\end{tikzcd}$$

First, by property (6), $\iota \circ \tilde{\Phi}_1$ induces an isomorphism of relative homotopy groups.

Since ι is a homotopy equivalence, ι_* is an isomorphism.

So $\tilde{\Phi}_1$ induces an isomorphism

Also Φ_1 induces an isomorphism, since it is a homotopy equivalence.

Now, since p' is assumed to be a q.f., also p induces isomorphisms of homotopy groups, so it is a quasi-fibration. □

Application:

Consider a family of spaces and maps

$$\begin{tikzcd}
F := E_0 \arrow[r, phantom, "\subseteq"] \arrow[d] & E_1 \arrow[r, phantom, "\subseteq \cdots \subseteq"] \arrow[d] & E_{k-1} \arrow[r, phantom, "\subseteq"] \arrow[d, "p_{k-1}"] & V_k \arrow[r, phantom, "\subseteq"] \arrow[d, "\tilde{p}_k"] & E_k \arrow[r, phantom, "\subseteq \cdots"] \arrow[d, "p_k"] & E \arrow[d, "p"] \\
\{pt\} = B_0 \arrow[r, phantom, "\subseteq"] & B_1 \arrow[r, phantom, "\subseteq \cdots \subseteq"] & B_{k-1} \arrow[r, phantom, "\subseteq"] & U_k \arrow[r, phantom, "\subseteq"] & B_k \arrow[r, phantom, "\subseteq \cdots"] & B
\end{tikzcd}$$

such that for each k, $U_k \subseteq B_k$ and $V_k \subseteq E_k$ are open and have a deformation retraction onto B_{k-1} resp. E_{k-1}. (In order to use criteria III) Also, $B_{k-1} \subseteq U_k$ and $E_{k-1} \subseteq V_k$ are closed.

Then $B_k = U_k \cup (B_k \setminus B_{k-1})$ and $E_k = V_k \cup (E_k \setminus E_{k-1})$, and both subsets are open.

Assume in addition that $E_k \setminus E_{k-1} \cong (B_k - B_{k-1}) \times F$, so p_k is a fibration (in particular a q.f.) when restricted to $E_k \setminus E_{k-1}$.

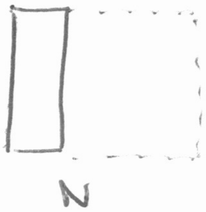
Then by induction, each p_k is a q.f.: ~~so~~ if p_{k-1} is then also $\tilde{p}_k$ by property III, so by property I p_k is as well.

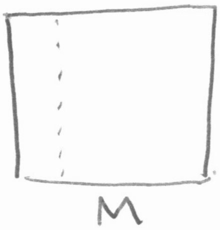
Then by II, p is also a q.f.

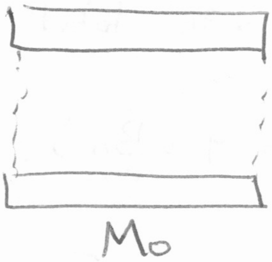
Now consider m-manifolds $M_0 \subseteq M$ and another m-manifold $N \subseteq M$ that has good intersection with M_0.

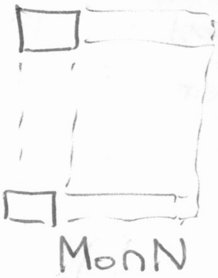
Let (X, x_0) be a based space.

Drawing:



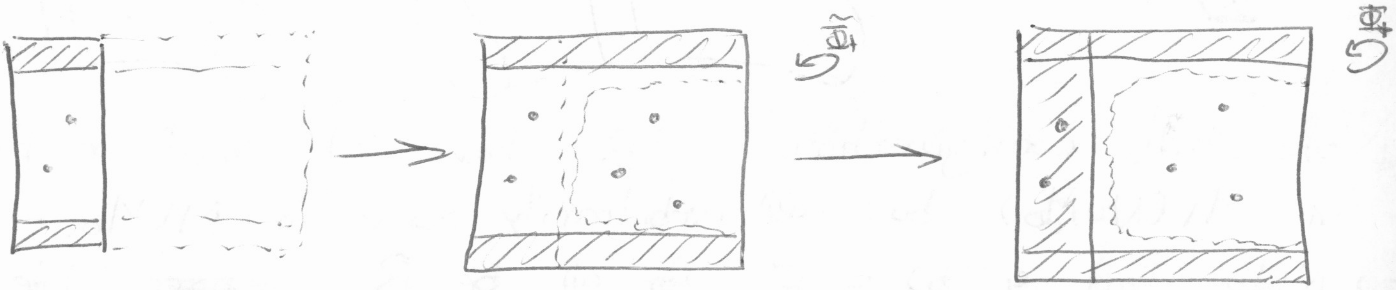






We get maps

$$C(N, N\cap M_0; X) \longrightarrow C(M, M_0; X) \xrightarrow{p} C(M, M_0 \cup N; X)$$



The final map p is surjective.

Our goal is to show that <u>p is a quasi-fibration</u>, using the method described before.

Note that the fibres of p are not all homotopy equivalent. If a point "enters" N, suddenly a point is gone. However if we restrict to $C_k \setminus C_{k-1}$ then everything is fine, since we will always have exactly k points; there can no points be added or removed. So the property of E_t and B_t mentioned before is satisfied.

We will need either one of the following conditions

1) $N \cap M_0$ is non-empty for all components of N

2) X is connected.

<u>Lecture 9</u> 07-05-18

<u>Proposition</u>: In the situation above, the sequence of embeddings of pairs

$$(N, N\cap M_0) \xrightarrow{\iota} (M, M_0) \xrightarrow{\pi} (M, N \cup M_0)$$

induces a quasi-fibration

$$F := C(N, N\cap M_0; X) \xrightarrow{j=\iota_*} E = C(M, M_0; X) \xrightarrow{q=\pi_*} B = C(M, N\cup M_0; X)$$

if ~~[illegible]~~ X is connected or $(N, N\cap M_0)$ connected.

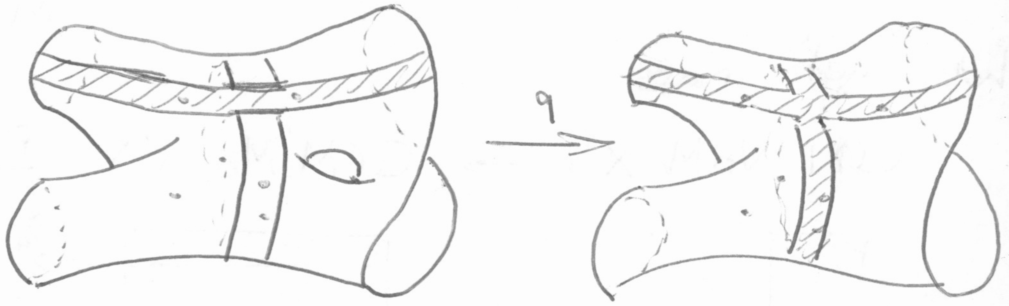
Proof: 1) Filter B by

$$B_n := C_n(M, N\cup M_0; X)$$

which is the space of configurations in $M\setminus(N\cup M_0)$ of at most n labels with label $\neq x_0$

Filter E by

$$E_n := q^{-1}(B_n).$$



Note that configurations in E_n have at most n particles in $M\setminus(N\cup M_0)$ but arbitrarily many in $N\setminus M_0$.

2) Note that $q^{-1}(b)\cong F$ for all $b\in B$. ~~where~~ Note that

$$E_n\setminus E_{n+1} = q^{-1}(B_n\setminus B_{n-1}) \cong (B_n\setminus B_{n-1})\times F.$$

$$D: \xi = [\underbrace{z_1,\dots,z_n}_{\text{in } M\setminus(N\cup M_0)}, \underbrace{z_{n+1},\dots,z_r}_{\text{in } N\setminus M_0}; x_1,\dots,x_n, x_{n+1},\dots,x_r] \longmapsto ([z_1,\dots,z_n; x_1,\dots,x_n], [z_{n+1},\dots,z_r; x_{n+1},\dots,x_r])$$

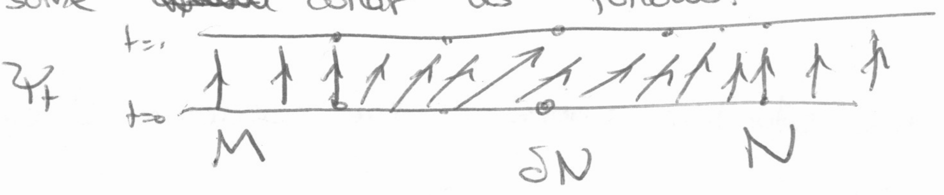
As an inverse, we have an 'addition map"

$$(\zeta', \zeta'') \longmapsto \zeta'+\zeta'' \in q^{-1}(B_n\setminus B_{n-1})$$
$$:= [z_1',\dots,z_n', z_{n+1}'',\dots,z_r''; x_1',\dots,x_n', x_{n+1}'',\dots,x_r'']$$

(Note that it is not true that $E = B\times F$.)

Addition is continuous, and D is continuous and they are an inverse of each other.

3) Let U be a closed neighborhood of N in M in form of some ~~[illegible]~~ collar as follows:



So $\psi_0 = id$ $\quad \psi_1(U) = N$

$\psi_t(N)\subseteq N$

$\psi_t(M_0)\subseteq M_0$ $\quad$ each ψ_t is an embedding.

The isotopy ψ_t induces maps

$$\begin{tikzcd}
F = C(N, N\cap M_0 ; X) \arrow[r, "\hat{\Phi}|_F"] \arrow[d, "j"'] & C(N, N\cap M_0 ; X) \arrow[d, "j"] \\
E = C(M, M_0 ; X) \arrow[r, "\hat{\Phi}_t"] \arrow[d, "q"'] & C(M, M_0 ; X) \arrow[d, "q"] \\
B = C(M, N\cup M_0 ; X) \arrow[r, "\tilde{\hat{\Phi}}_t"] & C(M, N\cup M_0 ; X)
\end{tikzcd}$$

4) Let $U_n := \{ b \in B_{n+1} \mid$ at least one particle lies in $\mathring{U} \}$ and let $V_n = q^{-1}(U_n)$, which are both open in B_{n+1} resp E_{n+1}.

Note that ~~$\Phi_1(B$~~ $\Phi_1(U_n) \subseteq B_n$, since the points in U will end up inside of N.

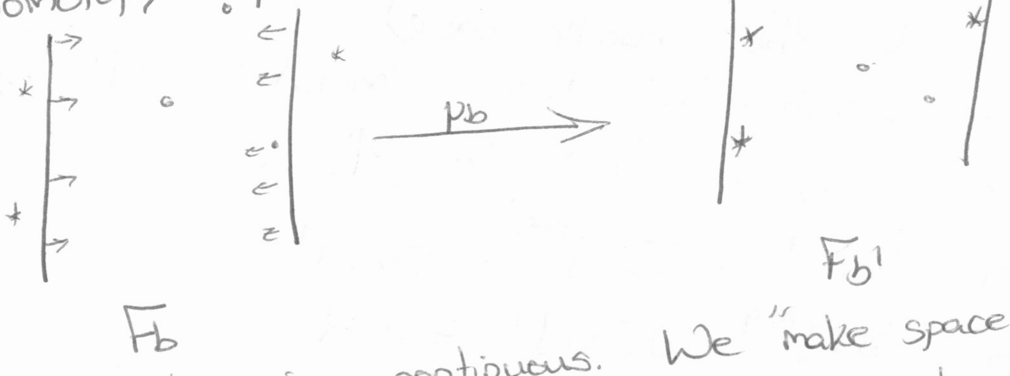
Also $\hat{\Phi}_1(V_n) \subseteq E_n$.

To apply the application mentioned before, we just need:

<u>Claim</u>: For each $b \in U_n$ the monodrome of the fibre

$$\begin{tikzcd}
F_b = q^{-1}(b) \arrow[r, "\mu_b"] \arrow[d, phantom, "\cong"] & F_{\Phi_1(b)} = q^{-1}(\Phi_1(b)) \arrow[d, phantom, "\cong"] \\
F = F \times \{b\} \arrow[r, "\mu_b'"] & F \times \{\overset{b'}{\overset{\shortparallel}{\Phi_1(b)}}\} = F
\end{tikzcd}$$

is a homotopy equivalence.



For fixed b, this is continuous. We "make space" for the extra points in $U\setminus N$ of b that are moved in by Φ_1.

(We cannot do this for all b at the same time, since the number of particles in b that we move in isn't continuous.)

So
$\mu_b(\xi) = \hat{\Phi}_1(\xi) +$ "particles of b in $U\setminus N$"

i.e. for $\xi = [z_1'', \dots, z_r''; x_1'', \dots, x_r'']$

$\mu_b(\xi) = [\psi_1(z_1''), \psi_1(z_2''), \dots, \psi_1(z_r''), \psi_1(z_1), \dots, \psi_1(z_s); x_1'', \dots, x_r'', x_1, \dots, x_r]$

where $[z_1, \dots, z_s, x_1, \dots, x_s]$ is the part of b in $U\setminus N$.

To show the claim, it is enough to show that $\mu_b' \simeq id$.
In other words we want to get rid of the added material from b. Here we use the extra condition:

1) In the case that every components of N intersects $N \cap M_0$, we can kill all extra particles by moving them along the boundary into $N \cap M_0$, so they're gone.

2) In the case that X is path-connected, we can kill the particles by moving their labels to x_0.

So indeed $\mu_b' \simeq id$, so $F_b \longrightarrow F_{b'}$ is a hpty equivalence.

We have thus satisfied all constraints of our application, so we can conclude that q is a quasi-fibration.

Proof of the approximation theorem

We want to prove that for M compact, the approximation map

$$\gamma : C(M, M_0; X) \longrightarrow \mathrm{Sect}(M, M_0, WM ; E_w(X))$$

is a weak homotopy equivalence if

1) (M, M_0) connected or

2) X is path connected.

Part 1 (Reduce to the handle-case)

Let M be constructed from a manifold M' by ~~glueing~~ attaching a handle (H, H') to M', with

$$H \cong [0,1]^m = [0,1]^{m-q} \times [0,1]^q$$
$$H' = [0,1]^{m-q} \times \partial [0,1]^q \subseteq \partial H$$

as $M = M' \cup_{H'} H$.

As M can be built by finitely many of such attachments (which is a difficult theorem) we can do induction: we assume the result for M' and prove it for M.

Example

Now we get a commutative diagram

$$\begin{tikzcd}
C(H, H' ; X) \arrow[r, "j"] \arrow[d, "\gamma_3"] & C(\overset{M' \cup_{H'} H}{\overset{\shortparallel}{M}} - \partial M ; X) \arrow[r, "q"] \arrow[d, "\gamma_2"] & C(M, M' ; X) \arrow[d, "\gamma_1"] \\
\begin{array}{c}\mathrm{Sect}(\sim) \\ (W \setminus H', W \setminus H)\end{array} \arrow[r] & \begin{array}{c}\mathrm{Sect}(\sim) \\ (W \setminus \partial \bar{M} - W \setminus (M \cup H))\end{array} \arrow[r] & \begin{array}{c}\mathrm{Sect}(\sim) \\ (W \setminus M' - W \setminus M)\end{array}
\end{tikzcd}$$

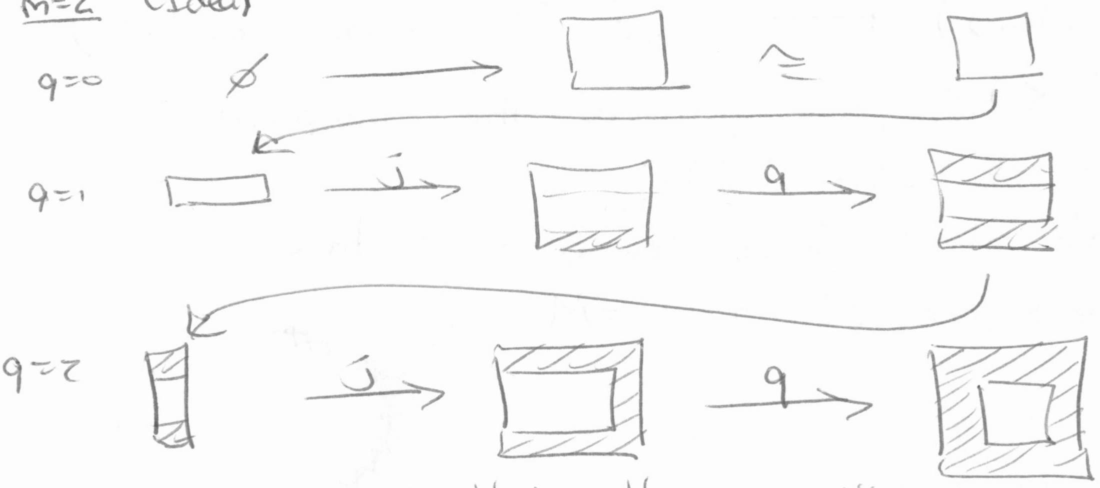
- The top is a quasi-fibration
- The bottom is a fibration.
- By the ~~ind~~ induction hypothesis, γ_1 is a weak homotopy equivalence.
- We will show that γ is a weak hpty equiv for all handles on Wednesday.

So by the l.e.s. of π-groups and the five-lemma also γ_3 is a weak hpty equiv.

So it remains to show:

Part 2 : The theorem is true for all handles.

m=2 (Idea)



We then note that the middle spaces are contractible for $q > 1$. So we can do induction. For $q = m$, we can show that it works, by an example we did.

We then go up using that all q are quasi-fibrations, so we get the result for the fibre $\simeq H_{q-1}$.

Lecture 10 - 09-05-18

Examples:

1) $M = D^m$, $M_0 = \emptyset$, $W = \mathbb{R}^m$, X connected

W is parallelizable, so $E(W;X) \cong W \times \Sigma^m X$

$$\begin{tikzcd}
C(\mathbb{R}^m, X) & \mathrm{map}(\mathbb{R}^m, \mathbb{R}^m \setminus D^m; \Sigma^m X) \cong \Omega^m \Sigma^m X \\
C(D^m; X) \arrow[u, hook] \arrow[r, "\gamma", "\simeq_w"'] & \mathrm{Sect}(\mathbb{R}^m, \mathbb{R}^m \setminus D^m; E_W(X)) \arrow[u, phantom, "\cong"]
\end{tikzcd}$$

2) $M = G$ = Lie group, compact, connected; X connected, $W = G$.

$C(G;X) \simeq_w \mathrm{map}(G, \Sigma^m X)$ $(m = \dim G)$

3) Special case $M = G = SO(2) = S^1$, $m=1$, $M_0 = \emptyset$, X connected.

Then

$C(S^1;X) \simeq_w \Lambda\Sigma X$ = free loop space of ΣX.

We can thus say something about the topology of $\Lambda \Sigma X$ by looking at $C(S^1;X)$.

$$C_n(S^1;X) - C_{n-1}(S^1;X) \cong (S^1 \times \Delta^{n-1})/\mathbb{Z}_n$$

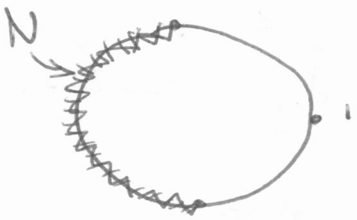
where $\mathbb{Z}_n$ acts on $S^1 \times \Delta^{n-1}$ by

$$(\zeta, (t_0, \dots, t_{n-1})) \xmapsto{T} (e^{2\pi i t_0}\zeta, (t_1, t_2, \dots, t_{n-1}, t_0))$$

Thus $C_n(S^1;X)/C_{n-1}(S^1;X)$ is the one-point compactification of $(S^1 \times \Delta^{n-1})/\mathbb{Z}_n$.

Remark: Some remark about Morse Theory + K-theory

Now consider a subspace N on $S^1 = M$



Let $W = S^1$

The pairs

$$(N, \emptyset) \longrightarrow (M, \emptyset) \longrightarrow (M, N)$$

gives

$$\begin{tikzcd}
C(N;X) \arrow[r, hook] \arrow[d, "\gamma", "\simeq_w"'] & C(M;X) \arrow[r] \arrow[d, "\gamma", "\simeq_w"'] & C(M;N;X) \arrow[d, "\gamma", "\simeq_w"'] \\
\mathrm{map}(S^1, S^1 \setminus N; \Sigma X) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{map}(S^1, \emptyset; \Sigma X) \arrow[r] \arrow[d, phantom, "="] & \mathrm{map}(S^1 \setminus N; \Sigma X) \arrow[d, "\mathrm{eval.}", "\simeq"'] \\
\Omega\Sigma X \arrow[r] & \Lambda \Sigma X \arrow[r, "\mathrm{eval.}"] & \Sigma X
\end{tikzcd}$$

Note that the top row is a quasi-fibration while the bottom row $\mathrm{eval}_1 : \Lambda\Sigma X \longrightarrow \Sigma X$ is a fibration.

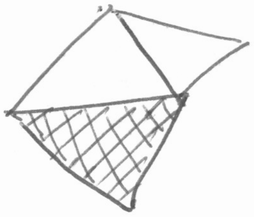
4) Let K be a finite complex, K_0 a subcomplex.

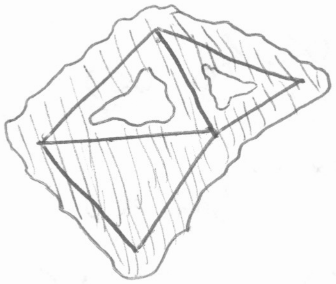
Assume

1) $K \subseteq \mathbb{R}^m$

2) There is a thickening M of K in $\mathbb{R}^m$, i.e. a smooth compact manifold (probably $\partial M \neq \emptyset$) containing K in its interior that (strongly) deformation retracts onto K.

$$\begin{tikzcd} K \arrow[r, hook] & M \arrow[l, dashed, bend right, "\rho_t"'] \end{tikzcd}$$





(Note that this is similar to the situation of a smooth k-submanifold $K \subseteq \mathbb{R}^m$ with M a closed tubular nbhd of K in $\mathbb{R}^m$.)

3) There is $M_0 \subseteq M$ that retracts to K_0 under the deformation retraction ρ_t.

Bödigheimer: this can always be arranged.

We will now see

$$C(M \setminus M_0, \partial M ; X) \xrightarrow[\simeq_w]{\sigma} \mathrm{map}(K, K_0 ; \Sigma^m X)$$

for X connected.

"exterior" "interior"

$(\mathbb{R}^m \setminus M) \sqcup \mathrm{int}\, M$

$\parallel$

Indeed with $W = \mathbb{R}^m$

$$C(M \setminus M_0, \partial M ; X) \xrightarrow[\sigma]{\simeq_w} \mathrm{map}(W \setminus \partial M, W \setminus (M \setminus M_0) ; \Sigma^m X)$$

$$\simeq \mathrm{map}(\mathbb{R}^m \setminus M \sqcup \mathrm{int}\, M, \mathbb{R}^m \setminus M \cup M_0 ; \Sigma^m X)$$

$$\simeq \mathrm{map}(\mathrm{int}(M), \mathrm{int}(M) \cap M_0 ; \Sigma^m X)$$

$$\simeq \mathrm{map}(K, K_0, \Sigma^m X).$$

Snaith splitting for $C(M, M_0 ; X)$

Recall for X connected

$$\begin{tikzcd}
\Sigma J(X) \arrow[r, phantom, "\overset{S'}{\simeq_w}"] \arrow[d, "\simeq_w"] & \Sigma \left(\bigvee_{n \geq 1} J_n(X) / J_{n-1}(X) \right) \arrow[d, phantom, "\cong" rotate=90] \\
\Sigma \Omega \Sigma X & \Sigma \left(\bigvee_{n \geq 1} X^{(n)} \right)
\end{tikzcd}$$

Similarly

$$\Sigma C(\mathbb{R} ; \Sigma X) \simeq_w \Sigma \left(\bigvee_{n \geq 1} C_n(\mathbb{R} ; X) / C_{n-1}(\mathbb{R} ; X) \right)$$

$$C_n \setminus C_{n-1} \cong \tilde{C}^n(\mathbb{R}) \times_{\mathfrak{S}_n} (X \setminus x_0)^n \longrightarrow C^n(\mathbb{R}) \simeq *$$

Let

$$V := \bigvee_{n \geq 1} J_n(X) / J_{n-1}(X)$$

$$V_k := \bigvee_{n \geq 1}^{k} J_n(X) / J_{n-1}(X)$$

To prove $\Sigma J(X) \simeq_w \Sigma V$, we want a map
$$J(X) \longrightarrow \Omega \Sigma V,$$
and since $J(V) \xrightarrow{\simeq_w} \Omega \Sigma V$, we just want to find a map

$$S : J(X) \longrightarrow J(V)$$

Slogan: "Take all subwords and organize them well".

Let $w = x_1 \cdot \ldots \cdot x_r$ be a word. The subwords of length n are given by

$$w_\alpha = x_{i_1} \cdot \ldots \cdot x_{i_n} \qquad \text{for } \alpha = \{ i_1 < \ldots < i_n \}$$

Order all these w_α lexicographically and concatinate them:

$$S_n(w) := w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_t} \in J \left(J_n(X) / J_{n-1}(X) \right)$$

Note that this is well-defined.

(I won't copy the argument of leaving out the letter x_0 etc.)

Concatenating all these S_n gives a map

$$S : J(X) \longrightarrow J(V) \simeq_w \Omega \Sigma V$$

§ - Note that $S(J_r(X)) \subseteq J(V_r)$
- Taking the adjoint gives
$$S' : \Sigma J(X) \longrightarrow \Sigma V.$$

It restricts to

$$S' : \Sigma J_n(X) \longrightarrow \Sigma V_n(X)$$

We get a homotopy commuting diagram

$$\begin{tikzcd}
\Sigma J_n(X)/J_{n-1}(X) \arrow[r, "id"] & \Sigma J_n(X)/J_{n-1}(X) \\
\Sigma J_n(X) \arrow[u, "\text{cofibration}"'] \arrow[r, "S'"] & \Sigma V_n(X) \arrow[u] \\
\Sigma J_{n-1}(X) \arrow[u] \arrow[r, "S'"] & \Sigma V_{n-1}(X) \arrow[u]
\end{tikzcd}$$

For $n=1$, $S' : \Sigma J_1(X) \longrightarrow \Sigma V_1(X)$ is homotopic to id. so it's a weak homotopy equivalence.

Then by induction, all S' are a weak hpty equivalence.

The question is now: how do we generalize this to $C(M, M_0 : X)$? We want an analog for the slogan. We will choose sub-configurations consisting of n particles. However, we won't have a natural ordering. We will see that for this reason, just one suspension won't be enough.

Lecture 11 : 14-05-18

We have seen

- $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma X \wedge Y$
- $\Sigma J(X) \simeq \Sigma(\bigvee_{n\geq 1} X^{(n)})$
- $J(X) \simeq \Omega \Sigma X$

We will see

$$\Sigma^\infty C(\mathbb{R}^\infty : X) \xrightarrow[\simeq_w]{\delta} \Sigma^\infty \Omega^\infty \Sigma^\infty \overset{\text{Snaith}}{\simeq} \Sigma^\infty\Big(\bigvee_{n\geq 1} C_n(\mathbb{R}^\infty : X)/C_{n-1}(\mathbb{R}^\infty : X)\Big)$$

Let

- ▷ (M, M_0) be a compact manifold pair
- ▷ (X, x_0) a connected based pair
- ▷ $C = C(M, M_0 : X)$ with the filtration $C_n(M, M_0 : X)$
- ▷ $D_n = C_n(M, M_0 : X)/C_{n-1}(M, M_0 : X)$

▷ $V = \bigvee_{k\geq 1} D_k$

▷ $V_n = \bigvee_{k=1}^{n} D_k$

Theorem (general Snaith splitting)
There is a weak homotopy equivalence of spectra

$$\gamma : \Sigma^\infty C \xrightarrow{\simeq_w} \Sigma^\infty V,$$

in other words: C and V are stably equivalent.
This means that there is a weak homotopy equivalence

$$\gamma : \Omega^\infty \Sigma^\infty C \longrightarrow \Omega^\infty \Sigma^\infty V$$

of topological spaces.

Proof: Recall

1) $\Omega^\infty \Sigma^\infty X = \underrightarrow{\mathrm{colim}}\, \Omega^k \Sigma^k Y$

2) $C(\mathbb{R}^\infty; Y) \simeq_w \Omega^\infty \Sigma^\infty Y$, if Y connected.

So we want a map

$$\begin{tikzcd}
\Phi : C(\mathbb{R}^\infty; C) \arrow[r] \arrow[d, phantom, "w\wr"] & C(\mathbb{R}^\infty; V) \arrow[d, phantom, "w\wr"] \\
\Omega^\infty \Sigma^\infty C & \Omega^\infty \Sigma^\infty V
\end{tikzcd}$$

We will now construct this map.

For a configuration $\zeta = [z_1, \dots, z_n; x_1, \dots, x_n] \in \tilde{C}_n(M, M_0; X)$ consider all subconfigurations of size at most k:
for $\alpha = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$, define

$\bar{\zeta}_\alpha := [z_{i_1}, \dots, z_{i_k}, x_{i_1}, \dots, x_{i_k}] \in C_k / C_{k-1} =: D_k$

Note that this is a well-defined map, since if one of the z_i is in M_0 or x_i is x_0, then $\bar{\zeta}_\alpha$ is the base point in D_k.

Also consider $z_\alpha := [z_{i_1}, \dots, z_{i_k}] \in \mathrm{Conf}^k(M)$

Choose an embedding $e^k : \mathrm{Conf}^k(M) \hookrightarrow \mathbb{R}^\infty$. Then we can define a map

$$\tilde{C}_n(M, M_0; X) \longrightarrow C(\mathbb{R}^\infty, D_k)$$

by

$$\zeta \longmapsto [e^k(z_{\alpha_1}), \dots, e^k(z_{\alpha_r}); \bar{\zeta}_{\alpha_1}, \dots, \bar{\zeta}_{\alpha_r}]$$

where $r = \binom{n}{k}$ and α_i runs over all $\{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$

Since we map into an unordered configuration space, we see that the order of the z_i in $\zeta = [z_1, \dots, z_n : x_1, \dots, x_n]$ doesn't matter for the final result. So the above map induces a map

$$\Phi'_{n,k} : C_n(M, M_0 ; X) \longrightarrow C(\mathbb{R}^\infty ; D_k) \subseteq C(\mathbb{R}^\infty ; V).$$

We can assume all $e^k : \mathrm{Conf}^k(M) \longrightarrow \mathbb{R}^\infty$ to be disjoint, so we can "sum" the maps $\Phi'_{n,k}$ for $1 \leq k \leq n$ to get a map

$$\Phi'_n : C_n(M, M_0 ; X) \longrightarrow C(\mathbb{R}^\infty, V)$$

Note that we can do this for all n and that we have the relation $\Phi'_{n+1}|_{C_n} = \Phi'_n$, so we have a map

$$\Phi' : C \longrightarrow C(\mathbb{R}^\infty, V).$$

Note that C is naturally embedded in $C(\mathbb{R}^\infty, C)$ by mapping ζ to $[0 ; \zeta]$. We now want to extend Φ' to a map $\Phi : C(\mathbb{R}^\infty, C) \longrightarrow C(\mathbb{R}^\infty, V)$. That we can do this is a separate lemma:

Lemma: Each map $f' : Y \longrightarrow C(\mathbb{R}^\infty, Z)$ has an extension $f : C(\mathbb{R}^\infty, Y) \longrightarrow C(\mathbb{R}^\infty, Z)$

Proof: By taking the ~~1pt-compactification~~ [illegible] inclusion $\mathbb{R}^\infty \cong \mathring{\mathbb{I}}^\infty \hookrightarrow \mathbb{I}^\infty$, we can regard f' as

$$f' : Y \longrightarrow C(\mathbb{I}^\infty, Z)$$

It is now clear that this extends to the little ∞-cubes operad $E^\infty(Y)$ consisting of finite tuples of disjoint affine embeddings $\mathbb{I}^\infty \hookrightarrow$ ~~$\mathbb{R}^\infty$~~ $\mathbb{I}^\infty$, giving

$$E^\infty(Y) \longrightarrow C(\mathbb{I}^\infty, Z) \simeq C(\mathbb{R}^\infty, Z)$$

The space $C(\mathbb{R}^\infty, Y)$ is a deformation retract of $E^\infty(Y)$ via some ε-map $C(\mathbb{R}^\infty, Y) \xhookrightarrow{\varepsilon} E^\infty(Y)$.

The composition gives the requiered map.

So we get a map $\Phi : C(\mathbb{R}^\infty, C) \longrightarrow C(\mathbb{R}^\infty, V)$, i.e. a map $\Phi : \Omega^\infty \Sigma^\infty C \longrightarrow \Omega^\infty \Sigma^\infty V$.

We consider

$$\begin{tikzcd}
C(\mathbb{R}^\infty; D_n) \arrow[r, "="] & C(\mathbb{R}^\infty; D_n) \\
C(\mathbb{R}^\infty; C_n) \arrow[u] \arrow[r, "\Phi_n"] \arrow[ur, phantom, "②"] & C(\mathbb{R}^\infty; V_n) \arrow[u] \\
C(\mathbb{R}^\infty; C_{n-1}) \arrow[u] \arrow[r, "\Phi_{n-1}"] \arrow[ur, phantom, "①"] & C(\mathbb{R}^\infty; V_{n-1}) \arrow[u]
\end{tikzcd}$$

Then ① commutes, as $\Phi_n'|_{C_{n-1}} = \Phi_n'$

Also ② commutes, as elements in C_n have only one non-trivial subconfiguration in D_n.

By naturality of $\gamma : C(\mathbb{R}^\infty, -) \longrightarrow \Omega^\infty \Sigma^\infty(-)$, we get

$$\begin{tikzcd}
\Omega^\infty \Sigma^\infty D_n \arrow[r, equal] & \Omega^\infty \Sigma^\infty D_n \\
\Omega^\infty \Sigma^\infty C_n \arrow[u] \arrow[r, "\Phi_n"] & \Omega^\infty \Sigma^\infty V_n \arrow[u] \\
\Omega \Sigma^\infty C_{n-1} \arrow[u] \arrow[r, "\Phi_{n-1}"] & \Omega^\infty \Sigma^\infty V_{n-1} \arrow[u]
\end{tikzcd}$$

Since $\Omega^\infty \Sigma^\infty(-)$ turns cofibrations into fibrations, we have long exact sequences of homotopy groups

Lemma: $\Phi_n : \Omega^\infty \Sigma^\infty C_n \longrightarrow \Omega^\infty \Sigma^\infty V_n$ induces an isomorphism on all π_q, $q \geq 1$ for all n.

Proof For $n=1$, $V_1 = D_1 = C_1 / C_0 = C_1$, so Φ_1 is ~~indu~~ an extension of $C_1 \longrightarrow \Omega^\infty \Sigma^\infty V_1 \cong \Omega^\infty \Sigma^\infty C_1$ to $\Omega^\infty \Sigma^\infty C_1$ which is homotopic to the identity.

For $n>1$, assume Φ_{n-1} induces isomorphisms in π_q. Then use the five-lemma

$$\begin{tikzcd}
\pi_q(\Omega^\infty \Sigma^\infty D_n) \arrow[r, equal] & \pi_q(\Omega^\infty \Sigma^\infty D_n) \\
\pi_q(\Omega^\infty \Sigma^\infty C_n) \arrow[u] \arrow[r] & \pi_q(\Omega^\infty \Sigma^\infty V_n) \arrow[u] \\
\pi_q(\Omega^\infty \Sigma^\infty C_{n-1}) \arrow[u] \arrow[r] & \pi_q(\Omega^\infty \Sigma^\infty V_{n-1}) \arrow[u]
\end{tikzcd}$$

Infinite Symmetric Product

For a pointed space, define the <u>n-th symmetric product of X</u> as

$$SP_n(X) := X^n / \mathfrak{S}_n$$

Elements are denoted either $[z_1, \dots, z_n]$ or as $\sum_{j=1}^{n} z_j = \sum_i k_i z_i$, where in the last sum, $k_i \in \mathbb{N}$ and all z_i are different.

<u>Examples</u>:

1) $\mathrm{Conf}^n(X) \subseteq SP_n(X)$

2) $$\begin{tikzcd} SP_n(S^1) \arrow[r, "\mu"] & S^1 \\ {[z_1, \dots, z_n]} \arrow[r, mapsto] & z_1 \cdot \dots \cdot z_n \end{tikzcd}$$

3) (Fundamental theorem of algebra)

$$\begin{tikzcd} SP_n(S^2) \arrow[r, phantom, "\simeq"] & \mathbb{CP}^n & {[1 : a_{n-1} : \dots : a_1 : a_0]} \\ \mathbb{C} \arrow[u, phantom, "\shortparallel"] & & \\ \zeta = [z_1, \dots, z_n] \arrow[r, mapsto] & P_\zeta(t) = \prod_{i=1}^{n} (t - z_i) = t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 \arrow[ur, mapsto] & \end{tikzcd}$$

Define the (infinite) <u>symmetric product</u> of X as

$$SP(X) = SP_\infty(X) := \mathrm{colim}\, SP_n(X)$$

using the embeddings

$$\begin{tikzcd} SP_n(X) \arrow[r, hook] & SP_{n+1}(X) \\ {[x_0, \dots, x_n]} \arrow[r, mapsto] & {[x_1, \dots, x_n, x_0]}. \end{tikzcd}$$

This is the topological free abelian monoid, strictly assoc., strictly commut. generated by X.

<u>Dold-Thom</u> There is a natural equivalence of functors

$$\pi_q(SP(-)) \xrightarrow{\simeq} H_q(X; \mathbb{Z})$$

for connected based spaces (X, x_0).

Lecture 12 : 16-05-18

<u>Remark</u>: Elements $\zeta = \sum k_i x_i$ are called divisors.

<u>Examples</u>: Meromorphic functions $f \rightsquigarrow$

- $\mathrm{div}^+(f)$ = divisor of its zeroes
- $\mathrm{div}^{-1}(f)$ = divisor of its poles
- $\mathrm{div}(f) = \mathrm{div}^+(f) - \mathrm{div}^-(f)$

Meromorphic forms: zeroes $Z(f)$, poles $P(f)$

If we are precise:

$$SP_\infty(X,x_0) = \lim \{ SP_n(X) \longrightarrow SP_{n+1}(X) \}$$
$$[x_1,\dots,x_n] \longmapsto [x_1,\dots,x_n,x_0]$$

This is the topological free abelian monoid generated by X, x_0 unit.

We also have

$$SP_\infty(X) := \coprod SP_n(X)$$

This is the topological free abelian monoid generated by X.

We also have

$SP\mathbb{Z}(X,x_0)$ = free abelian group, generated by $X \setminus \{x_0\}$ (or by X modulo $\mathbb{Z}<x_0>$) plus topology.

And for an abelian group (or monoid) $\mathbb{A}$:

$$SP\mathbb{A}(X,x_0) = \{ \sum_{i=1}^{n} a_i x_i \,,\ a_i \in \mathbb{A},\ x_i \in X \},$$

in other words, it is $(\coprod_{n \geq 0} X^n \times \mathbb{A}^n)/\sim$ with

1) $(x_1,\dots,x_n : a_1,\dots,a_n) \sim (x_{\pi(1)},\dots,x_{\pi(n)} : a_{\pi(1)},\dots,a_{\pi(n)})$

2) $(x_1,\dots,x_n : a_1,\dots,a_n) \sim (x_1,\dots,\hat{x_i},\dots,x_n : a_1,\dots,\hat{a_i},\dots,a_n)$
 if $a_i = 0$ or $x_i = x_0$

3) $(x_1,\dots,x_i,\dots,x_j,\dots,x_n : a_1,\dots,a_i,\dots,a_j,\dots,a_n)$
 $\sim (x_1,\dots,x_i,\dots,\hat{x_j},\dots,x_n : a_1,\dots,a_i+a_j,\dots,\hat{a_j},\dots,a_n)$
 if $x_i = x_j$

We can use this in particular for topological groups $\mathbb{A}$.

It turns out that $SP\mathbb{A}(X,x_0)$ is a product of Eilenberg Maclane spaces.

Segel's Γ-category

Objects: $\underline{n} = \{0,1,\dots,n\}$

Morphisms: $\varphi : \underline{n} \longrightarrow \underline{m}$ based maps

ex.: $\{0,1,\dots,n\}$ ↻ permutations

ex.:

$$d_i(j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

ex.:

Think of multiplication.
Inherently invariant under permutation of 1 and 2.

Also we have

so it is inherently associative.

Let $\mathbb{A} : \Gamma \longrightarrow Top_*$ be a Γ-functor. So we have maps

$$\mathbb{A}(d_i) : \mathbb{A}(n) \longrightarrow \mathbb{A}(1)$$

combined giving maps

$$d : \mathbb{A}(n) \longrightarrow \mathbb{A}(1)^n$$

Also we have in particular $\mu : \mathbb{A}(2) \longrightarrow \mathbb{A}(1)$

We can look at covariant Γ-functors $\mathbb{A}$ s.t.

1) $\mathbb{A}(0) \simeq *$
2) $d : \mathbb{A}(n) \xrightarrow{\simeq} \mathbb{A}(1)^n$ is a homotopy equivalence.

Letting $\mathbb{A}$ be $SP\mathbb{A}(X)$, we note that it has these above properties.

Theorem:
(i) The functors $X \longmapsto h_q(X) := \pi_q(SP\mathbb{A}(X))$ are homology theories, defined on CW-complexes.
(ii) Any connective homology theory h_* is of this form, i.e.

$$h_*(X) \cong \pi_q(SP\mathbb{A}(X))$$

for some Segel-space $\mathbb{A}$

(h_* connective if $h_*(X) = 0$ for $* < 0$.)

Note: For general Γ-functors $\mathbb{A}$, we get a space $SP\mathbb{A}(X)$ as

$$SP\mathbb{A}(X) := \Big(\coprod_{n \geq 0} X^n \times \mathbb{A}(n) \Big) / \sim$$

with similar identifications.

For $\mathbb{A}$-functors with properties 1) and 2), we have the theorem

Examples: 1) $SP(X) = SP\mathbb{N}(X) \rightsquigarrow H_*(X; \mathbb{Z})$
2) $SP\mathbb{A}(X) \rightsquigarrow H_*(X; \mathbb{A})$ ($\mathbb{A}$ abelian grp. discret
3) $SP\mathbb{A}(X) \rightsquigarrow \bigoplus_{i \geq 0} H_{*+i}(X; \pi_i(\mathbb{A}))$
($\mathbb{A}$ abelian topological group)

$$SP/A(\mathbb{S}^n) \simeq \prod_{i\geq 0} K(\pi_i(/A), n+i)$$

Dold-Thom theorem

For a connected space X, there is a natural isomorphism
$$\Theta : \pi_q(SP(X,x_0)) \longrightarrow \tilde{H}(X;\mathbb{Z})$$

Proof Main ingredient is the following proposition

Proposition: Let $A \overset{\iota}{\hookrightarrow} X \xrightarrow{q} (X,A)$ be a based cofibration (X,A) connected with cofiber $C_\iota \simeq X/A$. Then the induced map $Q = SP(q)$ is a quasi-fibration with fibre $SP(A,x_0)$:

$$\begin{tikzcd} SP(A,x_0) \arrow[r] \arrow[d, equal] & SP(X,x_0) \arrow[r, "Q"] \arrow[d, equal] & SP(X/A, \bar{x}_0) = SP(X,A) \arrow[d, equal] \\ F & E & B \end{tikzcd}$$

Proof Filtration $B_n := SP_n(X,A)$ = at most n points outside A
$E_n := Q^{-1}(B_n)$

③ $E_n \setminus E_{n-1}$ = exactly n points outside A, some more inside A.

The image under Q is $B_n \setminus B_{n-1}$ = exactly n points outside A (the ones in A are killed)

Then
$$\begin{tikzcd} E_n\setminus E_{n-1} \arrow[rr, "\psi", "\simeq"'] \arrow[dr, "Q"'] & & (B_n\setminus B_{n-1}) \times F \arrow[dl, "proj"] \\ & B_n \setminus B_{n-1} & \end{tikzcd}$$

where ψ exists as the number of points in $X\setminus A$ is constant n. We can represent elements $B_n\setminus B_{n-1}$ only by n points outside A, and elements of F only by some points inside A, so we have some decomposition map $\psi : E_n \setminus E_{n-1} \xrightarrow{\simeq} (B_n \setminus B_{n-1}) \times F$.

In particular, $E_n \setminus E_{n-1} \xrightarrow{Q} B_n \setminus B_{n-1}$ is a q.f.

④ There is an open neighborhood U of A in X with a strong def. retr. $\rho_t : (X,A,x_0) \longrightarrow (X,A,x_0)$ s.t.
$\rho_0 = id_X$, $\rho_1(U) \subseteq A$.

Define
$U_n := \{\xi \in B_{n+1} \mid$ at least one point of ξ is in $U\}$
= open nbhd of B_n in B_{n+1} that def. retr. onto B_n
$V_n := Q^{-1}(U_n)$.

Let $R_t := SP(P_t)$ the retraction of U_n onto B_n and $\hat{R}_t$ the retraction of V_n onto E_n.

We can almost use the lemma about quasi-fibrations. What is left to show is what happens with a single fiber $F_b = Q^{-1}(b)$ for $b \in U_n$ under R_t. Let $b' := R_1(b) \in B_n$.

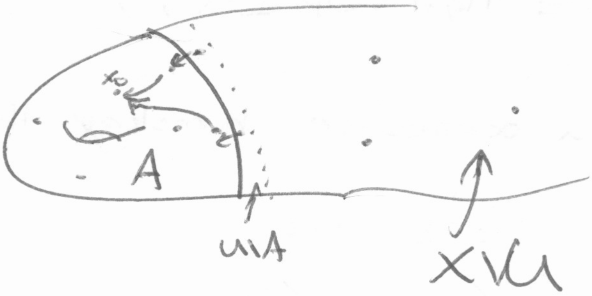
We have

$$\begin{tikzcd}
F_b \arrow[r, "\hat{R}_1"] \arrow[d, hook] & F_{b'} \arrow[d, hook] \\
E_n \setminus E_{n-1} & E_{n-1} \setminus E_{n-2}
\end{tikzcd}$$

$$\begin{gathered} E_n \setminus E_{n-1} \\ \cong \\ (B_n \setminus B_{n-1}) \times F \longrightarrow (B_{n-1} \setminus B_{n-2}) \times F \end{gathered}$$

$$\begin{gathered} (\xi_{X \setminus A}, \xi_A) \\ \cong \\ (\xi_{X \setminus U}, \xi_{U \setminus A}, \xi_A) \longmapsto (\hat{R}_1(\xi_{X \setminus U}), \tilde{R}_1(\xi_{U \setminus A}) + \hat{R}(\xi_A)) \end{gathered}$$

But for fixed b we have $\xi_{X \setminus A}$ fixed, so $\xi_{U \setminus A}$ is fixed, and we can move $\xi_{U \setminus A}$ to the base point as (X, A) is connected.



So $F_b \simeq F_{b'}$ and we can use the lemma to conclude that Q is a quasi-fibration.

Proposition: We have a homology theory for connected spaces:

$$h_q(X) := \pi_q(SP(X, x_0))$$

Proof: 1) functor of pair (X, A)

2) Homotopy invariant for maps of pairs and hpts of pairs

3) Long exact homology sequence for pair (X, A):

$$(A, x_0) \xhookrightarrow{i} (X, x_0) \xrightarrow{q} (X, A)$$

So by previous prop

$$SP(A, x_0) \xrightarrow{SP(i)} SP(X, x_0) \xrightarrow{Q = SP(q)} SP(X, A)$$

Q is a quasi-fibration, so we get a l.e.s.

$$\begin{tikzcd}
\arrow[r] & \pi_{q+1}(SP(X,A)) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_q(SP(A,x_0)) \arrow[r] & \pi_q(SP(X,x_0)) \arrow[r] & \pi_q(SP(X,A)) \arrow[r] & \cdots \\
& h_{q+1}(X,A) \arrow[r] & h_q(A) \arrow[r] & h_q(X) \arrow[r] & h_q(X,A) \arrow[r] & \cdots
\end{tikzcd}$$

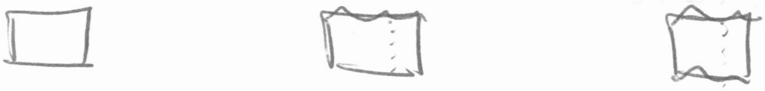
4) Excision: Let $W \subseteq \bar{W} \subset \mathring{A} \subset A \subseteq X$. We have

$$SP(X\setminus W, A\setminus W) \xrightarrow{\cong} SP(X,A)$$

~~trivial as we~~

5) Suspension

$$X \xrightarrow{i} CX \xrightarrow{q} SX$$



$$X \longrightarrow (X\times I, X\times\{1\}) \longrightarrow (X\times I, X\times\partial I,$$

Then

$$SP(X) \xrightarrow{\mathfrak{Z}} SP(X\times I, X\times\{1\}^{\cup x_0\times I}) \xrightarrow{Q} Sp(X\times I, X\times\partial I \cup x_0\times I)$$

is a q.f. with contractible total space, so from l.e.s. we get

$$h_q(X) \coloneqq \pi_q(Sp(X)) \cong \pi_{q+1}(Sp\,\Sigma(X)) = h_{q+1}(\Sigma X) \quad \square$$

Now note that h_q is even a connective homology theory as we take homotopy groups.

We can show ~~h_q~~ $\hat{h}_q(S^0_0) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q\neq 0 \end{cases}$

and by the suspension property

$$\hat{h}_q(S^n) = \begin{cases} \mathbb{Z} & q=n \\ 0 & q\neq n. \end{cases}$$

Now we want to have a <u>transformation</u>

$$\Theta : h_q(X) \longrightarrow H_q(X)$$

<u>Existence</u>: Compare cellular of h_* with cellular of H_*.

$$\begin{tikzcd}
Ch_n(X) := h_n(X_n, X_{n-1}) \arrow[d, dashed] & \cong Frab(\mathfrak{X}_n) \arrow[d, Leftarrow, "\Theta"] \\
CH_n(X) = H_n(X_n, X_{n-1}) & \cong Frab(\mathfrak{X}_n)
\end{tikzcd}$$

This gives $\Theta : h_n(X) \longrightarrow H_n(X)$.

Hurewicz-map: The inclusion $j: X \longrightarrow SP(X, x_0)$ induces

$$\begin{tikzcd}
\pi_q(X, x_0) \arrow[r, "j_*"] \arrow[d, "hur^X"'] & \pi_q(SP(X, x_0)) = h_q(X) \arrow[d, "hur^{SP(X,x_0)}"] \\
H_q(X, x_0) \arrow[ur, "\cong"] \arrow[r, "j_*"] & H_q(SP(X, x_0))
\end{tikzcd}$$

Note that $SP(X, x_0)$ is an h-space, so $\pi_q(SP(X, x_0))$ is indeed abelian.

Note that $\pi_q(SP(S^n)) = \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq 0. \end{cases}$

so $SP(S^n)$ is a ~~KG(~~ $K(\mathbb{Z}, n)$.

Similarly we have in general

$$SP(M(G, n)) = K(G, n)$$

for general Moore-spaces.

Lecture 13 28-05-18

Recall we constructed a functor

$$(X, x_0) \longmapsto SP(X, x_0)$$

We can extend this to pairs

$$(X, A) \longmapsto SP(X, A) := SP(X/A, A/A)$$

This is a covariant homotopy invariant functor.

Dold-Thom theorem: For X a connected CW-complex

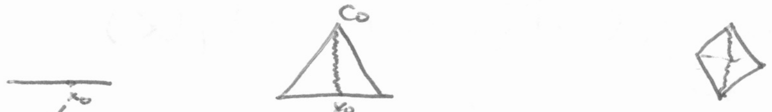
$$\Theta: \pi_i(SP(X, x_0)) \xrightarrow{\cong} H_i(X, x_0; \mathbb{Z})$$

Corollary 1: For any connected CW-complex X, the space $SP(X, x_0)$ is an infinite loop space, i.e. there is a sequen $SP(X) = X_0, X_1, X_2, \ldots$ and for each i a (weak) homotopy equivalence $X_i \xrightarrow{\simeq_w} \Omega X_{i+1}$ ("X_{i+1} is a delooping of X_i")

There are many obstructions to be deloopable

- ▷ abelian fund group
- ▷ h-space with inverse
- ▷ action of little cubes operad

<u>Proof</u>: the sequence of maps



$$X \xrightarrow{\iota} CX \longrightarrow \Sigma X$$

is a cofibration, so $SP(-)$ turns this into a quasi-fibration

$$\begin{tikzcd} SP(X, x_0) \arrow[r] & SP(CX, c_0) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & SP(\Sigma X, \bar{x}_0) \\ & * & \end{tikzcd}$$

so the inclusion of the fibre $SP(X, x_0)$ into the homotopy fibre of $SP(CX) \longrightarrow SP(\Sigma X)$, but as $SP(CX) \simeq *$, this is just the loop space $\Omega SP(\Sigma X)$, i.e. we have a weak hpty equivalence

$$SP(X) \xrightarrow{\simeq_w} \Omega SP(\Sigma X)$$

<u>Corollary 2</u>: $SP(S^n, \infty) \simeq \Omega^1 SP(S^{n+1}, \infty)$

<u>Example</u>:

$$S^1 = \underbrace{SP(S^1, \infty)}_{\text{eigenval. of unit. matrices}} \simeq \Omega \underbrace{SP(S^2, \infty)}_{\mathbb{C}P^\infty} \simeq \Omega^2 SP(S^3, \infty)$$

<u>Corollary 3</u>:

$$\pi_i(SP(S^n, \infty)) \cong \tilde{H}_i(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

So

$$[S^i, SP(S^n, \infty)] \cong \tilde{H}_i(S^n, \mathbb{Z})$$

<u>Definition</u>: A connected based space (= CW-complex) with exactly one non-trivial homotopy group is called an <u>Eilenberg MacLane space.</u>

Notation: $K(G, n)$ is an EML-space of type (G, n) iff

$$\pi_i(K(G, n)) = \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

For G discrete, $K(G, 1) \simeq BG$, where $BG = EG/G$, where EG is a contractible free G-space.

A $K(G, n)$ is not unique, only up to homotopy equivalence as we will see.

Corollary 4: If $M = M(G, n)$ is a Moore-space of type (G, n), then $K = SP(M, *)$ is an EML-space of type (G, n).

Example: As $M(\mathbb{Z}/_2, 1) = \mathbb{R}P^2$, we see

$$SP(\mathbb{R}P^2, \infty) = K(\mathbb{Z}/_2, 1) \simeq \mathbb{R}P^\infty$$

Exercise: $SP_n(S^2, \infty) \triangleq \mathbb{C}P^n$

$SP_n(\mathbb{R}P^2, \infty) \triangleq \mathbb{R}P^{2n}$

$$\begin{tikzcd} SP_n(G) \arrow[r, "\pi_*"] \arrow[d] & SP_n(X/_G) \arrow[dl, "\tilde{\pi}"'] \arrow[d, "\triangleq"] \\ SP_{n(G)}(X) & im(\tilde{\pi}) \triangleq SP^G_{n(G)}(x) \arrow[l] \end{tikzcd}$$

$$SP(X/_G) = SP(x)^G \longleftarrow \text{fixed points of induced action}$$

Example: $K(G, n) \simeq \Omega K(G, n+1)$

So each EML-space is an infinite loop space.

Goal: $[X, K(G, n)] \cong H^n(X; G)$

for G abelian, $n \geq 1$.

Chapter: Brown representability theorem

Let Top_0 be the category of based topological spaces

CW_0^{conn} = cat. of conn. based CW-complexes with basepoint a vertex

Set_0 = category of based sets.

Consider contravariant functors $F: CW_0^{conn} \longrightarrow Set_0$

Example: K a fixed CW-complex with base point k_0.

$F(X) := [X, K]_0$ = based hpty classes of maps, with distinctive element $[c]$ for $c: X \longrightarrow \{k_0\} \subseteq K$.

Definition: We call a functor F representable if

$F(X) \triangleq [X, K]$

for some K, i.e. an equivalence of functors $F(-) \to [-, K]$. The space K is called a <u>representing space</u>

Examples 1) $F(X) := H^1(X, \mathbb{Z}) \triangleq [X, S^1]$ $K(\mathbb{Z}, 1)$

2) $F(X) := H^2(X, \mathbb{Z}) \triangleq [X, \mathbb{C}P^\infty]$ $K(\mathbb{Z}, 2)$

3) $F(X) := K^0_{\mathbb{C}}(X)$ = 0-th K-group/$\mathbb{C}$ $\triangleq [X, BU]$ (complex vector bundles over X.) ($BU = \lim BU(n) = \lim Gr_n(\mathbb{C}^\infty$

<u>Notation</u> For $\iota : A \hookrightarrow X$, we denote $F(\iota)$ by $F(A) \xleftarrow{\iota^*} F(x)$ (note direction of arrow!)

written as $\xi \longmapsto \iota^*(\xi) = \xi|_A$

We can now formulate our <u>extended goal</u>:

Given <u>any</u> reduced cohomology theory $h^* = (h^n)_{n\in\mathbb{Z}}$ we want to find spaces K_n such that

$$h^n(X) \cong [X, K] \qquad \text{for all } n \in \mathbb{Z}$$

such that there are maps $K_n \longrightarrow \Omega K_{n+1}$ inducing the suspension homomorphisms - i.e.

$$\begin{tikzcd}
h^n(X) \cong [X, K] \arrow[rd] \arrow[d, "\text{suspension}"'] & \\
& {[X, \Omega K_{n+1}]} \\
h^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}] \arrow[ru, "\cong"] &
\end{tikzcd}$$

commutes.

The question now is: under what conditions is a functor F representable?

Consider the following three axioms

(BR1) <u>homotopy invariant</u>: $f \underset{\text{rel } x_0}{\simeq} g \Longrightarrow f^* = g^*$

(BR2) <u>Mayer-Vietoris</u>: For a push-out of subcomplexes

$$\begin{tikzcd}
A \arrow[r] & X = A \cup B \\
x_0 \in C = A \cap B \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd}$$

(a "co-cartesian square"), we get a pullback of based sets (i.e "cartesian square")

$$\begin{tikzcd}
F(A) \arrow[d] & F(x) \arrow[l] \arrow[d] \\
F(C) & F(B) \arrow[l]
\end{tikzcd}$$

i.e. if α, β have $\alpha|_C = \beta|_C$ there is a unique ξ s.t. $\alpha = \xi|_A$, $\beta = \xi|_B$.

(BR3) Wedge Axiom: For a wedge $X = \bigvee X_i$, the inclusions $X_i \xrightarrow{\iota_i} X$ induce a bijection

$$F(\bigvee_i X_i) \longrightarrow \prod_i F(X_i)$$

Examples: $h^* = (h^n)_{n\in\mathbb{Z}}$ = reduced cohomology with wedge ax.
Then every h^n satisfies (BR 1-3)

Remark: For all such F, $\boxed{F(pt) = \{0\}}$, since

$$F(pt) \cong F(pt \vee pt) \cong F(pt) \times F(pt)$$

by the wedge axiom.

Also $\boxed{F(\Sigma X) \text{ is an abelian group}}$ via

$$\nabla : \Sigma X \longrightarrow \Sigma X \vee \Sigma X$$

(∇ has a co-unit and co-inverse and is co-associative.)

Then $\nabla^* : F(\Sigma X) \times F(\Sigma X) \cong F(\Sigma X \vee \Sigma X) \longrightarrow F(\Sigma X)$ defines a group structure on $F(\Sigma X)$.

Brown Representability theorem:
Let $F : CW_0^{conn} \longrightarrow Set_0$ be a contravariant functor satisfying Br1 - Br3, then there is a connected based CW-complex K and an element $w \in F(K)$ such that

$$\Theta_w^X : [X, K] \longrightarrow F(X)$$

$$[f : X \to K] \longmapsto f^*(w) \in F(X)$$

is a bijection of based sets for all $X \in CW_0^{conn}$
(It is obviously natural by the way it is defined and thus is an equivalence of functors $[-, K] \longrightarrow F(-)$)

Lecture 14: 30-05-18

Remarks before the proof:
1) We call w universal for K. K is a representing space of F.
We say (K, w) is a universal pair for F.
2) We call $w \in F(K)$ n-universal if
$\Theta_w(S^i) : \pi_i(K) = [S^i, K] \longrightarrow F(S^i)$
is surjective for $i \leq n$ and has trivial kernel for $i < n$.

<u>Warning</u>: The kernel of a morphism

$$\varphi : (S, s) \longrightarrow (S, s')$$

in Sets is $\varphi^{-1}(s')$. It contains at least s.
The statement "$\ker(\varphi)$ is trivial" means $\varphi^{-1}(s') = \{s\}$.
This is <u>less</u> than injectivity.

3) If (K, ω) is n-universal for all $n \geq 1$, we call it ∞-universal.

4) For X a CW-complex, (K, ω) is ∞-universal $\Longleftrightarrow$ it is universal.

<u>Lemma 1</u>: For any connected Z and $\zeta \in F(Z)$ there is a ∞-universal pair (K, ω) with $Z \subseteq K$ and $\omega|_K = \zeta$, i.e. the map $F(K) \longrightarrow F(Z)$ induced by $Z \hookrightarrow K$ sends ω to ζ.

<u>Proof</u>: We start ~~by~~ with Z and construct K by attaching cells.

① $K_1 := Z \vee \bigvee_\alpha \mathbb{S}^1_\alpha$, where $\alpha \in F(\mathbb{S}^1)$

By axiom BR3 there is a bijection

$$\begin{array}{ccc} F(K_1) & \xrightarrow{\cong} & F(Z) \times \prod_\alpha F(\mathbb{S}^1_\alpha) \\ \omega_1 & \longleftarrow\!\shortmid & (\zeta, (\alpha)_\alpha) \end{array}$$

For each index α we have an element $(\alpha)_\alpha \in F(\mathbb{S}^1_\alpha)$. We define $\omega_1 \in F(K_1)$ to be the inverse image of $(\zeta, (\alpha)_\alpha)$.
Thus we have an element $\omega \in F(K_1)$ with

$$\omega_1|_Z = \zeta, \qquad \omega_1|_{\mathbb{S}^1_\alpha} = \alpha$$

Obviously it is 1-universal, as

$$\Theta_{\omega_1}(\mathbb{S}^1) : \pi_1(K_1) \longrightarrow F(\mathbb{S}^1)$$

reaches all $\alpha \in F(\mathbb{S}^1)$

② Suppose we have constructed $Z \subseteq K_1 \subseteq \ldots \subseteq K_{n-1} \subseteq K_n$, and we have found $\omega_n \in F(K_n)$ which is n-universal and $\omega_n|_Z = \zeta$.

Consider the kernel of $\Theta_{\omega_n}(\mathbb{S}^n) : \pi_n(K_n) \longrightarrow F(\mathbb{S}^n)$ and represent each element $\alpha \in \ker(\Theta_{\omega_n}(\mathbb{S}^n))$ by some map $f_\alpha : \mathbb{S}^n \longrightarrow K_n$. Set

$$f : \bigvee_\alpha \mathbb{S}^n_\alpha \longrightarrow K_n \qquad f|_{\mathbb{S}^n_\alpha} = f_\alpha$$

Form the mapping cylinder $Cyl(f)$. Since it deformation retracts onto K_n we have $K \xhookrightarrow[i]{\simeq} Cyl(f)$, so we have a bijection

$$F(K_n) \overset{\iota^*}{\cong} F(Cyl(f))$$

There is a unique $\omega_n' \in F(cyl(f))$ with $\omega_n = \iota^*(\omega_n') = \omega_n'|_K$.
On S^n_α it restricts to the trivial element: $\omega_n'|_{S^n_\alpha} = 0$.

$$\begin{tikzcd}
K_n \arrow[r, hook, "\iota"] & Cyl(f) \\
& S^n_\alpha \arrow[u, hook, "j_\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(K_n) & F(Cyl(f)) \arrow[l, "\iota^*"', "\cong"] \arrow[d, "j_\alpha^*"] \\
\omega_n & \omega_n' \arrow[l, mapsto] \arrow[d, mapsto] \\
& 0 \quad F(S^n_\alpha)
\end{tikzcd}$$

③ The (reduced) mapping cone $C(f)$ has in $F(C(f))$ (using BR2) an element $\gamma \in F(C(f))$ with $\gamma|_{K_n} = \omega_n$
Indeed, we have

$$\begin{tikzcd}
\bigvee_\alpha S^n_\alpha \arrow[r, "f"] & K_n \arrow[r] \arrow[d, hook] & C(f) \arrow[d, equal] \\
& Cyl(f) \arrow[r] & Cyl(f)/\bigvee_\alpha S^n_\alpha
\end{tikzcd}$$

cofibration

By BR2 we have an exact sequence

$$\begin{tikzcd}
F(\bigvee_\alpha S^n_\alpha) & F(K_n) \arrow[l] & F(C(f)) \arrow[l] \\
0 & \omega_n \arrow[l, mapsto] \arrow[d, leftrightarrow] & \gamma \arrow[l, mapsto] \\
& \omega_n' \arrow[ul, dotted, "F."] &
\end{tikzcd}$$

④ Set $K_{n+1} := C(f) \vee \bigvee_\beta S^{n+1}_\beta$

where β ranges over all $\beta \in F(S^{n+1})$

By BR3 we find via

$$\begin{tikzcd}
F(K_{n+1}) \arrow[r, "\cong"] & F(C(f)) \times \prod_\beta F(S^{n+1}_\beta) \\
\omega_{n+1} & (\gamma, (\beta)_\beta) \arrow[l, mapsto]
\end{tikzcd}$$

an element $\omega_{n+1} \in F(K_{n+1})$ with

$$\omega_{n+1}|_{C(f)} = \gamma, \qquad \omega_{n+1}|_{S^{n+1}_\beta} = \beta$$

So

$$\omega_{n+1}|_{K_n} = \omega_n$$

Note that $C(f)$ can be written as

$$K_n \cup_f (\bigcup_\alpha e^{n+1}_\alpha)$$

where the e^{n+1}_α are non-trivially attached by f. So

$$K_{n+1} = K_n \cup \underbrace{(\bigcup_\alpha e^{n+1}_\alpha)}_{\text{non-trivially attached}} \cup \underbrace{(\bigcup_\beta e^{n+1}_\beta)}_{\text{trivially attached}}.$$

⑤ To see that (K_{n+1}, ω_{n+1}) is $(n+1)$-universal, we consider

$$K_n \overset{j}{\hookrightarrow} K_{n+1}.$$

For $1 \leq i \leq n+1$, we get

$$\begin{tikzcd} \pi_i(K_n) \arrow[r, "j_*"] \arrow[d, "\theta_{\omega_n}"] & \pi_i(K_{n+1}) \arrow[d, "\theta_{\omega_{n+1}}"] \\ F(S^i) \arrow[r, equal] & F(S^i) \end{tikzcd}$$

Note that j_* is an iso for $i<n$ and epi for $i=n$, since $K_{n+1} = K_n \cup$ (some $(n+1)$-cells). Use l.e.s. of π_*-groups + quotient theorem + bouquet theorem.

By the inducton hypothes, θ_{ω_n} has trivial kernel for $i<n$ and is surjective for $i \leq n$.

Thus the same is true for $\theta_{\omega_{n+1}}$.

Thus (K_{n+1}, ω_{n+1}) is n-universal.

Furthermore, the kernel of $\theta_{\omega_{n+1}}$ is trivial for $i=n$, since any element $\sigma \in \pi_n(K_{n+1})$ with $\theta_{\omega_{n+1}}(\sigma) = 0$ must be hit by some $\sigma' \in \pi_n(K_n)$: $\sigma = j_*(\sigma')$, since j_* is epi in this degree.

But this would mean $\theta_{\omega_n}(\sigma') = 0$, and thus σ' is in the kernel of θ_{ω_n}. But we attached to K_n ~~[illegible]~~ ~~these~~ $(n+1)$-cells corresponding exactly to these σ' so all these σ' lie in the kernel of j_* and thus $\sigma = 0$.

Lastly, $\Theta_{\omega_{n+1}}$ is surjective for $i = n+1$ by construction, since for each $\beta \in F(\mathbb{S}^{n+1})$ we have attached an $(n+1)$-cell $\mathbb{S}^{n+1}_{\beta}$ and $\omega_{n+1}|_{\mathbb{S}^{n+1}_{\beta}} = \beta$

⑥ Set $K := \bigcup_{n\geq 1} K_n = \underrightarrow{\mathrm{colim}}_{n} K_n$ (Boedigheimer writes $\underrightarrow{\lim} K_n$)

So

$$Z \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_n \subseteq \dots \subseteq K$$

⑦ Recall the Milnor's exact sequence for cohomology. This holded for all cohomology theories h^* with

1) Homotopy invariance
2) Mayer-Vietoris
3) Wedge axiom

which are precisely axioms BR1 - BR3.
So ~~#~~ it applies to F and we get the following s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim^1 F(\Sigma K_n) \arrow[r] & F(K) \arrow[r] & \varprojlim F(K_n) \arrow[r] & 0 \\ & & \omega \arrow[r, mapsto] & (\omega_n) & \end{tikzcd}$$

The family (ω_n) is an element of $\varprojlim F(K_n)$ since it is coherent: $\omega_{n+1}|_{K_n} = \omega_n$.
So there must be some (not necessarily unique) element $\omega \in F(K)$ with $\omega|_{K_n} = \omega_n$.

Recall: we have the "shift-map"

$$\begin{tikzcd} sh: \prod F(K_n) \arrow[r] & \prod F(K_n) \\ (\varphi_n) \arrow[r, mapsto] & (\varphi_{n-1}) \end{tikzcd}$$

and we defined

$$\varprojlim F(K_n) = \ker(1 - sh)$$
$$\varprojlim^1 F(K_n) = \mathrm{coker}(1 - sh)$$

⑧ To see that (K, ω) is ∞-universal, consider the map $j: K_n \hookrightarrow K$, inducing for each i a diagram

$$\begin{tikzcd} \pi_i(K_n) \arrow[r, "j_*"] \arrow[d, "\Theta_{\omega_n}"] & \pi_i(K) \arrow[d, "\Theta_{\omega}"] \\ F(\mathbb{S}^i) \arrow[r, Rightarrow] & F(\mathbb{S}^i) \end{tikzcd}$$

<u>Note</u>: j_* is iso for $i < n-1$.
Also Θ_{ω_n} is surjective with trivial kernel for $i < n-1$.
So the same is true for Θ_ω.
This holds for all Θ_ω so (K, ω) is ∞-universal.
This finishes the proof of the lemma.

<u>Lemma 2</u>: Let (K, ω) be ∞-universal and let (X, A) be any connected CW-pair with $x_0 \in A \subseteq X$.
For each $\xi \in F(X)$ and each $f : A \longrightarrow K$ with $f^*(\omega) = \xi|A$, there exists an extension $g: X \longrightarrow K$ with $g|_A \simeq f$ and $g^*(\omega) = \xi$.

<u>Proof</u>: Consider

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & K \\
X \arrow[ur, dashed, "\exists g"'] \arrow[ur, phantom, "\simeq", near end, shift left=3] &
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) & & & F(K) \arrow[lll, "f^*"'] \arrow[dddlll, dashed, "g^*"] \\
 & \xi|A & \omega \arrow[l, mapsto] \arrow[dl, dotted, mapsto] & \\
 & \xi \arrow[u, mapsto] & & \\
F(X) \arrow[uuu, "i^*"] & & &
\end{tikzcd}$$

First replace K by the (reduced) mapping cylinder of f. We can thus assume that $f: A \hookrightarrow K$ is the embedding of a subcomplex.
Write $Z := X \cup_A K$

By BR2, ~~there is sq.~~ we get the Cartesian square

$$\begin{tikzcd}
\xi \arrow[d, mapsto] & F(X) \arrow[d] & & F(Z) \arrow[ll] \arrow[d] & \\
\xi|_A & F(A) & & F(K) \arrow[ll] & \\
 & \xi|A & & & \omega \arrow[lll, mapsto]
\end{tikzcd}$$

and since ξ and ω are both sent to $\xi|A$ there is an element $\zeta \in F(Z)$ with $\zeta|_X = \xi$ and $\zeta|_K = \omega$.

By lemma 1, we can embed Z into some complex K' and $w' \in F(K')$ s.t. (K', w') is ∞-universal.

Thus the inclusion $K \hookrightarrow K'$ (with $w'|K = \omega$) induces an isomorphism on homotopy groups since both ω and ω' are ∞-universal:

$$\begin{tikzcd} \pi_i(K) \arrow[rr, "\cong"] \arrow[dr, "\Theta_\omega"'] & & \pi_i(K') \arrow[dl, "\Theta_{\omega'}"] \\ & F(S^i) & \end{tikzcd}$$

Lecture 15 04-06-18

Proof of Brown-representability theorem

It is enough to show that an ∞-universal pair (K, ω) is actually universal.

We apply lemma 2 with $A = \{x_0\}$ and see:

(1) $\Theta_\omega : [X, K] \longrightarrow F(X) : [f] \longmapsto f^*(\omega)$ is surjective.

Proof: Any given $\xi \in F(X)$ restricts to the ~~g~~ trivial element $0 = \xi|_{x_0} \in F(\{x_0\}) = \{0\}$.
Clearly $0 = f^*(\omega)$ for $f: A = \{x_0\} \longrightarrow K$ the constant map. By lemma 2 there is some based map $g: X \to K$ (for which $f = g|A$ holds) and for which $\xi = g^*(\omega)$.

(2) Θ_ω is injective.

Proof: Suppose $\Theta_\omega([f_0]) = \Theta_\omega([f_1])$ for two based maps $f_0, f_1 : X \longrightarrow K$. We apply lemma 2 to the pair $(X \times [0,1], X \times \{0,1\})$. the map $X \times \{0,1\} \longrightarrow K$ is just

$$f_0 \sqcup f_1 : X \times \{0,1\} \longrightarrow K$$

As element of $F(X \times [0,1])$ we take $\xi = p^*(f_0^*(\omega)) = p^*(f_1^*(\omega))$ with $p: X \times [0,1] \longrightarrow X$ the projection.

(Note: we should take reduced product $X \wedge [0,1]_+$ and the wedge $X \vee X$.)

Then lemma 2 garantuees the existence of a

$$g : X \wedge [0,1]_+ \longrightarrow K$$

with $g|_{X \times \{0,1\}} = f$, and $g^*(\omega) = \xi$. Then g homotopy so $f_0 \simeq f_1$

Application: Assume we have a reduced cohomology theory $h^* = (h^n)_{n\in\mathbb{Z}}$ satisfying the Wedge Axiom.

1) Each h^n satisfies BR1-3, so there exist representing spaces K_n and universal elements $\omega_n \in h^n(K_n)$ s.t.

$$\Theta_{\omega_n} : [X, K_n] \xrightarrow{\cong} h^n(X), \quad [f] \longmapsto f^*(\omega_n).$$

2) Our functors h^n come with a natural ~~transformation~~ equivalence

$$\sigma : h^n \longrightarrow h^{n+1} \circ \Sigma,$$

in other words, for every connected space X

$$\begin{tikzcd}
h^n(X) \arrow[r, "\cong"] & h^{n+1}(\Sigma X). \\
{[X, K_n]} \arrow[u, "\Theta_{\omega_n}", "\cong"'] \arrow[rd, dotted, "(s_n)_\#"'] & {[\Sigma X, K_{n+1}]} \arrow[u, "\cong", "\Theta_{\omega_{n+1}}"'] \arrow[d, phantom, "\cong"] \\
 & {[X, \Omega K_{n+1}]}
\end{tikzcd}$$

So we have a natural equivalence of functors

$$[-, K_n] \xrightarrow{\cong} [-, \Omega K_{n+1}]$$

By Yoneda, there must be an element in $[K_n, \Omega K_{n+1}]$ corresponding to id_{K_n}.

So we have a map $s: K_n \longrightarrow \Omega K_{n+1}$.

Since the natural transformation is even a natural equiv., this must be a weak homotopy equivalence.

Result: We have an "equivalence of categories"

$$\{\text{reduced cohomology theories}\} \longleftrightarrow \{\Omega\text{-spectra}\}/_\sim$$

$$h^* = (h^n)_{n\in\mathbb{Z}} \longleftrightarrow \underline{K} = (K_n)_{n\in\mathbb{Z}} \text{ together with } s_n : K_n \xrightarrow{\simeq_w} \Omega K_{n+1}$$

Definition: A spectrum $\underline{E}$ is a sequence of spaces $(E_n)_{n\in\mathbb{Z}}$ and structure maps $e: \Sigma E_n \longrightarrow E_{n+1}$.

Remark: There are no further assumptions on the e_n, apart from the technical assumption that all E_n are CW-complexes, based, and the e_n are cellular.

Examples: 1) For any space Z, we can take

$$E_n := \Sigma^n Z, \quad e_n = id_{\Sigma^{n+1} X}$$

It is called the suspension-spectrum of Z.

2) In general, call E a suspension-spectrum (Σ-spectrum) if all e_n are weak homotopy equivalences.

3) Dually, call E an Ω-spectrum if all dual maps $e'_n : E_n \longrightarrow \Omega E_{n+1}$ are weak homotopy equivalence.

4) $E_n := K(\mathbb{Z}, n)$ is an Ω-spectrum, where $E_0 = E_1 = \dots = pt$.

Then

$$\begin{tikzcd} K(\mathbb{Z}, n) \arrow[r, "e_n'"] \arrow[d, phantom, "\parallel"] & \Omega K(\mathbb{Z}, n+1) \arrow[d, phantom, "\parallel"] \\ Sp(\$^n, \infty) \arrow[r, "\simeq"] & \Omega Sp(\Sigma \$^n, \infty) \end{tikzcd}$$

Note that for an Ω-spectrum K we get a cohomology theory

$$h^n(X) := [X, K_n]$$

with suspension isomorphism

$$h^n(x) = [x, K_n] \xrightarrow{\simeq} [x, \Omega K_{n+1}] = [\Sigma X, K_{n+1}] = h^{n+1}(\Sigma X)$$

For a general spectrum E, we can define

$$h^n(x) := \lim_k \left\{ [\Sigma^k X, E_{n+k}] \xrightarrow{\Sigma(\cdot)} [\Sigma^{k+1} X, \Sigma E_{n+k}] \xrightarrow{(e_{n+k})_\#} [\Sigma^{k+1} X, E_{n+k+1}] \right\}$$

and this is indeed a cohomology theory

- homotopy invariance is clear
- MV-sequence by Blakers-Massey and taking limit.
- Wedge axiom for free, since we take based maps.

Example: Let $F(X) = H^1(X)$ for connected X.

Let $K_0 = pt$.

$K_1 = \bigvee_\beta \$^1$ for each $\beta \in H^1(\$^1) \cong \mathbb{Z} \ni u$ generator

$$\omega_1 = (u)_\beta \in H^1(K_1) = \prod_\beta \mathbb{Z}$$

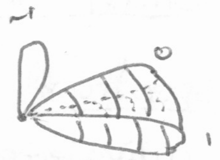
$$\Theta_{\omega_1} : \pi_1(K_1) \longrightarrow H^1(\$^1)$$

$$[f] \longmapsto f^*(\omega_1)$$

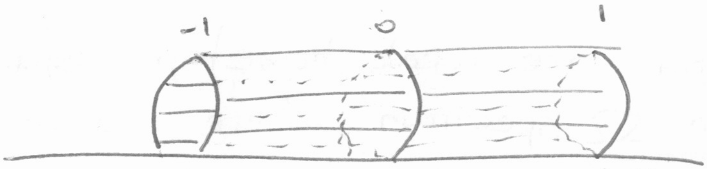
Denote by f_β the composition $\$^1 \longrightarrow K_1 \xrightarrow{proj_\beta} \$^1_\beta$.

Each $f : \$^1 \longrightarrow K_1$ corresponds to a word in integers.

Write
$$n_f = \sum_\beta \deg(f_\beta) = \text{total degree of } f.$$
To kill the elements in the kernel we attach cells



or differently represented



$\simeq S^1$

Let $K_1' := C(\varphi)$ where $\varphi : \bigvee S^1 \longrightarrow K_1$
the attachements of α-type cells.

Then $K_2 := K_1' \vee \bigvee S^2_\beta$

where β runs over $\beta \in F(S^2) = H^1(S^2) = \{0\}$

So
$$K_2 = K_1' \vee S^2 \simeq S^1 \vee S^2.$$
Then $K_2' \simeq S^1$. $K_3 \simeq S^1 \vee S^3$ . . . , $K_n \simeq S^1 \vee S^n$.

so $K = S^1 \vee S^\infty \simeq S^1$

Remarks: We observe that

1) If $F(S^i) = \{0\}$ for $i<n$, then one can assume K to be $(n-1)$-connected.

2) If $F(S^i) = \{0\}$ for $i>n$, then $\pi_i(K) = 0$ for $i>n$.

3) If $F(S^i) = \begin{cases} G & i=n \\ 0 & i \neq n \end{cases}$ then K is an ~~KG~~ EML-space $K(G,n)$.

4) $F(X) = \hat{H}^1 \cong [X, S^1]$

5) $F(X) = \hat{H}^2(X) \cong [X, \mathbb{CP}^\infty]$

6) $F(X) = \hat{H}^n(X;G) \cong [X, K(G,n)]$.

Note: $K(\mathbb{Z},n) = SP(S^n)$

$K(G,n) = SP(M(G,n))$

$K(\mathbb{Z}_2,n) = SP\mathbb{Z}_2(S^n)$

$\hat{H}^n(X;G) \cong [X, K(G,n)] \cong [X, SP(M(G,n))]$

$\hat{H}^1(X;\mathbb{Z}_2) \cong [X, \mathbb{RP}^\infty]$.

Exercise: $\Phi^{(2)}(X)$ = isomorphism classes of 2-fold coverings. This satisfies the axioms and thus is representable.

$$\Phi(X) \cong [X, ?].$$

It turns out that $? = \mathbb{RP}^{\infty}$, so

$$\Phi(X) \cong \tilde{H}^1(X; \mathbb{Z}/2).$$

Lecture 16 06-06-18

Last time we have seen the following application of BR:

Theorem A: A reduced cohomology theory satisfying the wedge axiom is representable by an Ω-spectrum $\underline{K} = (K_n)_{n\in\mathbb{Z}}$.

Theorem B: Reduced ~~coh~~ singular cohomology $\tilde{H}^*(-;G)$ with coefficients in an abelian group G is represented by the Ω-spectrum of Eilenberg MacLane spaces $\underline{K} = (K(G,n))_{n\geq 0}$ (set $K_0 = pt$)

Example: 1) Reduced complex ~~$\tilde{K}\mathcal{U}$~~ K-theory $\tilde{K}\mathcal{U}$ is represented by the spectrum

$$\begin{array}{cccccccc} K_{-2} & K_{-1} & K_0 & K_1 & K_2 & K_3 & \cdots \\ \| & \| & \| & \| & \| & \| & \\ BU & U & BU & U & BU & U & \cdots \end{array}$$

of period 2.

The equivalences

$$BU \xrightarrow{\simeq} \Omega U$$
$$U \xrightarrow{\simeq} \Omega BU$$

are two forms of Bott periodicity.

2) Reduced real K-theory $\hat{K}O$ is represented by an Ω-spectrum of period 8.

Example: Let G_1, G_2 be abelian groups, $K_i = K(G_i, n)$ for $i = 1,2$.

We have

$$\begin{array}{ccc} [K_1, K_2] & \xrightarrow{\simeq} & [G_1, G_2] \\ [f] & \longmapsto & \pi_n(f): \pi_n(K_1) \longrightarrow \pi_n(K_2) \end{array}$$

This is a bijection.

For abelian groups G, G', and a map $\varphi: G \to G'$ and given filtrations

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r] \arrow[d, dashed] & F_0 \arrow[r] \arrow[d, dashed] & G \arrow[d, "\varphi"] \\ 0 \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & G' \end{tikzcd}$$

We get a lift of φ.

This can be used for constructing Moore spaces.

Example: Let $F : CW_0^{conn} \longrightarrow Ab$ be a functor with the usual properties. We have

$$F^2 : CW_0^{conn} \longrightarrow Ab : X \longmapsto F(X)^2 \cong F(X \vee X).$$

By Brown-representability, we have spaces K_1, K_2 s.t.

$$F^1(X) \cong [X, K_1]$$
$$F^2(X) \cong [X, K_2].$$

But notice we have the natural ~~equivalence~~ transformation

$$\mu : F^2 \longrightarrow F$$

where $\mu_X : F^2(X) = F(X)^2 \longrightarrow F(X)$ is multiplication.

Notice that

$$[X, K_2] \cong F^2(X) \cong F(X) \times F(X) \cong [X, K_1] \times [X, K_1]$$
$$\cong [X, K_1 \times K_1]$$

so by Yoneda $K_2 \cong K_1 \times K_1$.

So $\mu : F^2 \longrightarrow F$ iduces a map $\underline{\mu} : K_1 \times K_1 \longrightarrow K_1$.

In fact, properties of F are reflected in properties of K_1

$$\Longrightarrow$$

is semi group-valued	is H-space
has neutral element	has h-unit
is associative	is h-associative
is commutative	is h-commutative
has inverse	has h-inverse

Remark: How is multiplication on h^* reflected in a representing Ω-spectrum $\underline{K}$?

$$\begin{tikzcd}
h^n(X) \otimes h^m(X) \arrow[r] \arrow[d, equal] & h^{n+m}(X) \arrow[d, equal] \\
{[X, K_n] \otimes [X, K_m]} \arrow[r] & {[X, K_{n+m}]}
\end{tikzcd}$$

We get maps

$$\mu_{n,m} : K_n \wedge K_m \longrightarrow K_{n+m}$$

For $h^n = H^n(-, \mathbb{Z})$ this gives

$$SP(S^n) \wedge SP(S^m) \longrightarrow SP(S^n \wedge S^m).$$

Chapter: Fibre bundles

Let G be a topological group (discrete! Lie-group, ...)

Let F be a space with a G-action, i.e. a continuous map

$$\rho : G \longrightarrow \mathrm{Homeo}(F).$$

Let $\xi : E \longrightarrow B$ be a continuous surjective map, with B path-connected.

Definition: We say ξ is a fibre bundle with fibre F and structure group G if the following two conditions hold:

1) There is an open cover $\mathcal{U} = (U_i \mid i \in I)$ of B and homeomorphisms $h_i : E_i \xrightarrow{\cong} U_i \times F$ for $E_i = \xi^{-1}(U_i)$ making the following commutative

$$\begin{tikzcd} E_i \arrow[rr, "h_i"] \arrow[dr, "\xi|_{E_i}"'] & & U_i \times F \arrow[dl, "\mathrm{proj}"] \\ & U_i & \end{tikzcd}$$

We write $h_i(e) = (\xi(e), h_i'(e))$, with $h_i'(e) \in F$.

In particular: All fibers $\xi^{-1}(b)$ are homeomorphic to F.

If we have a basepoint $b_0 \in B$, we could set

$$F := \xi^{-1}(b_0) \overset{j}{\hookrightarrow} E.$$

2) For i, j with $U_i \cap U_j =: U_{ij} \neq \emptyset$, $E_{ij} = \xi^{-1}(U_{ij})$, the composition $h_{ij} := (h_i)|_{E_{ij}} \circ (h_j|_{E_{ij}})^{-1}$

$$\begin{tikzcd} U_{ij} \times F \arrow[r, "h_j|_{E_{ij}}^{-1}"] \arrow[d] & E_{ij} \arrow[r, "h_i|_{E_{ij}}"] \arrow[d] & U_{ij} \times F \arrow[d] \\ U_{ij} \arrow[r, equal] & U_{ij} \arrow[r, equal] & U_{ij} \end{tikzcd}$$

has the form

$$(b, x) \longmapsto e \longmapsto (b, T_{ij}^{b}(x)).$$

The adjoint $\rho_{ij} : U_{ij} \longrightarrow \mathrm{Homeo}(F) : b \longmapsto T_{ij}^{b}$ must factor over G.

$$\begin{tikzcd} U_{ij} \arrow[rr] \arrow[dr] & & \mathrm{Homeo}(F) \\ & G \arrow[ur, "\rho"'] & \end{tikzcd}$$

Examples 1) **Coverings** $\xi : E \longrightarrow B$

Then $F =$ discrete.

$G = \mathrm{Homeo}(F) \cong \mathfrak{S}_n$ (for $n = |F|$)

2) **Vector bundles** : F is an n-dim vector space over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$).

$G = GL_n(\mathbb{F})$ or $G_1 = O_n(\mathbb{F})$ or $G_2 = SL_n(\mathbb{F})$

Terminology

$E =$ total space

$B =$ base space

$F =$ fibre

$h_i =$ coordinates

h_{ij} resp. $\rho_{ij} =$ coordinate changes

$\mathfrak{A} = (U_i, h_i)_{i \in I} =$ atlas.

Definition : We call ξ : a principal G-bundle if $F = G$ and G acts on $F = G$ by left multiplication.

Example : Let $E \longrightarrow B$ be a two fold covering.

Then $F = \{0,1\}$, $G = \mathbb{Z}/2 \ni T$

T acts on F via $T(0) = 1$, $T(1) = 0$.

We can regard this as the action of $\mathbb{Z}/2$ on $\mathbb{Z}/2$.

Example : Let $E \longrightarrow B$ be a complex line bundle and $S \longrightarrow B$ be the corresponding $\mathbb{S}^1$- bundle.

The action of $\mathbb{S}^1$ on $F = \mathbb{S}^1$ is induced by scalar multiplication. So we can restrict the action to $\mathbb{S}^1 \subseteq \mathbb{C}$ and to the sphere in each fiber of $E \longrightarrow B$.

So $S \longrightarrow B$ can be regarded as a principal $\mathbb{S}^1$- bundle.

New definition of principal G-bundle :

$\xi : E \to B$ is a principal G-bundle if there is a free action

$\gamma : G\times E \to E : (g.e) \mapsto g.e$

satisfying

1) $\xi(g.e) = \xi(e)$

2) There is an open cover $\mathcal{U} = (U_i \mid i\in I)$ and G-homeomorphisms $h_i : E_i \xrightarrow{\cong} G\times U_i$ where G acts on $G\times U$ as $g(h,b) = (gh,b)$ s.t.

$$\begin{tikzcd}
E_i \arrow[rr, "h_i", "\cong"'] \arrow[dr, "\xi"'] & & G\times U_i \arrow[dl, "\mathrm{proj}"] \\
& U_i &
\end{tikzcd}$$

i.e.

$$h_i(e) = (h_i'(e), \xi(e))$$

The fact that h_i is G-linear means

$$h_i(g.e) = g\cdot h_i(e) = g\cdot(h_i'(e), \xi(e)) = (g h_i'(e), \xi(e))$$

Morphisms between two (F,G)-bundles ξ and ξ' are maps $f : B \to B'$, $\hat{f} : E \to E'$ s.t. $\xi'\circ\hat{f} = f\circ\xi$:

$$\begin{tikzcd}
E \arrow[r, "\hat{f}"] \arrow[d, "\xi"] & E' \arrow[d, "\xi'"] \\
B \arrow[r, "f"] & B'
\end{tikzcd}$$

For principal bundles $\hat{f}$ is assumed to be G-equivariant.

We thus have the notion of isomorphisms of (F,G)-bundles.

In particular we have isomorphism classes for bundles over B where $\xi : E \to B$, $\xi' : E' \to B$ and $\hat{f} : E \xrightarrow{\cong} E'$ st.

$$\begin{tikzcd}
E \arrow[rr, "\hat{f}"] \arrow[dr, "\xi"'] & & E' \arrow[dl, "\xi'"] \\
& B &
\end{tikzcd}$$

in other words we restrict to $f = id_B$.

Thus we have the isomorphism classes

$$[\xi] \in Bun_G^F(B).$$

The trivial bundle over B is $F \times B \xrightarrow{\xi} B$.
Any (F,G) bundle isomorphic to it will also be called '
a trivial bundle.

Application of Brown Representability

Consider the functor

$$F: CW_0^{conn} \longrightarrow Sets_0$$
$$X \longmapsto Bund_G^F(X)$$

where the trivial bundle gives the distinct element.

By Brown Representability, there exists a space $K = BG$ and a universal bundle $\omega \in Bund_G^F(K)$, i.e. a (F,G)-bundle $\omega: E \longrightarrow K$, such that for all X

$$[X, K] \xrightarrow{\cong} Bund_G^F(X)$$
$$[f] \longmapsto f^*(\omega).$$

This is called the Classification theorem

We see that F seems to disappear. ~~Indeed every (F,G) bundle is isomorphic to~~

Indeed the (F,G)-bundles are in bijection (~~or~~ : in equivalence) with the principal G-bundles:

- Given (F,G)-bundle we can make principal G-bundle.
- Given principal G-bundle, fibre F, we can make an (F,G)-bundle.
- These are inverses up to isomorphism.

Lecture 17 11-06-18

Pull-backs of bundles: let $\xi: E \longrightarrow B$ be an (F,G)-bundle, and $f: B' \longrightarrow B$ a map. Define

$$E' = B' \times_B E = \{ (b', e) \in B' \times E \mid f(b') = \xi(e) \}$$

the pull-back of f and ξ

$$\begin{tikzcd} E' \arrow[r, "\hat{f} = pr"] \arrow[d, "pr = \xi'"'] & E \arrow[d, "\xi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Proposition: Pull-backs of (F,G)-bundles are (F,G) bundle.

Proof Ex. 8.i

(Idea: atlas will be "pull-back" of old atlas on B.)

Corollary: A pull-back of a principal G-bundle is a principal G-bundle.

Corollary: A restriction of an (F,G)-bundle (resp. principal G-bundle) is an (F,G) bundle (resp. principal G-bundle.)

Example: Assume that B' is the total space of another (F',G) bundle, i.e. $B' = E_2$ and $f = \xi_2 : E_2 \longrightarrow B$.

Write $\xi = \xi_1 : E_1 = E \longrightarrow B$.

$$\begin{tikzcd}
 & & & F_1 \arrow[d] \\
F_2 \arrow[rr] & & E_2 \times_B E_1 \arrow[r, "\tilde{\xi}_2"] \arrow[d, "\tilde{\xi}_2"] & E_1 \arrow[d, "\xi_1"] \\
 & \hat{E} \arrow[ur, equal] & & \\
F_2 \arrow[rr] & & E_2 \arrow[r, "\xi_2"] & B
\end{tikzcd}$$

Now $\tilde{\hat{E}} := E_2 \times_B E_1$ is both $\xi_1^*(E_2)$ as well as $\xi_2^*(E_1)$.

The fibre of $\tilde{\xi}_2 : \hat{E} \longrightarrow E_1$ is F_2, the fibre of $\hat{\xi}_1 : \hat{E} \longrightarrow E_2$ is F_1.

Proposition: The composition of an (F,G)-bundle with an (F',G)-bundle is an $(F \times F', G)$-bundle.

Above we see an example: $\xi_2 \circ \hat{\xi}_1 = \xi_1 \circ \hat{\xi}_2$ is an $(F_1 \times F_2, G)$-bundle:

$$\begin{tikzcd}
F_1 \times F_2 \arrow[dr, dashed] & F_1 \arrow[d] & F_1 \arrow[d] \\
F_2 \arrow[r] & \tilde{\hat{E}} \arrow[r, "\hat{\xi}_2"] \arrow[d, "\hat{\xi}_1"'] \arrow[dr, dashed] & E_1 \arrow[d, "\xi_1"] \\
F_2 \arrow[r] & E_2 \arrow[r, "\xi_2"] & B
\end{tikzcd}$$

Subexample: What would we get for

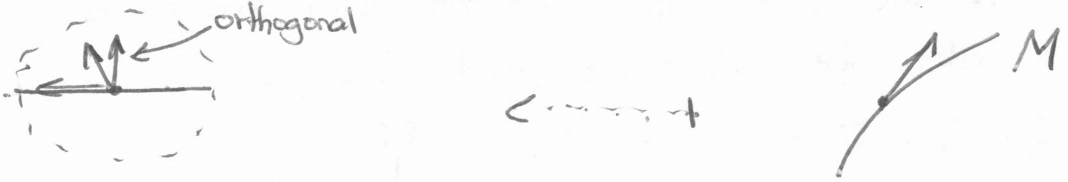
$$\begin{tikzcd}
? \arrow[r] \arrow[d] & TM \arrow[d, "\tau_M"] \\
M \times N \arrow[r, "proj"] & M
\end{tikzcd}$$

for manifolds M, N.

More generally, consider a bundle $p : E \longrightarrow M$ with the fibre also being a manifold.

$$\begin{tikzcd}
p^*TM \arrow[r] \arrow[d] & TM \arrow[d, "\pi_M"] \\
E \arrow[r, "p"] & M
\end{tikzcd}$$

We can regard p^*TM as a subset of TE consisting of tangent vectors "~~Frea~~orthogonal to the fibre".



(Bödigheimer meant this more as a sketch than as a formal discussion)

We now want to define <u>sub-bundles</u>

Let $G' \leq G$ be a subgroup of G and F a G-space (a space with a G-action). Let F' be a G'-invariant subspace of F.

<u>Example</u>: $G' = GL_n(\mathbb{R}) \leq G = GL_{n+m}(\mathbb{R})$

$F' = \mathbb{R}^n \qquad F = \mathbb{R}^{n+m}$

(This will correspond to sub vector bundles.)

We say that $\xi' : E' \longrightarrow B$ is an <u>(F', G')-sub-bundle</u> of the (F, G)-bundle $\xi : E \to B$ if there is a map φ,

$$\begin{tikzcd}
F' \arrow[d] & F \arrow[d] \\
E' \arrow[r, hook, "\varphi"] \arrow[d, "\xi'", "(G')"'] & E \arrow[d, "\xi"', "\xi\ (G)"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

s.t. all "obvious" conditions on the atlas are satisfied which Bödigheimer didn't want to spell out, things like

$$\begin{tikzcd}
E_i' \arrow[rr, "\varphi"] \arrow[dr, "h_i'"] \arrow[dd] & & E_i \arrow[dr, "h_i"] \arrow[dd] & \\
& U_i \times F' \arrow[rr] \arrow[dl] & & U_i \times F \arrow[dl] \\
U_i \arrow[rr] & & U_i &
\end{tikzcd}$$

commuting.

<u>Example</u>: Let $j : \underset{m}{M} \hookrightarrow \underset{w=m+n}{W}$ be an embedding of manifolds ($\dim M = m$ $\dim W = m+n$). We have two bundles

$$\tau_M : T(M) \xrightarrow{\tau_M} M \qquad F = \mathbb{R}^m$$

$$\nu_M : N(M) \xrightarrow{\nu_M} M \qquad F' = \mathbb{R}^n.$$

↑ depends on j and w

(Other notation is $N(M,W)$.)

(At this point, Bödigheimer realizes we need some condition on the previous proposition.)

Then the Whitney sum of $T(M)$ and $N(M)$ is

$$T(M) \oplus N(M) \cong T(W)|_M.$$

<u>Example</u>: Let F be discrete. Then a sub-bundle is a sub-covering which is a union of components of the total space.

<u>Sometimes</u>: Decomposition questions:

Is ξ the product ("sum") of two or more (non-trivial) sub-bundles (which are "complementary" to each other).

<u>Ex</u> ① Can one decompose a vector bundle ξ as a sum of line bundles? <u>No</u>, not necessarily.

But the pull-back of ξ to a certain space X (a product of Graßmannians) along a certain map <u>can</u> be decomposed.

$$\begin{tikzcd} f^*(E) = E' \arrow[r] \arrow[d, "f^*(\xi) = \xi'"'] & E \arrow[d, "\xi"] \\ X = X(\xi) \arrow[r, "f = f_\xi"] & B \end{tikzcd}$$

Then

1) $\xi' \cong$ sum of line bundles

2) $f^* : KU^*(B) \longrightarrow KU^*(X)$ is mono.

$\rightsquigarrow$ Adams operations "Remember the seminar about K-theory"

② Decomposition of coverings = decomposition into components of total space.

"For the advanced reader": Burnside ring $A(G)$ of a group G,

Segal conjecture: "Look this up if you want"

The completion of $A(G)$ is precisely $\pi_*^{stab}(BG)$. (Proved in 80's.)

Quotient bundles

If we have an equivalence relation on F that is G-equivariant, we get a G-equivariant map

$$F \twoheadrightarrow \overline{F} = F/\sim .$$

This gives a ~~g~~ G-equivariant ~~G-e~~ equiv. rel. on E:

1) $a \sim b \implies \xi(a) = \xi(b) \in B$ (equiv. elmts lie in same fibre)

2) $a \sim b \implies g\cdot a \sim g\cdot b$,

where the action of G is induced via the trivialisations

$$\begin{tikzcd}
a \arrow[r, mapsto, "h_i"] & (\xi(a), \alpha) \in U_i \times F \arrow[dd, bend left, "\cdot g"] \\
b \arrow[r, mapsto, "h_i"] & (\xi(b), \beta) \arrow[dd, bend left, "\cdot g"] \\
ga & (\xi(a), g\alpha) \arrow[l, mapsto] \\
g\cdot b & (\xi(b), g\cdot\beta) \arrow[l]
\end{tikzcd}$$

Then we get an $(\overline{F}, G)$-bundle $\overline{E} = E/\sim \longrightarrow B$, called a <u>quotient bundle</u>

Sections:

A section of the (E,G)-bundle $\xi: E \longrightarrow B$ is a map $s: B \longrightarrow E$ s.t. $\xi \circ s = id_B$.

Examples i) E = Vector bundle over manifold B

sections = vector field (if $\xi = T_M$)

Important question: how many lin independent v.f. do there exist in T_M for fixed M. ("Vector field problem")

If we have n lin. ind. v.f., then we have a decomposition $T_M \cong \lambda_1 \oplus \dots \oplus \lambda_n \oplus \rho$, where λ_i are line bundles and ρ is the "complementary" rest-bundle. We must have $n \leq m = \dim M$

$n = m$ iff M is parallelizable $\iff T_M \cong M \times \mathbb{R}^m$

It can happen that $n = 0$; that there are no non-vanishing vector fields. For example for $M = S^2$.

Lemma: A principal G-bundle is trivial iff there is one section.

Proof: Given $s: B \longrightarrow E$, regard $s(b) \in \xi^{-1}(b) \cong F = G$ as the ~~utt~~ neutral element $1 \in G$. More precisely, define

$$\begin{aligned} E &\longrightarrow B \times G \\ e &\longmapsto (\xi(e), \gamma) \end{aligned}$$

where γ is the unique $\gamma \in G$ s.t. $\gamma \cdot s(\xi(e)) = e$.

Recall that $F = F_b$ is a free and transitive G-space, so this $\gamma = \gamma(e)$ is uniquely determined.

$e \mapsto \gamma(e)$ is continuous because we can check this on the E_i, where we can use the trivialization.

The inverse is (by def. of γ) given by $(b, \gamma) \mapsto \gamma \cdot s(b)$.

Maps between (F,G)-bundles

First remark: one could define maps of bundles where the fibre or structure group differ, but we won't go into that.

Let $\xi: E \longrightarrow B$, $\xi': E' \longrightarrow B'$ be (F,G)-bundles.

Definition A pair $(f, \hat{f})$ of $f: B \longrightarrow B'$, $\hat{f}: E \longrightarrow E'$ is called a map of (F,G)-bundles if

1) $\xi' \circ \hat{f} = f \circ \xi$, i.e. the diagram on the right commutes.

$$\begin{tikzcd}
F \arrow[d] & F \arrow[d] \\
E \arrow[r, "\hat{f}"] \arrow[d, "\xi"] & E' \arrow[d, "\xi'"] \\
B \arrow[r, "f"] & B'
\end{tikzcd}$$

2) There is an atlas $\{U_i\}$ for ξ and an atlas $\{U_j'\}$ for ξ' s.t.:

- $U_i \subseteq B \Rightarrow \exists j$ s.t. $f(U_i) \subseteq U_j' \subseteq B'$
- $\left.\begin{array}{l} E_i = \xi^{-1}(U_i) \\ E_j' = (\xi')^{-1}(U_j') \end{array}\right\} \Rightarrow \hat{f}(E_i) \subseteq E_j'$
- The following commutes

$$\begin{tikzcd}
E_i \arrow[rr, "\hat{f}"] \arrow[dr, "h_i", "\cong"'] \arrow[dd, "\xi"'] & & E_j' \arrow[dr, "h_j'", "\cong"'] \arrow[dd, "\xi'"'] & \\
& U_i \times F \arrow[rr, "(f, \varphi_{ij})"] \arrow[dl, "proj"] & & U_j' \times F \arrow[dl, "proj"] \\
U_i \arrow[rr, "f"] & & U_j' &
\end{tikzcd}$$

where $\varphi_{ij}: U_i \times F \longrightarrow F$ is a G-equivariant map, (where $U_i \times F$ has G-action only on the F-component.)

Now: • We have a category of (F,G)-bundles $\boxed{Bun(F,G)}$

• We have: isomorphisms of two bundles over $B = B'$ and $f = id_B$

Exercise: $\hat{f}$ is then always a homeomorphism

- Let $\mathrm{Bun}^F_G(B)$ be the isomorphism classes of (F,G)-bundles over B.

Is $\Phi : X \longmapsto \mathrm{Bun}^F_G(X)$ a representable functor?

Br1) Homotopy invariance?

Lemma: If $f_0 \simeq f_1 : X \longrightarrow B$ and $\xi : E \longrightarrow B$ an (F,G)-bundle, then $f_0^*(\xi) \cong f^*(\xi)$

Br2) MV-axiom? Almost obvious (exercise)

Br3) Wedge-axiom? obvious.

Also $\mathrm{Bun}^F_G(X)$ is a set with $[E = B \times F]$ as ~~distinguished~~ trivial element.

Theorem: There is a representing space $K = BG$ with a distinguished element $[\omega] \in \mathrm{Bun}^F_G(K)$ s.t.

$$\begin{aligned} [X,K] &\xrightarrow{\cong} \mathrm{Bun}^F_G(X) \\ [f] &\longmapsto f^*(\omega) \end{aligned}$$

is a bijection. □

Bödigheimer: "I'm against Set Theory"
(In response to a question about why $\mathrm{Bun}^F_G(X)$ is a set.)

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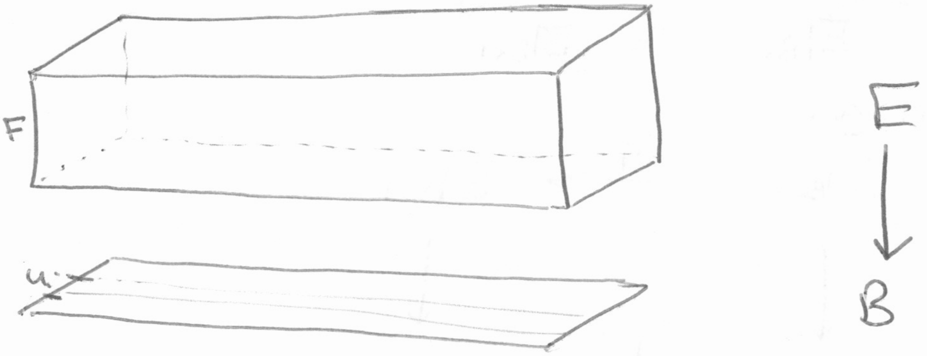
Homotopy-properties of (F,G)-bundles

Assume that all spaces are paracompact, or somewhat weaker that all bundles are numerable: they have an atlas $\mathcal{A} = ((U_i, h_i) \mid i \in I)$ s.t. there exists a partition of unity subordinate to the covering $(U_i \mid i \in I)$

Lemma: Let $\xi : E \longrightarrow B \times I$ be a fibre bundle with fibre F and structure group G. Then there is a bundle map $(\hat{R}, R)$

$$\begin{tikzcd} E \arrow[r, "\hat{R}"] \arrow[d, "\xi"] & E \arrow[d, "\xi"] \\ B \arrow[r, "R"] & B \end{tikzcd}$$

with $R(b,t) = (b,1)$ for all $b \in B$, and s.t. the diagram is a pull-back and $\hat{R}|_{B\times 1} = id : E|_{B\times 1} \longrightarrow E|_{B\times 1}$



<u>Proof</u>: 1) We can choose a locally finite covering $(U_j | j \in J)$ of B such that ξ is trivial over $U_j \times [0,1]$. (I is compact.) We can assume that J is well-ordered.

2) Choose a partition of unity f_j $(j \in J)$ subordinate to $(U_j | j \in J)$. Possible since B paracompact.

3) Define

$$f(b) := \max \{ f_j(b) \mid j \in J \}$$

$$f : B \longrightarrow [0,1] \quad \text{continuous positive.}$$

Define $v_j : B \longrightarrow [0,1]$ by $v_j(b) = \frac{f_j(b)}{f(b)}$.

- $\operatorname{supp}(v_j) \subseteq U_j$ (as $\operatorname{supp}(f_j) \subseteq U_j$)
- $\max \{ v_j(b) \mid j \in J \} = 1$.

4) Thus

$$R : B \times [0,1] \longrightarrow B \times [0,1] \text{ is given by}$$
$$(b,t) \longmapsto (b, \max \{ v_j(b) \mid j \in J \})$$

5) Define $\hat{R}_j : E \longrightarrow E$ as the identity outside $\xi^{-1}(U_j \times I)$.
On $\xi^{-1}(U_j \times I)$, use a trivialization $\xi^{-1}(U_j \times I) \xrightarrow[\cong]{h_j} U_j \times I \times F$ to define $\hat{R}_j$ here as

$$\hat{R}_j(b,t,x) = [illegible]$$
$$= (b, \max \{ v_j(b), t \}, x)$$

Define $R_j : B \times I \longrightarrow B \times I$ in a similar fashion.
It's clear that $R_j|_{B \times 1} = id|_{B \times 1}$
Also $(\hat{R}_j, R_j)$ is a bundle map forming a pull-back square.

6) Define $\hat{R} := \underbrace{\ldots \circ \hat{R}_j \circ \ldots \circ \hat{R}_2 \circ \hat{R}_1}_{\text{the composition of all } \hat{R}_j \text{, only finitely many are not the identity for every } e \in E.}$

Note that $R = \ldots \circ R_j \circ \ldots \circ R_2 \circ R_1$.

7) Then $(\hat{R}, R)$ is a bundle map forming a pull-back square.

Corollary 1:

$$E|_{B\times 0} \cong E|_{B\times 1}$$

Proof: Restrictions give

$$\begin{tikzcd} & E|_{B\times 0} \arrow[r, "\hat{R}|"] \arrow[d] & E|_{B\times 1} \arrow[d] & \\ (B\times 0)\cong & B \arrow[r, "id"] & B & (\cong B\times 1) \end{tikzcd}$$

and bundle maps over the identity are homeomorphisms.

Corollary 2: If $f_0 \simeq f_1 : B' \longrightarrow B$ and $\xi : E \longrightarrow B$ is an (F,G)-bundle then $f_0^*(\xi) \cong f_1^*(\xi)$.

Proof: Let $f : B'\times I \longrightarrow B$ be a homotopy.

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d, "\xi"'] & E \arrow[d, "\xi"] \\ B'\times I \arrow[r, "f"] & B \end{tikzcd}$$

Then by corollary 1, $f_0^*(\xi) \cong f^*(E)|_{B'\times 0} \cong f^*(E)|_{B'\times 1} \cong f_1^*(\xi)$.

From principal bundles to fibre bundles

Let G be a structure group and F a fibre.

Let a principal bundle $\xi : P \longrightarrow B$ be given. (fibre is G).

Then we have a free action of G on P fibrewise and $B = G/P$.

Now define the (F,G)-bundle $\xi : E \longrightarrow B$ as

$$E := P\times_G F \longrightarrow P/G \cong B$$

We can use the same atlas as for ξ as

$$U_j \times F \cong (U_j \times G)\times_G F$$

We can similarly lift isomorphisms, thus seeing that this construction behaves well on isomorphism classes.

$$\begin{tikzcd} P \arrow[rr, "\cong"] \arrow[dr, "\xi"'] & & P' \arrow[dl, "\xi'"] \\ & B & \end{tikzcd}$$

So we get a map $\Phi_{F,G}$ (sometimes denoted Φ_F by Bödigheimer)

$$\begin{tikzcd} \Phi_{F,G} : \mathrm{Prin}_G(B) \arrow[r] \arrow[d, equal] & \mathrm{Bun}_G^F(B) \\ \mathrm{Bun}_G^G(B) & \end{tikzcd}$$

This is functorial in B: for $f: B \longrightarrow B'$:

$$\begin{tikzcd} \mathrm{Prin}_G(B') \arrow[r, "\Phi_{F,G}"] \arrow[d, "f^*"] & \mathrm{Bun}_G^F(B') \arrow[d, "f^*"] \\ \mathrm{Prin}_G(B) \arrow[r, "\Phi_{F,G}"] & \mathrm{Bun}_G^F(B). \end{tikzcd}$$

so $\Phi_{F,G}$ is a natural transformation of functors.

It's clear that it sends trivial bundles to trivial bundles.

Example: $G = GL_n(\mathbb{R})$, $F = \mathbb{R}^n$. Then

$$\mathrm{Prin}_{GL_n}(B) \longrightarrow \mathrm{Vect}^n(B)$$

Note that $\Phi_{F,G}$ is not necessarily injective if F is not complicated enough.

From (F,G)-bundles to principal bundles

Let $\xi : E \longrightarrow B$ be an (F,G) bundle.

Fix the trivial bundle $t_F : F \longrightarrow *$ over a point.

Let $P = \mathrm{map}(t_F, \xi) \subseteq \mathrm{map}(F, E) \times \mathrm{map}(*, B)$ consisting of pairs $(\hat{f}, f)$ s.t.

$$\begin{tikzcd} F \arrow[r, "\hat{f}"] \arrow[d, "t_F"'] & E \arrow[d, "\xi"] \\ * \arrow[r, "f"] & B \end{tikzcd}$$

1) $\xi \circ \hat{f} = f \circ t_F$

i.e. if $b := f(*)$ then

$\xi(\hat{f}(x)) = b$ for all $x \in F$.

i.e. $\hat{f} : F \longrightarrow F_b = \xi^{-1}(b) \subseteq E$ maps into the fibre over b

2) $\hat{f}$ must be a homeomorphism $F \xrightarrow{\cong} F_b$.

3) For each i with $b \in U_i$ we have commuting diagrams

$$\begin{tikzcd} F_b \arrow[r, hook] \arrow[d, "\cong"] & \xi^{-1}(U_i) \arrow[d, "h_i", "\cong"'] \\ F \cong b \times F \arrow[r, hook] & U_i \times F \end{tikzcd}$$

Then define $\zeta : P \longrightarrow B$ by $(\hat{f}, f) \longmapsto f(*)$.

The fibre of ζ is homeomorphic to $\rho(G) \leq \mathrm{Homeo}(F)$ for the action map $\rho : G \longrightarrow \mathrm{Homeo}(F)$.

So we have a "new" or "actual" structure group $G' = \rho(G)$.

Then P is a principle G'-bundle:

$$\xi^{-1}(b) \cong \{ \hat{f} : F \longrightarrow F_b \text{ s.t. } \underbrace{h_i|_{F_b} \circ \hat{f}}_{\gamma} \in G' \}$$

$$\begin{tikzcd} \xi^{-1}(b) \arrow[d, "\cong"] & \hat{f} \arrow[d, mapsto] \\ G & \gamma = h_i \circ \hat{f} \end{tikzcd}$$

We could give an atlas, so it's indeed a principal bundle. It behaves well on isomorphism classes and behaves functorially and thus induces a natural transformation of functors

$$\Psi_{F,G} : \text{Bund}_G^F(B) \longrightarrow \text{Princ}_{G'}(B)$$

Theorem If $\rho : G \longrightarrow \text{Homeo}(F)$ is faithful (i.e. injective) then there is a bijection of isomorphism classes

$$\text{Bund}_G^F(B) \underset{\Phi_{F,G}}{\overset{\Psi_{F,G}}{\rightleftarrows}} \text{Princ}_G(B)$$

"So the fibre is kind of fake... It doesn't really matter." –CFB.

Proof: Check that the compositions are the identity on isomorphism classes.

Classifying spaces BG

Let G be a topological space (may or may not be discrete).

Goal: We construct a space EG (possibly CW-complex) s.t.

1) $EG \simeq *$ (contractible)
2) G acts freely on EG.

Then define $BG := EG/G$, $\omega : EG \twoheadrightarrow BG$ principal G-bundle.

Theorem: BG is a classifying space of G, i.e. a representing space (with universal element ω) of the functor $B \longmapsto \text{Princ}_G(B)$, in the sense of Brown Representability.

Spelled out this means: for every principal bundle $\xi : P \to B$ there is a (up to homotopy unique) map $f_\xi : B \to BG$ such that $\xi \cong f^*(\omega)$.

Milnor construction

Let $E_nG := \underbrace{G * \cdots * G}_{\text{join of } n+1 \text{ copies.}}$

This has the topology inherited by the topology on G.
(E_nG is CW-complex, simplicial complex, Lie ... etc.
if G has the corresponding property.)

We can include $E_nG \hookrightarrow E_{n+1}G$ by

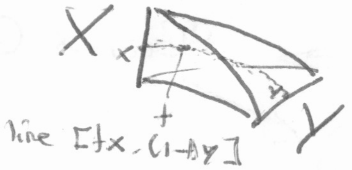
$$[g_0 . t_0 : \dots g_n . t_n] \longmapsto [g_0 . t_0 : \dots : g_n . t_n , 1 . 0]$$

Define

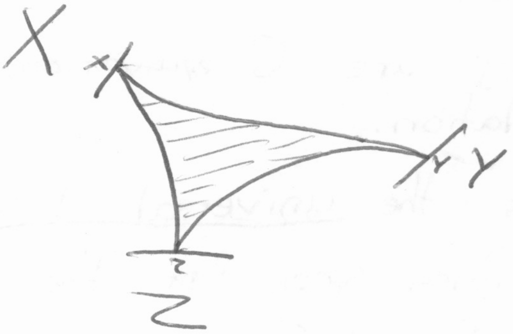
$$EG := \varinjlim_n E_nG \quad (= \underrightarrow{\mathrm{colim}}\, E_nG) \mathrel{\hat=} \mathop{\ast}_{j=1}^{\infty} G$$

Points look like $e = [t_0 . g_0 : t_1 . g_1 , \dots , t_n . g_n , \dots]$ s.t. almost all $t_i = 0$, the sum of the t_i is 1

$X * Y$:



$X * Y * Z$:



plane $[t_0 . x : t_1 . y : t_2 . z]$

Recall that the equivalence relation defining EG is given by $(t_0 . g_0 , t_1 . g_1 : \dots) \sim (t_0' . g_0' : t_1' . g_1' : \dots)$ iff $t_i = t_i'$ for all i and $g_i = g_i'$ if $t_i = t_i' > 0$.

To describe the topology on EG, define the maps

$$\tau_i : EG \longrightarrow [0,1]$$
$$[t.g] = e \longmapsto t_i$$

and

$$\gamma_i : \tau_i^{-1}((0,1]) \longrightarrow G$$
$$[t.g] = e \longmapsto g_i$$

which is well-defined. Let $EG_{(i)} := \tau_i^{-1}((0,1])$

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Note that G acts on EG by

$$h \cdot [t, g] := [t, hg] = [t_0, hg_0; t_1, hg_1; \dots, t_i; hg_i, \dots]$$

- This action is free
- $E_n G$ is invariant
- $EG(i)$ is invariant
- π_i is invariant
- γ_i is equivariant. (as $\gamma_i(h \cdot [t, g]) = hg_i = h\, \gamma_i([t, g])$)

Define $BG = EG/G$, with projection

$$\omega_G\ \xi : EG \twoheadrightarrow BG \quad : [t, g] \longmapsto [[t, g]]$$

By freeness of the ξ action, the fibres are G.

Theorem: $EG \twoheadrightarrow BG$ is a principle G-bundle.

Proof Take atlas $U_i = BG(i) := EG(i)/G$, with coordinates

$$\begin{tikzcd} EG(i) \arrow[r, "h_i"] \arrow[dr, "\omega_G\ \xi"'] & BG(i) \times G \arrow[d] \\ & BG(i) \end{tikzcd} \quad : [t, g] \longmapsto ([[t, g]], g_i)$$

The compositions $(h_j) \circ (h_i)^{-1}$ are G-equivariant, since the action is just by left translation.

Goal: $\omega_G : EG \longrightarrow BG$ is the universal bundle:

(1) BG is a classifying space for the functor

$$\Phi(X) = \mathrm{Prin}_G(X),$$

with $\omega_G \in \Phi(BG)$ the universal element.

$\Updownarrow$

(2) Every principle G-bundle ζ over any space X is the pull-back $\zeta \cong f^*(\omega_G)$ of ω_G under a certain map $f : X \longrightarrow BG$ that is unique up to homotopy.

This f is called a classifying map for ζ.

~~We with prove (2) using~~

Theorem: EG is contractible.

Lemma 1: Let Z be any G-space. Then any G-maps $f, f' : Z \longrightarrow EG$ are G-homotopic. We can write

$$f(z) = [t_0(z), g_0(z), t_1(z), g_1(z), \dots]$$
$$f'(z) = [t_0'(z), g_0'(z), t_1'(z), g_1'(z), \dots]$$

Step 1: Bring f and f' of the form

$$f_0(z) = [t_0(z)\, g_0(z),\ 0,\ 1,\ t_1(z)\, g_1(z),\ 0,\ 1,\ t_2(z)\, g_2(z),\ \dots]$$
$$f_0'(z) = [0,\ 1,\ t_0'(z)\, g_0'(z),\ 0,\ 1,\ t_1'(z)\, g_1'(z),\ 0,\ 1,\ \dots]$$

This can be achieved by composing infinitely many homotopies of the form

$$[t_0 g_0,\ (1-s)t_1, g_1,\ st_1, g_1,\ (1-s)t_2, g_2,\ st_2, g_2, \dots]$$

and versions in which the first s appears in $(1-s)t_i$.

~~Note~~ For each $[t, g]$, only finitely many of these homotopies are non-zero, so we can put them all in the time interval $[0, 1]$.

Step 2: We have $f_0 \simeq f_0'$ via

$$[(1-u)t_0 g_0,\ ut_0', g_0',\ (1-u)t_1, g_1,\ ut_1', g_1', \dots]$$

So $f \simeq f'$.

Proof of Theorem. id_{EG} is homotopic to any constant map by using the above proof, so EG is contractible.

Example: $G = \mathbb{Z}/_2$, $E\mathbb{Z}/_2 = S^\infty$, $B\mathbb{Z}/_2 = \mathbb{R}P^\infty$.

$G = S^1$, $ES^1 = S^\infty$, $BS^1 = \mathbb{C}P^\infty$.

Note: Same total space

Let $H \leq G$ be a subgroup.

$$\begin{tikzcd} & & G/H \arrow[d] & = \text{fibre, homogeneous space} \\ H \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & E/H & = BH \quad \text{principle } H\text{-bundle} \\ G \arrow[r] & E \arrow[r] & E/G \arrow[from=u, d] & = BG \quad \text{principle } G\text{-bundle} \end{tikzcd}$$

This works since $G \circlearrowright E$ acts freely and E is contractible, so it's also an EH.

Example: $G = O(n) \underset{\simeq}{\overset{\leq}{\longrightarrow}} GL_n(\mathbb{R})$

$E = V_n(\mathbb{R}^\infty)$ = orthonormal n-frames in $\mathbb{R}^\infty$

$\underset{\text{Orth. maps}}{\cong} (\mathbb{R}^n, \mathbb{R}^\infty) \ni f \mapsto (f(e_1), \dots, f(e_n))$

Then $G = O(n)$ acts on E by $f \mapsto f \circ A : \mathbb{R}^n \longrightarrow \mathbb{R}^\infty$

Note that this action leaves the span $(f(e_1), \dots, f(e_n))$ invariant, so we get $E/G = Gr_n(\mathbb{R}^\infty)$.

So $Gr_n(\mathbb{R}^\infty) = BO(n) = BGL_n(\mathbb{R})$

Now apply the previous remark about subgroups $H \leq G$.

We can regard all group ~~sa~~ as a subgroup of $GL_n(\mathbb{R})$ (well... not quite)

Example: Let $G = \mathfrak{S}_n$. ~~For~~ We have a G-bundle

$$\begin{tikzcd} \tilde{C}^n(\mathbb{R}^m) \arrow[d] \\ \tilde{C}(\mathbb{R}^m)/\mathfrak{S}_n \simeq C(\mathbb{R}^m) \end{tikzcd}$$

for all $1 \leq m \leq \infty$.

For $m = \infty$, $E = \tilde{C}^n(\mathbb{R}^\infty) \simeq *$ (we proved this by showing that $\pi_k(\tilde{C}^n(\mathbb{R}^\infty)) = 0$ for all k and using the unproved claim that $\tilde{C}^n(\mathbb{R}^\infty)$ is a CW-complex.)

So $\tilde{C}^n(\mathbb{R}^m)$ is $E\mathfrak{S}_n$ and $C^n(\mathbb{R}^\infty)$ is $B\mathfrak{S}_n$.

~~Theorem: A principal G-bundle $\xi: E \to B$~~

Theorem: $\omega_G: EG \to BG$ is a universal principal G-bundle.

Proof: Let $\xi: P \to X$ be a principal G-pundle. We want $f: X \to BG$ s.t. $\xi \cong f^*(\omega_G)$:

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, "\xi"'] & EG \arrow[d, "\omega_G"] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

up to hpty

By lemma 1, this $\hat{f}$ is unique if it exists, so f is unique up to homotopy.

For existence, let $\mathcal{U} = (U_i, h_i)_{i \in \mathbb{N}}$ be an atlas of X with $\xi|_{U_i} \xrightarrow{\cong} U_i \times G$ via h_i, and a partition of unity $\{t_i : X \to [0,1]\}$.

Let γ_i be the composition $P_i = \xi^{-1}(U_i) \xrightarrow[\cong]{h_i} U_i \times G \to G$.

Define $\hat{f}: P \to EG$ by

$$\hat{f}(y) = [t_0(\xi(y)), \gamma_0(y), t_1(\xi(y)), \gamma_1(y), \dots]$$

which is a well-defined element of EG by the properties of the partition of unity.

(For some reason. I forgot to use this page.)

As $\hat{f}$ is G-equivariant, we get an induced map

$$f : X \longrightarrow BG$$

So we would have to prove that indeed $\xi \cong f^*(\omega_G)$, but this follows from theory about principal G-bundles.

So $\omega_G : EG \longrightarrow BG$ is universal

Theorem: Every principal G-bundle $\omega : E \longrightarrow X$ with contractible total space E is universal.

Proof: Let $f : X \longrightarrow BG$ be a classifying map of ω, and $\hat{f} : E \longrightarrow EG$ the map on total spaces. Then $\hat{f}$ is a weak homotopy.

Well, we'll do the rest on Wednesday.

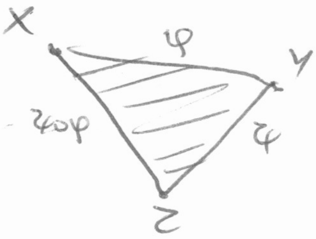
Classifying spaces of a category C

Let C be a category. ~~with~~ We allow the collection of objects to have a topology, and the collection of morphisms $Mor(X,Y)$ to have a topology.

Idea: We construct a ~~simplicial~~ cell-structure space with one vertex for each object in C (for now, assume C has no topology).
For each morphism $\varphi : X \longrightarrow Y$, take a new edge between X and Y

Take a triangle for each pair of composable maps

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\psi\circ\varphi"'] & Y \arrow[d, "\psi"] \\ & Z \end{tikzcd}$$



Define the simplicial set $\mathbb{B}_\cdot C$ by

$$\mathbb{B}_q C = \{(\varphi_0, \dots, \varphi_{q-1}) : \text{composable morphisms } X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \to \dots \to X_{q-1} \xrightarrow{\varphi_n} X_q \}$$

We have maps $d_i : \mathbb{B}_q C \longrightarrow \mathbb{B}_{q-1} C$ for $0 \leq i \leq q$ by "forgetting about X_i".

More precisely:

$$d_i(\varphi_0, \dots, \varphi_{q-1}) = \begin{cases} (\varphi_1, \dots, \varphi_{q-1}) & i=0. \\ (\varphi_0, \dots, \varphi_{i+1}\circ\varphi_i, \dots, \varphi_n) & 0<i<q \\ (\varphi_0, \dots, \varphi_{q-2}) & i=q \end{cases}$$

Similarly, we have $s_i : \mathbb{B}_{q-1} C \longrightarrow \mathbb{B}_q C$ via

$$s_i(\varphi_0, \dots, \varphi_{q-2}) = \text{"sneak in } id_{X_i}\text{"}$$

These satisfy the simplicial identities, so $\mathbb{B}_\cdot C$ is a simplicial set.

Now define the classifying space of C as the geometric realization of $\mathbb{B}_\cdot C$:

$$BC := |\mathbb{B}_\cdot C| = \Big(\coprod_{q\geq 0} \mathbb{B}_q C \times \Delta^q\Big)/\sim$$

Now what if $\mathrm{Hom}(X,Y)$ has a topology for all $X,Y \in C$? Then all tuples $(\varphi_0, \dots, \varphi_{q-1}) \in \mathbb{B}_q C$ for fixed objects $X_0, \dots, X_q$ have a topology.

The class of all morphisms $\mathrm{Mor}(C)$ gets a topology by taking the disjoint union and in particular $\mathrm{Mor}(C)^q$ has a topology.

Now also give $\mathrm{Ob}(C)$ a topology. Then one can regard $\mathbb{B}_q(C)$ as a subset of
$$\mathrm{Ob}(C)^{q+1} \times \mathrm{Mor}(C)^q$$
so it has a topology.

Examples of categories with a topology

1) Fundamental Groupoid of the space X:
$$\mathrm{Obj}(X) = X$$
$$\mathrm{Mor}(X) = \{\text{hpty classes of paths between } x_0 \text{ and } x_1 \text{ in } X\}$$
Here $\mathrm{Mor}(X)$ is discrete

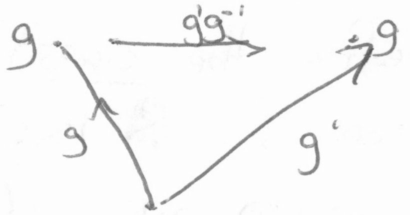
2) Same, but now $\mathrm{Mor}(X)$ has all paths in X. This has a topology.

Question: what does BC classify?

Example ① Let G be a group. ~~We can regard it as a category C with one object and $mor(C) = G$.~~

Let C be the category with G as objects and morphisms $mor(g, g') = \{g' \cdot g^{-1}\}$. ($C =$ "EG")

So all morphisms are isomorphisms.



Claim: $BC \cong EG$.

② Let $C =$ "BG", i.e. only one object $* = 1$, morphisms $mor(1, 1) = G$.

Claim $BC \cong BG$.

In these cases, it is clear what the B classifies. This is a consequence of the homegenity.

"Functor from category C to D, gives a map $BC \longrightarrow BD$, since it gives a simplicial map.

In the case of "EG" $\longrightarrow$ "BG", this gives us $EG \longrightarrow BG$, the universal bundle.

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Theorem: EG is contractible.

Proof: The identity is homotopic to the map

$$f([t, g]) = [t_0, g_0, 0, 1, t_1, g_1, t_2, g_2, \dots],$$

which is homotopic to a constant map $[0, 1, 1, 1, 0, 1, \dots]$ by

$$[(1-s)t_0, g_0, s, 1, (1-s)t_1, g_1, (1-s)t_2, g_2, \dots].$$

Example: (Conjugation with γ on EG induces identity on BG)

For $\gamma \in G$, define $A_\gamma : G \to G$ by $g \mapsto \gamma g \gamma^{-1}$.

This induces $a_\gamma : EG \longrightarrow EG : [t, g] \mapsto [t, \gamma g \gamma^{-1}]$.

Then $a_\gamma \simeq b_\gamma$, where $b_\gamma([t, g]) = [t, g\gamma^{-1}]$, and b_γ induces the identity on BG.

We have two fibrations with contractible total space over BG

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d] & \Omega BG \arrow[d] \\
* \simeq EG \arrow[r, "\tilde{\varphi}"] \arrow[d] & PBG \simeq * \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

- $\tilde{\varphi}$ exists (by homotopy lifting property?)
- We have a map $\varphi : G \longrightarrow \Omega BG$ given by mapping $g \in G$ to the loop $w_g : I \longrightarrow BG$ defined by

~~[illegible]~~

$$w_g(s) = \begin{cases} [\![(1-2s)_1 1 _ 2s _ 1 _ 0 _ 1 _ 0 _ 1 \ldots]\!] & 0 \leq s \leq \frac{1}{2} \\ [\![(2s-1)_ g _ 2-2s _ 1 _ 0 _ 1 _ 0 _ 1 \ldots]\!] & \frac{1}{2} \leq s \leq 1. \end{cases}$$

i.e. we go from

$[1 _ 1 , 0 _ 1 , 0 _ 1 , 0 _ 1 , \ldots]$

to

$[0 _ 1 , 1 _ 1 , 0 _ 1 , 0 _ 1 , \ldots]$

$= [0 _ g , 1 _ 1 , 0 _ 1 , 0 _ 1 , \ldots]$

to

$[1 _ g , 0 _ 1 , 0 _ 1 , 0 _ 1 , \ldots]$.

and the begin- and endpoint are the same in BG.

Proposition: $\varphi : G \longrightarrow \Omega BG$ is a weak hpty equivalence.

Corollary: $\pi_n(BG) \cong \pi_{n-1}(G)$.

Corollary: If G is discrete,

$$\pi_n(BG) = \begin{cases} 0 & n \neq 1 \\ G & n = 1. \end{cases}$$

so BG is a $K(G,1)$.

Example: For $G = O(n) \leq GL_n(\mathbb{R})$ we have

$BO(n) \simeq Gr_n(\mathbb{R}^\infty)$.

so

$\pi_k Gr_n(\mathbb{R}^\infty) \cong \pi_{k-1}(O(n))$

$\pi_k Gr_n(\mathbb{C}^\infty) \cong \pi_{k-1}(U(n))$

Back to classifying space of a category

A functor $\Phi : \mathcal{C} \rightarrow \mathcal{C}'$ induces a simplicial map

$$\mathbb{B}_\bullet \Phi : \mathbb{B}_\bullet \mathcal{C} \longrightarrow \mathbb{B}_\bullet \mathcal{C}'$$

$$(\varphi_0 , \ldots , \varphi_{n-1}) \longmapsto (\Phi(\varphi_0) , \ldots , \Phi(\varphi_{n-1})).$$

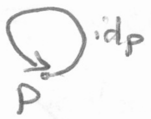
This induces a map

$$B\Phi : B\mathcal{C} \longrightarrow B\mathcal{C}'$$

- A natural transformation $\eta : \Phi \longrightarrow \Psi$ between two functors $\Phi, \Psi : \mathcal{C} \longrightarrow \mathcal{C}'$ induces a homotopy from $B\Phi$ to $B\Psi$.

To see this, we observe:

▷ For $\mathcal{C}$ = trivial category ⊛



we have $B\mathcal{C} \simeq *$

(The simplices $(id_p, \dots, id_p)$ are all degenerate.)

▷ Similarly for $\mathcal{C} = \mathbb{I}$ = "interval category" = $\mathcal{J}$

we have $B\mathbb{I} \simeq [0,1]$ $(\simeq B\mathcal{J})$

▷ A natural transformation η from Φ to Ψ is the same as a functor

$$\begin{aligned} \eta' : \mathcal{C} \times \mathcal{J} &\longrightarrow \mathcal{C}' \\ (x, 0) &\longmapsto \Phi(x) \\ (x, 1) &\longmapsto \Psi(x) \\ (\varphi, id_0) &\longmapsto \Phi(\varphi) \\ (\varphi, id_1) &\longmapsto \Psi(\varphi) \\ (id_x, t) &\longmapsto \eta(x) : \Phi(x) \longrightarrow \Psi(x) \end{aligned}$$

Since every morphism in $\mathcal{C} \times \mathcal{J}$ can be written as a composition of these three types, this fully determines η'.

So we get a natural transformation

$$\begin{tikzcd}
B\eta' : B(\mathcal{C} \times \mathcal{J}) \arrow[r] & B\mathcal{C}' \\
B\mathcal{C} \times B\mathcal{J} \arrow[u, phantom, "\wr"] & \\
B\mathcal{C} \times [0,1] \arrow[u, phantom, "\wr"] \arrow[ruu] &
\end{tikzcd}$$

Recall that for a group G, we had the categories

▷ $\tilde{\mathcal{C}}_G$: objects G, morphisms $mor(g, g') = \{(g, g')\}$

▷ $\mathcal{C}_G$: object $\{\mathbb{1}\}$, morphisms $mor(\mathbb{1}, \mathbb{1}) = G$.

We have a functor

$$V : \hat{C}_G \longrightarrow C_G$$

on objects $g \longmapsto \mathbb{1}$

on morphisms $(g,g') \longmapsto g'g^{-1}$

We have ~~BG~~ $B\hat{C}_G \simeq EG$

$$\begin{tikzcd} B\hat{C}_G \arrow[r, phantom, "\simeq"] \arrow[d, "BV"'] \arrow[dr, phantom, "\simeq"] & EG \arrow[d, "\omega_G"] \\ BC_G \arrow[r, phantom, "\simeq"] & BG \end{tikzcd}$$

Furthermore we have a functor

$T : \hat{C}_G \longrightarrow \circledast \longleftarrow$ (trivial category)

objects $g \longmapsto \bullet\ P$

morphisms $(g,g') \longmapsto id_P$

In the other direction we have $S : \circledast \longrightarrow \hat{C}_G$

~~$S(\mathbb{1})$~~ ~~=~~ $S(P) = 1$

$S(id_P) = id_1 = (1,1)$

Clearly $T \circ S = id_\circledast$. We want to find a natural equivalence $S \circ T \simeq id_{\hat{C}_G}$.

Define $\eta : S \circ T \longrightarrow Id_{\hat{C}_G}$ by

$$\begin{tikzcd} g \arrow[r, "\varphi=(g,g')"] & g' & {} \arrow[r, mapsto, "S\circ T"] & {} & 1 \arrow[r, "(1,1)"] \arrow[d, "(g,1)"'] & 1 \arrow[d, "(g',1)"] \\ g \arrow[r, "\varphi"] & g' & {} \arrow[r, mapsto, "Id."] & {} & g \arrow[r, "(g,g')"] & g' \end{tikzcd}$$

i.e. $\eta_g : 1 = S\circ T(g) \longrightarrow id(g) = g$

is just $(g,1)$ (or $(1,g)$, I'm not sure about his notation)

This shows that $\hat{C}_G$ is equivalent to the trivial category, and this produces a different proof of

$$EG \simeq B\hat{C}_G \simeq B\circledast \simeq *$$

<u>Example</u>: Let $\mathcal{C}$ be the category of vector spaces over $\mathbb{R}$ and linear maps.

Let $D(n)$ be the category of all $\mathbb{R}$-vector spaces of dimension n. all linear maps.

This is equivalent to the category $\textcircled{\mathbb{R}^n}$ with one object $\mathbb{R}^n$ and all linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ as morphisms

$$T : D(n) \longrightarrow \textcircled{\mathbb{R}^n}$$
$$V \longmapsto \mathbb{R}^n$$

By fixing a basis for all V

$$(M : V \to W) \longmapsto Mat_{B_V, B_W}(M)$$

Similarly, the category $ISO(n)$ of n-dimensional vector spaces over $\mathbb{R}$ with only the linear <u>isomorphisms</u> is equivalent to $\textcircled{\mathbb{R}^n + \text{isom}}$. $=$ $CGL_n(\mathbb{R})$

[Bödigheimer explains in words what are some classical examples of this.]

"Branched coverings"

"Divide out a properly discontinuous <u>non-free</u> action!"

<u>Chapter: Characteristic classes</u>

Let G be a topological group and F a (faithful) G-space.

<u>Definition</u>: A <u>characteristic class</u> c is an assignment

$$c: Bun_G^F(X) \longrightarrow H^k(X; R)$$

such that
$$c(f^*(\xi)) = f^*(c(\xi))$$
for all (F,G)-bundles $\xi : E \to B$, $f : B' \to B$

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f^*(\xi)"'] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

$$f^* : H^k(B; R) \longrightarrow H^k(B'; R).$$

Here R is a commutative ring with unit.
The integer k is called the *degree* of c.
In the same way we have characteristic classes

$$c : \mathrm{Prin}_G(X) \longrightarrow H^*(X;R).$$

▷ *Sums* For fixed R and c_1, c_2 of the same degree k, we have the sum

$$(c_1+c_2)(\xi) := c_1(\xi) + c_2(\xi)$$

▷ *Products* ; For fixed R, c_1, c_2 of arbitrary degree, we have the product

$$(c_1 \cup c_2)(\xi) = c_1(\xi) \cup c_2(\xi).$$

▷ *Scalar multiplication* ; For $\lambda \in R$

$$(\lambda c)(\xi) = \lambda \cdot c(\xi).$$

▷ *Constant class* ; $c(\xi) = 1$ for all ξ (degree $= 0$).

So the characteristic classes form an R-algebra, called *$\mathrm{CharCl}(G,R)$*

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Theorem : There is an isomorphism

$$\mathrm{CharCl}(G,R) \cong H^*(BG,R)$$

Proof : follows from Yoneda, since $\mathrm{Bun}^F_G(X) \cong [X, BG]$ is representable, so

$$\begin{aligned} \mathrm{CharCl}(G,R) &= \mathrm{NatTrans}(\mathrm{Bun}^F_G, H^*(-,R)) \\ &\cong \mathrm{Nat\,Trans}([-,BG], H^*(-,R)) \\ &\cong H^*(BG,R) \end{aligned}$$

Examples : (1) $G = O(1)$, $F = \mathbb{R}$.
Then Bun^F_G = real line bundles

$$BO(1) = Gr_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$$

For $R = \mathbb{Z}/2$, we get

$$\mathrm{CharCl}_*(O(1) ; \mathbb{Z}/2) \cong H^*(\mathbb{RP}^\infty ; \mathbb{Z}/2) = \mathbb{Z}_2[W_1]$$

($W_1 \in H^1(\mathbb{RP}^\infty ; \mathbb{Z}_2)$ a generator)

(2) $G = O(n)$, $F = \mathbb{R}^n$, $R = \mathbb{Z}/2$

Bun^F_G = n-dim real vector bundles

$$\mathrm{Char\,Cl}_*(O(n), \mathbb{Z}/2) \cong H^*(BO(n); \mathbb{Z}/2)$$
$$\cong \mathbb{Z}/2[\underbrace{W_1, \dots, W_n}_{\text{generators.}}]$$

(Note $O(n) \xrightarrow[\simeq]{\simeq} GL_n(\mathbb{R})$, Gram-Schmidt)

The $W_1, \dots, W_n$ are the universal Stiefel-Whitney classes

(3) $G = U(n)$, $F = \mathbb{C}^n$, $R = \mathbb{Z}$

Bun^F_G = n-dim complex vector bundles

So

$$\mathrm{Char\,Cl}_*(U(n); \mathbb{Z}) \cong H^*(BU(n); \mathbb{Z}) = \mathbb{Z}[Z_1, \dots, Z_n]$$

As $U(n) \xrightarrow[\simeq]{\simeq} GL_n(\mathbb{C})$, we have

$$\begin{tikzcd} V_n(\mathbb{C}^\infty) \arrow[r] \arrow[d, phantom, "\parallel"] & Gr_n(\mathbb{C}^\infty) \arrow[d, phantom, "\parallel"] & \\ EGL_n(\mathbb{C}) \arrow[r] & BGL_n(\mathbb{C}) & = BU(n) \end{tikzcd}$$

The $Z_i \in H^{2i}(BU(n); \mathbb{Z})$ are the universal Chern classes

More examples: Given any $\alpha \in$ ~~B~~ $H^k(BG; R)$, we can find a characteristic class $cl_\alpha : \mathrm{Prin}_G(X) \longrightarrow H^k(X, R)$ by

$$\begin{array}{ccccc} \mathrm{Prin}_G(X) \cong & [X, BG] & \longrightarrow & H^k(X, R) \\ [\xi] \longmapsto & [f_\xi] & \longmapsto & f_\xi^*(\alpha) \end{array}$$

Note:

- ▷ $cl_{\alpha+\beta} = cl_\alpha + cl_\beta$
- ▷ $cl_{\lambda\alpha} = \lambda\, cl_\alpha$
- ▷ $cl_{\alpha\cup\beta} = cl_\alpha \cdot cl_\beta$
- ▷ $cl_0 = 0$
- ▷ $cl_1 = 1$

So we get a ring map

$$H^*(BG; R) \xrightarrow{cl.} \mathrm{CharCl}_*(G, R)$$

The theorem says that every characteristic class can be written in such a way, cl_α.

Little survey: Stiefel Whitney classes

1) Given any real vector bundle $\xi : E \longrightarrow X$ of dimension n, we get for each generator ~~W~~ $_i \in H^i(BO(n);\mathbb{Z}/_2)$ a characteristic class $w_i(\xi) := f_\xi^*(W_i)$, called the <u>i-th Stiefel Whitney class of ξ</u>, in $H^i(X;\mathbb{Z}/_2)$. Here $f_\xi : X \to BO(n)$ is a classifying map for ξ.

Formulas

(1) $W_0(\xi) = 1 \qquad (W_0 := 1 \in H^0(BO(n);\mathbb{Z}/n))$

(2) $w_i(\xi) = 0$ if $i > n$

(3) <u>Cartan formula</u>: Set $w(\xi) := \sum_{i=0}^{\infty} w_i(\xi)$ (fin. sum. by (2)). Then

$$w(\xi_1 \oplus \xi_2) = w(\xi_1) \cup w(\xi_2)$$

(4) For the canonical line bundle

$$\mathbb{R} \longrightarrow L \xrightarrow{\gamma^1} \mathbb{R}P^\infty = Gr_1(\mathbb{R}^\infty)$$

(constructed from $O(1) \longrightarrow V_1(\mathbb{R}^\infty) \longrightarrow Gr_1(\mathbb{R}^\infty)$ by $L = V_1(\mathbb{R}^\infty) \times_{O(1)} \mathbb{R}$.)

we have $w_1(\gamma^1) = W_1$

Similarly $W_n(\gamma^n) = W_n \in H^n(BO(n);\mathbb{Z}/_2)$ for the canonical n-dim vector bundle on $BO(n)$:

$$O(n) \longrightarrow V_n(\mathbb{R}^\infty) \longrightarrow Gr_n(\mathbb{R}^\infty) = BO(n)$$

gives

$$\gamma^n : \mathbb{R}^n \longrightarrow V_n(\mathbb{R}^\infty) \times_{O(n)} \mathbb{R}^n \longrightarrow Gr_n(\mathbb{R}^\infty)$$

2) Same for the Chern classes. For a complex n-dim. vector bundle $\xi : E \longrightarrow X$, we define the <u>i-th Chern class</u> of ξ by

$$c_i(\xi) := f_\xi^*(Z_i) \in H^{2i}(X;\mathbb{Z})$$

Geometric definitions

(a) Originally one defined the $w_i(\xi)$ as obstructions for certain sections in certain bundles coming from ξ.

(b) For $c_i(\xi)$: via forms in certain bundles

(c) For the Chern-classes: a n-dim complex v.b. ξ gives a orientable (2n-1)-sphere bundle ξ', which gives us a (Thom class and thus a) euler class $e(\xi') \in H^{2n}(X;\mathbb{Z})$

We could define $c_n(\xi) = e(\xi')$, $c_i(\xi) = 0$ for $i > n$.

So how do we define $c_i(\xi)$ for $i<n$?

$$\begin{tikzcd}
\mathbb{C}^n \arrow[d] & E' \arrow[dl] \arrow[d, "f^*(\xi)"] \arrow[r, "\cong"] & E'\oplus L \arrow[d, "\xi'\oplus\lambda"] \\
E \arrow[d, "\xi"'] & S' = E(\xi') \arrow[l] \arrow[r] \arrow[dl, "f"] & S' \\
X_{x} & \begin{matrix}(x,e)\\ \xi(e)=x\\ x\neq 0\end{matrix} &
\end{tikzcd}$$

We can write $f^*(\xi)$ as a direct sum of an $(n-1)$-vector bundle vector bundle ξ' with a line bundle λ.

We define $f^*(c_{n-1}(\xi)) = c_{n-1}(\xi')$, and by iterating this procedure we can define all $c_i(\xi)$ for $i<n$.

Part 2 of the survey

We have the equivalences

$w_1(\xi) = 0 \iff \xi$ is orientable

$\iff \xi$ has $SO(n)$ as structure group

$c_1(\xi) = 0 \iff \xi$ has $SU(2)$ as structure group

Also

~~$\Leftrightarrow$~~ $w_2(\xi) = 0 \iff \xi$ has a Spin-structure.

What does this mean? As $\pi_1(SO(n)) = \mathbb{Z}/2$ for $n>2$, we have a universal covering space

$$\mathbb{Z}/2 \longrightarrow \widehat{SO(n)} \longrightarrow SO(n).$$

Then $\widehat{SO(n)}$ is a ~~topological~~ Lie-group! group, called $Spin(n)$.

For $n=3$

$$\begin{tikzcd}
\widehat{SO(3)} \arrow[r, dashed] \arrow[d] & S^3 = SU(2) = S(\mathbb{H}) \arrow[d] \\
SO(3) \cong & \mathbb{R}P^3
\end{tikzcd}$$

Let $BSO(n) \xrightarrow{w_2} K(\mathbb{Z}/2, n)$ correspond to $w_2 \in H^2(BSO(n); \mathbb{Z}/2)$

Then we get

$$B\widehat{SO}(n) \longrightarrow SO(n) \xrightarrow{w_2} K(\mathbb{Z}/2, n)$$

and $B\widehat{SO}(n)$ is the homotopy fibre of w_2.

Part 3

Question: if ξ_1, ξ_2 are $\mathbb{R}$ vector bundles over B, and $w_i(\xi_1) = w_i(\xi_2)$ for all i, is $\xi_1 \cong \xi_2$?

Unfortunately not.

Counter example: For $n>0$, let $\tau = TS^n$ = tangent bundle of S^n.
Then $w_0(\tau)=1$, $w_1(\tau) = \cdots = w_{n-1}(\tau) = 0$, obviously.
But also $w_n(\tau) = 0 \in H^n(S^n) \cong \mathbb{Z}/_2$:
If $\mathbb{1}$ is the normal bundle to S^n in $\mathbb{R}^{n+1}$, consider

$$TS^n \oplus \mathbb{1}^n \{ \cong T\mathbb{R}^{n+1}|_{S^n} = \mathbb{1}^{n+1}$$

Then by the Cartan formula

$$0 = w_n(\tau \oplus \mathbb{1}) = \sum_{j=0}^{n} w_j(\tau) \cup w_{n-j}(\mathbb{1})$$

~~So~~

$$\overset{\oplus}{=} w_n(\tau)$$

So $w_i(\tau) = w_i(\mathbb{1}^n)$ for all i, but $\tau \not\cong \mathbb{1}^n$ as
TS^n not trivial for many n!!

Part 4: Characteristic classes of manifolds M
= characteristic classes of T_M.

$$W(\mathbb{R}P^n) = (1+\alpha)^{n+1}, \quad \alpha \in H^*(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/_{\alpha^{n+1}}$$
$$|\alpha| = 1$$
$$W_i(\mathbb{R}P^n) = \binom{n+1}{i}\alpha^i$$
$$c(\mathbb{C}P^n) = (1+\alpha)^{n+1} \quad \alpha \in H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/_{\alpha^{n+1}}$$
$$|\alpha| = 2$$

Part 5: If $\mathbb{R}P^n$ is parallellizable then $n = 2^k-1$ for some k.
(If $n \neq 2^k-1$ then there is some i with $\binom{n+1}{i} \neq 0$)

$$\underset{(m)}{M} \xhookrightarrow{\iota} \mathbb{R}^{m+k}$$

Let ν be a normal bundle of ι. Then

$$T_M \oplus \nu(\iota) \cong T\mathbb{R}^{m+k}|_M = \mathbb{1}^{m+k}$$

So

$$1 = w(T_M \oplus \nu(\iota)) = w(T_M)\cdot w(\iota)$$

Lecture 22 27-06-18

Chapter: Spectral Sequences

☺ finally!

Subchapter 1: Exact couples

Let R be a commutative ring with 1. Let A and E be modules over R.

Definition: The modules A and E, together with the homomorphisms i, j, k as in the diagram

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

form an exact couple if the sequence

$$\begin{tikzcd} E \arrow[r, "k"] & A \arrow[r, "i"] & A \arrow[r, "j"] & E \arrow[r, "k"] & A \end{tikzcd}$$

is exact at all places.

Remark: Instead of R-modules, we can in principle work in any Abelian category.

Consequences:

Consequence I: E is a differential object, i.e. we have an endomorphism $d := jk$ with $d^2 = j\underbrace{kj}_{0}k = 0$.
So $\operatorname{im}(d) \subseteq \ker(d) \subseteq E$.

Consequence II: We can form the homology

$$H(E,d) := \frac{\ker(d)}{\operatorname{im}(d)} = \frac{\text{cycles}}{\text{boundaries}}$$

Note: At the moment, there is no grading.
If A and E are graded and i, j, k are graded (preserving), we want: i, j have degree 0, k has degree -1.
Then d has degree -1 and $H(E,d)$ is again graded.
This is called homological type
If the degree of k is $+1$ instead, then d has also degree $+1$ and we speak of cohomological type

Example 1: Bockstein exact couple

$A = H_*(X;\mathbb{Z})$ $\quad i: A \longrightarrow A$ multiplication by $p \in \mathbb{Z}\setminus\{0\}$

$E = H_*(X;\mathbb{Z}/p)$ $\quad j: A \longrightarrow E$ reduction mod p

$k: E \longrightarrow A$ Bockstein connecting homomorphism

In the background, this comes from

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

giving a l.e.s.

$$\begin{tikzcd}
\cdots \arrow[r] & H_*(X;\mathbb{Z}) \arrow[r, "p_*", "i"'] & H_*(X;\mathbb{Z}) \arrow[r, "red_p", "j"'] & H_*(X;\mathbb{Z}/p) \arrow[dll, "\beta_p"', "k"] \\
 & H_{*-1}(X;\mathbb{Z}) \arrow[r] & \cdots &
\end{tikzcd}$$

<u>Example 2</u> We can do the same for the s.e.s. of coefficients

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

See seminar.

The Bockstein is now: $H^*(-;\mathbb{Z}_p) \longrightarrow H^{*+1}(-;\mathbb{Z}_p)$

For $p=2$, this is Sq^1

<u>Derived exact couple</u>

Given an exact couple $\mathcal{C} = (A, E, i, j, k)$, we get a derived exact couple $\mathcal{C}' = (A', E', i', j', k')$ by

- $A' := im(i) \subseteq A$
- $i' : A' \longrightarrow A'$ is $i' := i|_{A'}$
- $E' := H(E, d)$ with $d = j \circ k$
- $j' : A' \longrightarrow E'$, $a' \longmapsto [j(a)]$, ~~ex~~ where $a' = i(a)$
- $k' : E' \longrightarrow A'$ $[e] \longmapsto k(e)$

$$\begin{tikzcd}
A \arrow[r, "i"] & A \\
im(i) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "i'"] & im(i) \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

<u>Proposition</u>: $\mathcal{C}'$ is an exact couple

<u>Proof</u> (Bödigheimer didn't prove this, I added ~~at~~ it)
We have to check that j' is independent of choice of a, which it is since if $i(a) = 0$, then $a = k(e)$ for some $e \in E$, so $[j(a)] = [jk(e)] = [d(e)] = 0$ in $H(E,d) = E'$.
Also k' is independent of representative e, since if $e = d(e')$ is a boundary, then $e = jk(e')$, so $k(e) = k(j(k(e'))) = 0$.
Exactness can be checked easily as well. □

This is now the start of a procedure

$$\mathcal{C} \rightsquigarrow \mathcal{C}' \rightsquigarrow \mathcal{C}'' \rightsquigarrow \cdots$$

In example 1, $\mathcal{C}$ says something about divisibility of elements of $H_*(X;\mathbb{Z})$ by p. Then $\mathcal{C}'$ says something about divisibility by p^2. In general $C^{(r)}$ says something about divisibility by p^{r+1}.

Definition: A spectral sequence $\mathcal{E}$ is a sequence $\{E^r\}$ of differential objects E^r with differential $d^r : E^r \longrightarrow E^r$, & such that $H(E^r, d^r) \cong E^{r+1}$

(Note: in principle, all differentials are allowed to be completely independent of each other.)

Example: Start with an exact couple $(A^0, E^0, i^0, j^0, k^0)$ and denote the iterated derived couples by $(A^r, E^r, i^r, j^r, k^r)$. Then with $d^r = j^r k^r$, $\{E^r\}$ forms a spectral sequence.

Example: Let X be a filtered space, i.e. there is a sequence of ascending subspaces

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_i \subseteq \ldots \subseteq X$$

We assume the sequence is exhaustive, i.e. $X = \bigcup_{i \geq 0} X_i$, and we assume X has the limit topology.

Define

$$A := \bigoplus_{p \geq 0} H_*(X_p; R) = \bigoplus_{p,q} H_{p+q}(X_p; R)$$

$$E := \bigoplus_{p} H_*(X_p, X_{p-1}; R) = \bigoplus_{p,q} H_{p+q}(\underbrace{X_p, X_{p-1}}_{\text{think of this as the filtration quotients}}; R)$$

Thus A and E are bi-graded modules.

Note the shift $p+q$ we used

Motivation for the shift: Often $H_{p+q}(X_p, X_{p-1}; R)$ vanishes for negative q. In the case of a filtration of a CW-complex by its skeleta, it is actually only non-zero for $q=0$.

If we have a fibre bundle $F \longrightarrow E \longrightarrow B$ and B is a CW complex with filtration $\{B_p\}$, we have

$B_p \setminus B_{p-1} \cong \coprod_{\alpha} \mathring{e}^p_\alpha$ $\qquad$ $B_p / B_{p-1} \cong \bigvee_\alpha S^p$ (only homology for $q = p$!)

$E_p \setminus E_{p-1} \cong \coprod_{\alpha} \mathring{e}^p_\alpha \times F$ $\qquad$ $E_p / E_{p-1} \cong \bigvee_\alpha S^p \wedge F_+$ (only homology for $q \geq p$!)

Define $A_{p,q} := H_{p+q}(X_p ; R)$

$E_{p,q} := H_{p+q}(X_p, X_{p-1} ; R)$

For the morphisms, we take:

- $i : A \longrightarrow A$ is induced by the inclusion $X_p \subseteq X_{p+1}$, giving

$$\begin{tikzcd} H_{p+q}(X_p) \arrow[r] \arrow[d, phantom, "\parallel"] & H_{p+q}(X_{p+1}) \arrow[d, phantom, "\parallel"] \\ A_{p,q} \arrow[r, "i"] & A_{p+1,q-1} \end{tikzcd}$$

- $j : A \longrightarrow E$ is induced by the inclusion $(X_p, \emptyset) \subseteq (X_p, X_{p-1})$ giving

$$\begin{tikzcd} H_{p+q}(X_p) \arrow[r] \arrow[d, phantom, "\parallel"] & H_{p+q}(X_p, X_{p-1}) \arrow[d, phantom, "\parallel"] \\ A_{p+q} \arrow[r, "j"] & E_{p,q} \end{tikzcd}$$

- $k : E \to A$ is the connecting homomorphism of the long exact homology sequence of the pair (X_p, X_{p-1}):

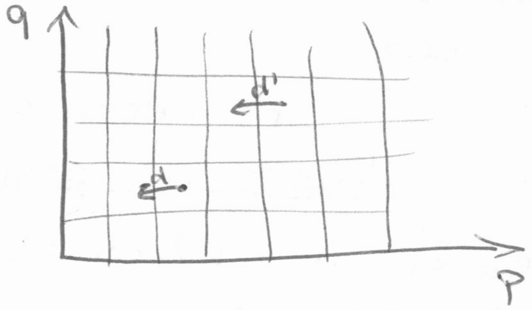
$$\begin{tikzcd} H_{p+q}(X_p, X_{p-1}) \arrow[r] \arrow[d, phantom, "\parallel"] & H_{p+q-1}(X_{p-1}) \arrow[d, phantom, "\parallel"] \\ E_{p,q} \arrow[r, "k"] & A_{p-1,q} \end{tikzcd}$$

Then taking $E^1 = E$ this gives the E^1-page with differential

$$d^1 : E^1_{p,q} \xrightarrow{k} A^1_{p-1,q} \xrightarrow{j} E^1_{p-1,q}$$

So the 1-page looks like

E^1:



In general, the direction of the derived differentials is:

E^0

$d^0 : E^0_{p,q} \to E^0_{p,q-1}$

E^1

$d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$

E^2

$d^2 : E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$

$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$

Example (Easy almost trivial)

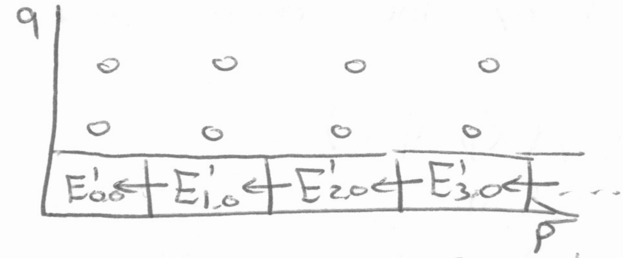
Let X be a CW-complex with i-skeleton X_i.

We can write $X_p / X_{p-1} \cong \bigvee_{\alpha} S^p_{\alpha}$. So

$$E^1_{p,*} = H_{p+*}(X_p, X_{p-1}) = H_{p+*}(\bigvee_{\alpha} S^p_{\alpha})$$

$$= \begin{cases} \bigoplus_{\alpha} \mathbb{Z} & * = 0 \\ 0 & * \neq 0. \end{cases}$$

So the E^1-term vanishes for (p,q) with $q \neq 0$.



The ~~boundary maps~~ ~~d~~ differential d^1 is

$$d^1 : H_p(X_p, X_{p-1}) \xrightarrow{k} H_{p-1}(X_{p-1}) \xrightarrow{j} H_{p-1}(X_{p-1}, X_{p-2}).$$

So $E^1_{p,0}$ is just the cellular chain complex with the usual cellular differential.

Thus the E^2 page is

$$E^2_{p,0} = H(E^1_{p,0}, d^1) \cong H_p(X)$$

$$E^2_{p,q} = 0 \quad \text{for} \quad q \neq 0.$$

It is clear that $E^r_{p,q} = 0$ for $q \neq 0$ for all $r \geq 2$ as well, so all differentials either start or end in a zero and thus are zero.

If all differentials $d^r, d^{r+1}, \dots$ are zero, one says that the spectral sequence collapses at page r

Thus in the above examples of a CW-complex, the spectral sequence collapses at the E^2-page.

Lecture 23 - 02-07-18

Recall we have a sequence $E = (E^r_{pq})_{r \geq 0}$ of bigraded R-modules with differentials $d^r : E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$, such that $E^{r+1}_{pq} \cong H(E^r)_{p,q}$

Also recall the spectral sequence we get for a filtered space $X_0 \subseteq X_1 \subseteq \dots \subseteq X$ with $A = \bigoplus_{p,q} H_{p+q}(X_q; R)$ $E = \bigoplus_{p,q} H_{p+q}(X_q, X_{q-1}; R)$.

Example 3: Similar to the previous example we can consider a filtered chain complex

Let $C_\bullet$ be a chain complex of R-modules with boundary operator $\partial: C_k \longrightarrow C_{k-1}$.

Assume that we have a filtration on $C_\bullet$, i.e. a sequence

$$\dots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \dots \subseteq C_\bullet$$

of subchain complexes $F_pC_\bullet$ such that ∂ preserves $F_pC_\bullet$, i.e. $\partial: F_pC_q \longrightarrow F_pC_{q-1} \subseteq C_{q-1}$.

An example of a filtered complex is the simplicial chain complex $S_\bullet(X)$ of a filtered space X ($\dots \subseteq X_p \subseteq X_{p+1} \subseteq \dots \subseteq X$), with filtration given by $F_pS_\bullet(X) = S_\bullet(X_p) \subseteq S_\bullet(X)$.

We call a filtration

- exhaustive if union over all F_p gives everything
- bounded below if F_{-1} is trivial
- bounded above if F_p is everything for large enough p.

Given a filtered complex we get an exact couple by

$$A := \bigoplus_p F_pC_\bullet = \bigoplus_{p,q} F_pC_q$$

$$E := \bigoplus_p F_pC_\bullet / F_{p-1}C_\bullet = \bigoplus_{p,q} F_pC_q / F_{p-1}C_q$$

The morphisms are

- $i: A \to A$ induced by inclusion $F_pC_\bullet \longrightarrow F_{p+1}C_\bullet$
- $j: A \to E$ is induced by $F_pC_\bullet \longrightarrow F_pC_\bullet / F_{p+1}C_\bullet$
- $k: E \to A$ is zero.

This is exact, since all sequences

$$0 \longrightarrow F_pC_\bullet \hookrightarrow F_{p+1}C_\bullet \twoheadrightarrow F_{p+1}C_\bullet / F_pC_\bullet \longrightarrow 0$$

are exact.

We have $d^0: F_pC_q / F_{p-1}C \longrightarrow F_p$

Definition: 1) A ss. is said to collapse at page k if $d^r_{pq} = 0$ for all p,q and all $r \geq k$. (So $E^{r+1}_{pq} = E^r_{pq}$ $\forall r \geq k$).

2) The ss. is converging to the graded R-module H with respect to a filtration $0 = F_0 H \subseteq F_1 H \subseteq \dots \subseteq H$ if there are isomorphisms $(F_n H_n = H_n)$

$$F_p H_n / F_{p-1} H_n \cong E^\infty_{p,n-p}$$

Definition: We have the following chain of ~~subobje~~ submodules of E^0:

$$B^0 \subseteq B^1 \subseteq \dots \subseteq B^r \subseteq \dots \subseteq B^\infty \subseteq Z^\infty \subseteq \dots \subseteq Z^r \subseteq \dots \subseteq Z^0 \subseteq E^0$$

Here $B^0 = \operatorname{im}(d^0)$, $Z^0 = \ker(d^0)$,

then $Z^1 = \ker(Z^0 \twoheadrightarrow \ker(d^0)/\operatorname{im}(d^0) = E^1 \xrightarrow{d^1} E^1) \subseteq Z^0$

and $Z^2 = \ker(Z^1 \twoheadrightarrow Z^1/B^1 \cong E^2 \xrightarrow{d^2} E^2) \subseteq Z^1$

We will see this in more detail later.

We have $Z^\infty = \bigcap_r Z^r$, called the permanent cycles.

$$B^\infty = \bigcup_r B^r$$

$$E^\infty := Z^\infty / B^\infty .$$

Theorem: Let C be a filtered chain complex as above. Assume that there are numbers $s(n)$, $t(n)$ s.t.

$$F_{s(n)} C_n = 0 \qquad F_{t(n)} C_n = C_n$$

Then there is a s.e.s. of homology type starting at page 1 with $E^1_{p,q} = H_{p+q}(F_p C_\bullet / F_{p-1} C_\bullet)$ and ending with

$$E^\infty_{p,q} = F_p H_{p+q}(C_\bullet) / F_{p-1} H_{p+q}(C_\bullet)$$

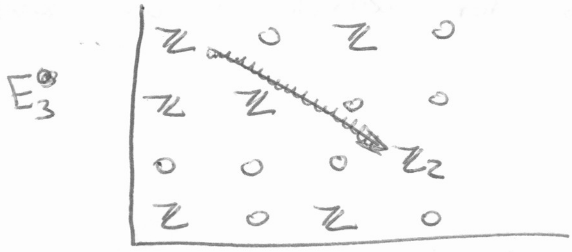
thus converging to the object $H_*(C_\bullet)$ with filtration

$$F_i H_*(C_\bullet) := \operatorname{im}(H_*(F_i C_\bullet) \longrightarrow H_*(C_\bullet)) \subseteq H_*(C_\bullet)$$

Example: Let $R = \mathbb{Z}$. Consider the following spectral sequence of cohomological type:

$$\begin{tikzcd}
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & & & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\
E_1^\bullet & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} & 0 & 0 & \Longrightarrow & & \mathbb{Z} & \mathbb{Z} \arrow[rrd, "0"] & 0 & 0 \\
 & 0 & 0 & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} & & E_2^\bullet & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} \\
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & & & \mathbb{Z} & 0 & \mathbb{Z} & 0
\end{tikzcd}$$

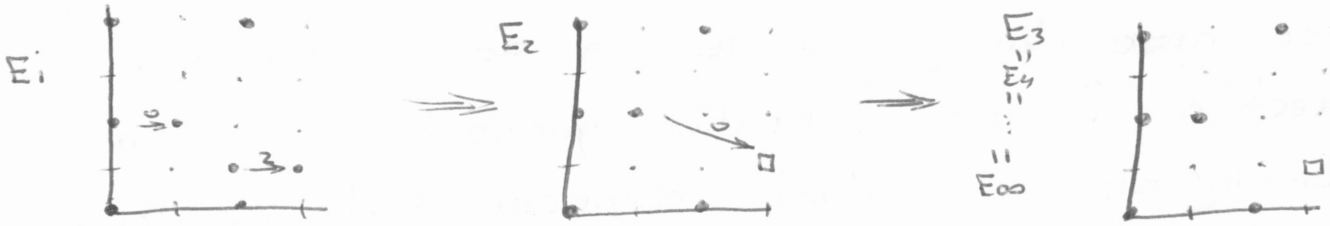
Then



Lecture 24 04-02-18

We continue the example of a spectral sequence.

$\bullet = \mathbb{Z}$. $\square = \mathbb{Z}/2$



There are no more differentials possible in $E_3, E_4, \dots$

Now collect all terms on the diagonal $p+q=n$.

$n=0$ Just one term $E_\infty^{0,0} = \mathbb{Z}$ $\qquad H^0 = \mathbb{Z}$

$n=1$ Nothing $\qquad H^1 = 0$

$n=2$ $0 = \underbrace{F_{-1}H^2 \leq F_0H^2}_{E_\infty^{0,2}=\mathbb{Z}} \leq \underbrace{F_1H^2}_{E_\infty^{1,1}=0} \leq \underbrace{F_2H^2}_{E_\infty^{2,0}=\mathbb{Z}} = H^2$ $\qquad H^2 = \mathbb{Z}\oplus\mathbb{Z}$

$n=3$ Just one term $E_\infty^{1,2} = \mathbb{Z}$ $\qquad H^3 = \mathbb{Z}$

$n=4$. This is more problematic.

$$0 = \underbrace{F_{-1}H^4 \leq F_0H^4}_{\mathbb{Z}} \leq \underbrace{F_1H^4}_{0} \leq \underbrace{F_2H^4}_{0} \leq \underbrace{F_3H^4}_{\mathbb{Z}/2} \leq \underbrace{F_4H^4}_{0} = H^4$$

This gives the following extension problem:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

This means that $H^4 \cong \mathbb{Z}$ or $H^4 \cong \mathbb{Z}\oplus\mathbb{Z}/2$. $H^6 = \mathbb{Z}$

$n=6$ ---

A trick that sometimes helps to solve these problems is to compare two spectral sequences, for example considering a subcomplex. For that we need the concept of a <u>morphism of spectral sequences</u> C' and C, given by a collection of maps $\{\varphi_{p,q}^r : C'^r_{p,q} \longrightarrow C^r_{p,q}\}$ such that

1) $d^r \circ \varphi^r = \varphi^r \circ d'^r$, so φ^r induces $H(\varphi^r)$

2) Under $H(E^r) \cong E^{r+1}$, $H(\varphi^r)$ corresponds to φ^{r+1}

We get a morphism of spectral sequences for example if we have a map of exact couples

$$\Phi : \mathcal{C}' \longrightarrow \mathcal{C}.$$

(Spell this out yourself.)

A useful theorem concerning morphisms of spectral sequences is the Zeeman Comparison theorem.

We might state this theorem later.

Confusion arose during the lecture about the use and direction of the filtration quotients of H^n in the cohomological case. Annika remarked that she thinks we probably have

$$H^n = F_0 \underbrace{H^n \geq F_1}_{E_\infty^{0,n}} H^n \geq \dots \geq F_{n-1} \underbrace{H^n \geq F_n}_{E_\infty^{n-1,1}} \underbrace{H^n \geq 0}_{E_\infty^{n,0}}$$

Let's look again at our previous example. Suppose d_2 is not zero:

So now we dont have non-trivial extension problems.

<u>Definition</u>: A spectral sequence $\{E^r_{pq}\}$ is called a first quadrant spectral sequence if $E^0_{pq} = 0$ for $p < 0$ or $q < 0$.

i.e. the non-trivial E^r_{pq} are contained in the first quadrant.

For example: the s.s. coming from $S_*(X)$ of a filtered space with $X_{-1} = \emptyset$.

Consider a chain complex $C_\bullet$ with filtration $\{F_p C_\bullet\}_{p\in\mathbb{Z}}$. Let $H_\bullet = H_\bullet C_\bullet$ its homology and

$$F_p H_\bullet = \operatorname{im}\left(H_\bullet(F_p C_\bullet) \longrightarrow H_\bullet(C_\bullet) \right)$$

Assume that there are for all $n\in\mathbb{Z}$ numbers $s(n) \leq t(n) \in \mathbb{Z}$ such that

$$F_{s(n)} C_n = 0$$

$$F_{t(n)} C_n = C_n.$$

Theorem: (i) There is a spectral sequence $E = (E^r_{pq})$ of homological type with $E^1_{pq} := H_{p+q}\left(F_pC_\bullet / F_{p-1}C_\bullet \right)$

(ii) There is a chain of submodules of E^1

$$E = E^1 \supseteq Z^1 \supseteq Z^2 \supseteq \dots \supseteq Z^\infty \supseteq B^\infty \supseteq \dots \supseteq B^2 \supseteq B^1$$

such that $E^{r+1} \cong Z^r/B^r$ $\quad r = 1, 2, \dots$

(iii) If we set $Z^\infty := \bigcap_r Z^r$, $B^\infty := \bigcup_r B^r$, and $E^\infty := Z^\infty/B^\infty$, then we have isomorphisms

$$E^\infty_{p,q} \cong F_p H_{p+q}(C_\bullet) \big/ F_{p-1} H_{p+q}(C_\bullet)$$

Proof: (I) We define an exact couple $\mathcal{C} = (E, A, i, \overset{\overset{j}{\shortparallel}}{\alpha}, \overset{\overset{k}{\shortparallel}}{\beta})$ as follows:

$$A = A^1 := \bigoplus_{p,q} H_{p+q}(F_{p-1}C_\bullet)$$

$$E = E^1 := \bigoplus_{p,q} H_{p+q}\left(F_pC_\bullet / F_{p-1}C_\bullet \right)$$

$i : A \to A$ by inclusion $F_{p-1}C_\bullet \hookrightarrow F_pC_\bullet$

$\alpha : A \to E$ by quotient $F_pC_\bullet \twoheadrightarrow F_pC_\bullet / F_{p-1}C_\bullet$

$\beta : E \to A$ as connecting homomorphism

$$H_{p+q}(F_pC_\bullet / F_{p-1}C_\bullet) \longrightarrow H_{p+q-1}(F_{p-1}C_\bullet)$$

So

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "\alpha"] \\ & E \arrow[ul, "\beta"] & \end{tikzcd}$$

The differential on E is $\partial := \alpha\beta$, via

$$H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet) \longrightarrow H_{p+q-1}(F_{p-1}C_\bullet) \longrightarrow H_{p+q-1}(F_{p-1}C_\bullet/F_{p-2})$$

(II) This leads via the derived exact couples to a bigraded spec. seq. with pages E^r and differential $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$.

Also $E^{r+1} = H(E^r, d^r) = \frac{\ker(d^r)}{\mathrm{im}(d^r)} = \frac{Z(d^r)}{B(d^r)}$

(III) <u>Note</u> E^{r+1} is a subquotient of E^r, so a subquotient of a subquotient of ... of E^1. We want to see cycles and boundaries "as" submodules of the first term E^1 we started with. We therefore ~~set~~ resolve this iterated subquotient as follows:

In the $(r-1)$th derived couple $\mathcal{C}^r = (A^r, E^r, i^r, \alpha^r, \beta^r)$, the subscript r is just an index not an iteration.

<u>But</u> in the case of i^r, we have something like an iteration, since $A^r = \mathrm{im}(i^{r-1} : A^{r-1} \longrightarrow A^{r-1})$, so

$$A^r = \mathrm{im}(\underbrace{i \circ \cdots \circ i}_{r-1 \text{times}}) = \mathrm{im}(i^{r-1})$$

Now define

$$Z^r := \beta^{-1}(\mathrm{im}(i^{r-1})) = \beta^{-1}(A^r) \subseteq E^1$$

$$B^r := \alpha(\ker(i^{r-1})) \subseteq Z^r \subseteq E^1$$

We have $Z^r \supseteq Z^{r+1}$, $B^r \subseteq B^{r+1}$ for all r

We set $Z^\infty = \bigcap_r Z^r$, $B^\infty := \bigcup_r B^r$, $E^\infty := Z^\infty/B^\infty$.

<u>Note</u> we're <u>not</u> redefining $E^{r+1} = Z^r/B^r$, since we want to prove this as part of the theorem.

So we want $E^{r+1} = \frac{\ker(\partial^r)}{\mathrm{im}(\partial^r)} = \frac{\ker(\alpha^r \circ \beta^r)}{\mathrm{im}(\alpha^r \circ \beta^r)} \cong \frac{Z^r}{B^r}$.

For $r = 1$, $Z^1 = \beta^{-1}(\mathrm{im}(i)) = \beta^{-1}(\ker(\alpha)) = \ker(\alpha\beta) = \ker(\partial)$

$B^1 = \alpha(\ker(i)) = \alpha(\mathrm{im}(\beta)) = \mathrm{im}(\alpha\beta) = \mathrm{im}(\partial)$.

Bödigheimer claims now that this proves that in general

$$E^{r+1} \cong Z^r/B^r$$

but I think that the general case is more involved...

(IV) So we have for fixed p,q

$$Z^{\infty}_{p,q} = \bigcap_r \beta^{-1}\big(\operatorname{im}(i^{r-1} : \underset{(p+q-1?)\ (p-r?)}{H_{p+q}(F_{p-r+1}C_\cdot)} \longrightarrow \underset{(p+q-1?)\ (p-1?)}{H_{p+q}(F_{p}C_\cdot)})\big)$$

By our assumption, for $n=p+q$, $H_{p+q}(F_{p-r}C_\cdot)=0$ for large r.

Hence

$$Z^{\infty}_{p,q} = \beta^{-1}\Big(\bigcap_r \operatorname{im}(i^{r-1})\Big) = \beta^{-1}(0) = \ker(\beta)$$

Likewise ~~ker(i^r~~

$$\ker(i^{r-1} : H_{p+q}(F_pC_\cdot \longrightarrow H_{p+q}(F_{p+r-1}C_\cdot))$$

becomes at some point

(V) Set $n=p+q$. Write $F_p = F_pC_{p+q}$

$$F_pZ_p = F_p \cap \ker(\partial)$$

$$F_pB_p = F_p \cap \operatorname{im}(\partial)$$

Thus $F_pZ_p/F_pB_p = F_pH$.

Observe $Z^{\infty}_{p,q} = \dfrac{Z_p + F_{p-1}}{F_{p-1}} \cong \dfrac{Z_p}{Z_p \cap F_{p-1}} =$

We <u>need</u> $E^{\infty}_{p,q} \cong \dfrac{Z_p/B_p}{Z_{p-1}/B_{p-1}}$.

Well.

Hopefully he'll do this again on Monday...

<u>Lecture 25 09-07-18</u>

Part V. Set $n=p+q$ and define $F_p = C_pC_n$

$$F_pZ = F_p \cap \ker(\partial)$$

$$F_pB = F_p \cap \operatorname{im}(\partial).$$

Then

▷ $F_pZ/F_pB = FH(C_\cdot) = \operatorname{im}(H_n(F_pC) \to H_n(C_\cdot))$

▷ $Z^{\infty}_{p,q} = (F_pZ + F_{p-1})/F_{p-1} \cong \dfrac{F_pZ}{F_pZ \cap F_{p-1}} = \dfrac{F_pZ}{F_{p-1}Z}$

We want to prove:

$$E^\infty_{p,q} \cong \frac{\mathrm{im}(H_{p+q}(F_pC_\cdot) \longrightarrow H_{p+q}(C_\cdot))}{\mathrm{im}(H_{p+q}(F_{p-1}C_\cdot) \longrightarrow H_{p+q}(C_\cdot))} = \frac{F_pZ/F_pB}{F_{p-1}Z/F_{p-1}B}$$

We have

$$\frac{F_pZ/F_pB}{F_{p-1}Z/F_{p-1}B} \cong \frac{F_pZ}{F_{p-1}Z + F_pB}$$

We <u>need</u> $$\pi: \frac{F_pZ}{F_{p-1}Z + F_pB} \longrightarrow \frac{F_pZ + F_{p-1}}{F_pB + F_{p-1}}$$

is an iso.

This is enough, since

$$\frac{F_pZ + F_{p-1}}{F_pB + F_{p-1}} \cong \frac{(F_pZ + F_{p-1})/F_{p-1}}{(F_pB + F_{p-1})/F_{p-1}} \cong \frac{Z^\infty_{p,q}}{B^\infty_{p,q}}.$$

(a) π is epi

(b) For the injectivity take $z \in F_pZ$ representing $[z] = z + (F_{p-1}Z + F_pB)$ with $\pi([z]) = 0$.

This means that $z = b + f \in F_pB + F_{p-1}$. We have

$$\partial(f) = \partial(b+f) = \partial(z) = 0$$

So $f \in F_{p-1}Z$.

Therefore $z \in F_{p-1}Z + F_pB$ hence $[z] = 0$ □

For more details we refer to

- John McCleary: User's guide to Spectral Sequences
- A. Hatcher.

<u>Examples</u> ① X CW-complex, filtered by skeleta $\ldots \subseteq X_{p-1} \subseteq X_p \subseteq \ldots$

Let $C_\cdot = S_\cdot(X)$ be the singular chain complex with induced filtration. We set

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \cong \bigoplus_{j \in J_p} \tilde{H}(S^p_j)$$

so in particular $E^1_{p,q} = 0$ if $q > 0$.

So E^1 is concentrated in $E^1_{\cdot,0}$, and d^1 is the cellular boundary operator

$$E^1_{0,0} \xleftarrow{d^1} E^1_{1,0} \xleftarrow{d^1} E^1_{2,0} \leftarrow \cdots$$

Example ② Serre's spectral sequence.

Let $F \to X \to B$ be a fibre bundle where B is a connected CW-complex.

Filter X by the "pull-back" of the skeletal filtration of B i.e.

$$X_p := \xi^{-1}(B_p)$$

This is an exhausting ascending filtration on X.

Let $C_* = \mathrm{Sing}_*(X)$, filtered via $\subseteq X_{p-1} \subseteq X_p \subseteq \ldots$

We get

$$E^1_{pq} = H_{p+q}(X_p, X_{p-1}) \cong \widehat{H}_{p+q}(X_p/X_{p-1})$$

Note: 1) Stratum $B_p - B_{p-1} \cong \coprod_{j \in J_p} e^p_j \Longleftarrow$ open p-cell.

2) $X_p \setminus X_{p-1} \cong \coprod_{j \in J_p} e^p_j \times F$

3) $X_p / X_{p-1} \cong \bigvee_j S^p_j \wedge F_+$.

So

$$E^1_{pq} = \widehat{H}_{p+q}(X_p/X_{p-1}) \cong \bigoplus_{j \in J_p} \widehat{H}_{p+q}(S^p_j \wedge F_+)$$

(suspension iso)

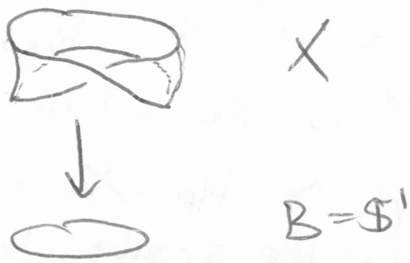
$$\cong \bigoplus_{j \in J_p} \widehat{H}_q(F_+) \cong \bigoplus_{j \in J_p} H_q(F) \cong C_p(B) \otimes H_q(F)$$

where $C_0(B)$ is the cellular chain complex of B.

What is $d^1 : E^1 \longrightarrow E^1$?

Example

$$\begin{tikzcd} F \arrow[r] & X = \text{mapping torus of } \gamma : F \to S^1 \arrow[d] \\ & B = S^1 \end{tikzcd}$$



This example shows that $d^1 : C_p(B) \otimes H_q(F) \to C_{p-1}(B) \otimes H_q(F)$ is not in general just $d^{cell} \otimes id$ given by $d^{cell} : C_p(B) \to C_{p-1}(B)$.

Example 3 (Atiyah - Hirzebruch Spectral sequence)

Let h_* be a generalized homology theory

(if h^* = gen. cohom theory, assume wedge axiom)

Let X be a CW-complex with skeletal filtration.

Instead of a filtered complex we define an exact couple, looking quite right:

$$\begin{tikzcd}
A^1 = \bigoplus_{pq} h_{p+q}(X_{p-1}) \arrow[rr, "i = \text{``inclusion''}"] & & \bigoplus_{pq} h_{p+q}(X_p) = A^1 \arrow[dl, "\alpha = j \text{ (reduction)}"] \\
& E^1 = \bigoplus_{pq} h_{p+q}(X_p, X_{p-1}) \arrow[ul, "\beta = k"', "\text{``connecting homom.''}"] &
\end{tikzcd}$$

Then $d = \alpha \circ \beta : E^1 \longrightarrow E^1$, a bigraded differential object.

We will also get a spectral sequence.

We have

$$E^1_{p,q} = h_{p+q}(X_p, X_{p-1}) \cong \tilde{h}_{p+q}(X_p / X_{p-1})$$
$$\cong \bigoplus_{j \in J_p} \tilde{h}_{p+q}(S^p_j)$$
$$\cong \bigoplus_{j \in J_p} \tilde{h}_q(S^0)$$
$$\cong C_p(X) \otimes h_q(*) \longleftarrow \text{coefficients of } h_*$$

Also $E^1_{p-1,q} = \cdots = C_{p-1}(X) \otimes h_q(*)$

Since there's no bundle involved now, the $d^1 : E^1 \to E^1$ just corresponds to

$$d^1 = d^{cell} \otimes id_{h_q(*)} : C_p(X) \otimes h_q(*) \longrightarrow C_{p-1}(X) \otimes h_q(*).$$

where d^{cell} is the cellular boundary operation.

Thus $E^2_{p,q} \cong H_p(X, h_q(*))$

is just the singular homology of X with coefficients in the coefficients of h.

The convergence of the spectral sequence is not clear, since it's half plane but not necessarily first-quadrant.

We only now $E^1_{pq} = 0$ for $p < 0$.

We do get convergence if $h_q(*) = 0$ for $q < L$.

This applies in particular to connective homology theories. For example π_i(Siegel-Γspace (X)) or $\pi_i^{stab}(X)$.

Theorem: For any generalized homology (cohomology) theory h_* and CW complex X, there is a spectral sequence

$$E^2_{pq} \cong H_p(X; h_q(*)) \quad (\Longrightarrow h_{p+q}(X))$$

If it converges, it converges to $h_{p+q}(X)$, with the filtration $F_q h_{p+q}(X) = \mathrm{im}\,(h_{p+q}(X_p) \longrightarrow h_{p+q}(X))$.

So if we know the coefficients $h_q(*)$ of a homology theory, we might be ~~apt~~ able to say something about $h_q(X)$ for arbitrary spaces.

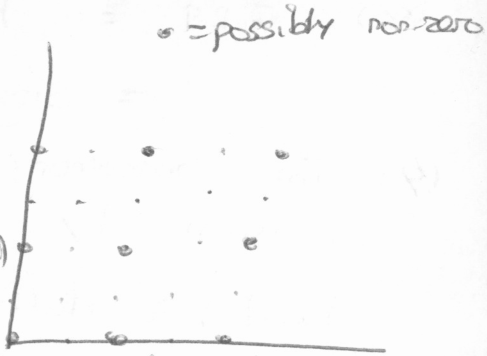
Application: $h^* =$ complex K-theory KU, with

$$KU^q(*) = \begin{cases} \mathbb{Z} & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$$

Let X be a CW-complex with only even-dim cells. eg. $X = \mathbb{C}P^n$.

Then $E_2^{pq} \cong H^p(X; KU^q(*)) = 0$ for $\cancel{p+}q \equiv 1 \bmod 2$ or $p \equiv 1 \bmod 2$.

Thus $d_2: E_2 \longrightarrow E_2$ is zero, since it either starts at or goes to something trivial.



In fact, all d_i are zero, since a diagonal $\{p+q \equiv 1 \bmod 2\}$ is always zero and d_i ~~thus~~ goes from one diagonal to the next one.

So $E_\infty^{p,q} \cong E_2^{p,q} = H_p(X; \cancel{h_q}\cancel{\oplus} KU^q(*))$

$\uparrow$ ~~Z~~ free $\mathbb{Z}$-module.

So ~~#~~ $KU^{2n}(X) \cong H^0(X;\mathbb{Z}) \oplus H^2(X;\mathbb{Z}) \oplus \dots \oplus H^n(X;\mathbb{Z})$.

Serre Spectral Sequence

Let $F \xrightarrow{i} X \xrightarrow{\xi} B$ be a fibre bundle (or more generally a fibration). Assume that B is a CW-complex, with B connected.

We have a bundle of graded groups over B, namely a local system of coefficients:

A family of functors

$$\underset{\text{fund. groupoid}}{\underset{\shortparallel}{\Pi(B)}} \xrightarrow{\mathcal{H} = (\mathcal{H}_q)} Ab$$

$$b \longmapsto H_q(\xi^{-1}(b))$$

fibre transport.

$$(w : b \rightsquigarrow b') \longmapsto T_w : H_q(\xi^{-1}(b)) \xrightarrow{\cong} H_q(\xi^{-1}(b'))$$

<u>Theorem</u>: There is a first quadrant spectral sequence $E = (E^r_{pq})$ converging to $H_*(X)$, with

(1) $E^1_{p,q} = \mathcal{C}_p(B) \otimes H_q(F) \Longrightarrow H_{p+q}(X)$

(2) The filtration $H_*(X)$ is as before by the filtration $X_p := \xi^{-1}(B_p)$, B_p = p-skeleton.

(3) $d^1 = d^{\text{cellular}}$ "twisted" with the local coefficient system $\mathcal{H}_q = H_q(F)$

in other words the group $H_q(F)$ with an ation of $\pi_1(B)$.

(We thus have no problems if $\pi_1(B) = 0$ or more generally if B is simple, i.e. $\pi_1(B)$ acts trivially.)

So $E^2_{p,q} \cong H_p(B ; \mathcal{H}_q(F))$

= singular homology of B with twisted coeff. syst.

(4) The homomorphism $i_*: H_n(F) \longrightarrow H_n(X)$ is given by

$$H_n(F) \cong H_0(B, H_n(F)) = E^2_{0,n} \twoheadrightarrow E^\infty_{0,n} = F_0 H_n(X) \hookrightarrow H_n(X)$$

(5) The homomorphism $\xi_* : H_n(X) \longrightarrow H_n(B)$ is given by

$$H_n(X) = F_n H_n(X) \twoheadrightarrow E^\infty_{n,0} \hookrightarrow E^2_{n,0} = H_n(B, H_0(F)) = H_n(B).$$

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<u>Remarks</u>: 1) If B is simply connected, $\pi_1(\pi) = 0$, then the local coefficient systems $\mathcal{H}_q$ are constant, so the E^2-page is just

"constant" module

$$E^2_{p,q} \cong H_p(B ; H_q(F))$$

By UCT

$$0 \to \bigoplus_{p+q=n} H_p(B) \otimes H_q(F) \longrightarrow \bigoplus_{p+q=n} H_p(B ; H_q(F)) \longrightarrow Tor(- , -) \twoheadrightarrow 0.$$

2) If B is simply connected and we work with <u>field</u> coefficients, then

$$H_p(B;H_q(F)) \cong H_p(B)\otimes H_q(F).$$

(We write $H_*(-) = H_*(-;A)$, where A is an abelian group)

3) If $\pi_1(B)$ acts trivially on all $H_q(F)$, we call the fibration $F \longrightarrow X \xrightarrow{\xi} B$ <u>simple</u>.
In this case, the local coefficient systems $\mathcal{H}_q(F)$ are <u>constant</u>, so again we have

$$E^2_{pq} = H_p(B;\mathcal{H}_q(F)) = H_p(B;H_q(F))$$

<u>Example</u>: Consider an $\mathbb{S}^1$-bundle over $\mathbb{S}^2$.

$$F=\mathbb{S}^1 \longrightarrow X \longrightarrow B=\mathbb{S}^2.$$

We take field coefficients so can apply 2):

$$E^2_{p,q} \cong H_p(\mathbb{S}^2)\otimes H_q(\mathbb{S}^1).$$

In fact, we can generalize 2):

2') If $H_*(B;\mathbb{Z})$ or $H_*(F;\mathbb{Z})$ is <u>torsion free</u>, the $\operatorname{Tor}(-,-)$-term also vanishes, so also

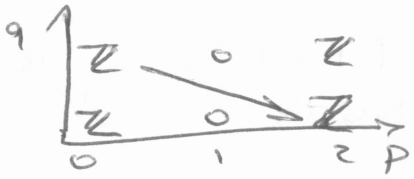
$$H_p(B;H_q(F;\mathbb{Z})) \cong H_p(B;\mathbb{Z})\otimes H_q(F;\mathbb{Z}).$$

Back to the example. We will now consider the <u>cohomology</u> spectral sequence with E_2-page

$$E_2^{pq} \cong H^p(\mathbb{S}^2;\mathbb{Z})\otimes H^q(\mathbb{S}^1;\mathbb{Z})$$

which thus looks like

E_2



There is only one possible non-trivial differential

$$\begin{tikzcd} d_2: E_2^{0,1} \arrow[r] \arrow[d, phantom, "\shortparallel"] & E_2^{2,0} \arrow[d, phantom, "\shortparallel"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Let's say that d_2 is multiplication by ℓ.

For $\ell = 0$ we have $E_2 = E_3 = \cdots = E_\infty$, so we see by looking at the diagonals that

$$H^*(X) = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{array}$$

Fact: Eulerclass $e(\xi) = \ell\omega_2$ is a multiple of a generator ω_2 of $H^2(\mathbb{S}^2)$

Fact: If $\ell = 0$, $X_0 \cong \mathbb{S}^1 \times \mathbb{S}^2$.

For $\ell = \pm 1$, the $E_3 = E_\infty$-page looks like

$$\begin{array}{c|ccc} q & & & \\ & 0 & 0 & \mathbb{Z} \\ & \mathbb{Z} & 0 & 0 \\ \hline & & & p \end{array}$$

so

$$H^*(X) = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \mathbb{Z} & 0 & 0 & \mathbb{Z} \end{array}$$

Fact: $X_{\pm 1} \cong \mathbb{S}^3$

If $\xi = \eta : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is the Hopf map, then $e(\eta)$ is a generator in $H^2(\mathbb{S}^2)$.

For $\ell > 1$, the $E_3 = E_\infty$-page looks like

$$\begin{array}{|ccc} 0 & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z}/\ell \\ \hline \end{array}$$

Again we don't have any extension problems and we can read off that

$$H^*(X) = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \mathbb{Z} & 0 & \mathbb{Z}/\ell & \mathbb{Z} \end{array}$$

Subexample if $\ell = 2$, **Fact**: $X_2 = \mathbb{S}T(\mathbb{S}^2)$, the sphere bundle in the tangent bundle of $\mathbb{S}^2$.

So $e(\xi_2) = 2 \cdot \omega_2$.

(To the reader: if you don't get the link between $d_2 : E_2^{0,1} \xrightarrow{\cdot \ell} E_2^{2,0}$ and $e(\xi) = \ell \cdot \omega_2$, don't worry, probably it's not that important. I also don't really get it...) That's very nice of you! ☺

Example Consider the unit sphere bundle of a closed orientable surface Σ_g of genus g.

$$S^1 \longrightarrow ST(\Sigma_g) \xrightarrow{\xi} \Sigma_g = B$$

Now B is not simply-connected, but it turns out that ξ is simple by orientability of Σ_g.

(The fibre-transports can only be translations, but no reflections by orientability, so on (co) homology, the fibre transports

$$\begin{tikzcd} H_q(\xi^{-1}(b)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_q(\xi^{-1}(b')) \arrow[d, phantom, "\cong" rotate=90] \\ H_q(F) \arrow[r] & H_q(F) \end{tikzcd}$$

give identity on (co)homology.)

The E_2-page thus looks like:

$$E_2 \quad \begin{tikzcd} \mathbb{Z} \arrow[rrd] & \mathbb{Z}^{2g} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} \end{tikzcd}$$

Again there is only one possible non-trivial differential

$$\begin{tikzcd} d_2 : E_2^{0,1} \arrow[r, "\ell"] \arrow[d, phantom, "\parallel" rotate=90] & E_2^{2,0} \arrow[d, phantom, "\parallel" rotate=90] \\ \mathbb{Z} \arrow[r, "\cdot \ell"] & \mathbb{Z} \end{tikzcd}$$

Fact: The ℓ appearing here is the Euler characteristic $\chi = 2-2g$. So d_2 is multiplying with χ.

So we have for the $E_3 = E_\infty$-page

$$\underset{\chi = 0}{\begin{tikzcd} \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} \end{tikzcd}} \qquad \underset{\chi \neq 0}{\begin{tikzcd} 0 & \mathbb{Z}^{2g} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z}/\chi \end{tikzcd}}$$

We could be worried because we might have non-trivial extension problems now. But for the extension problems we go "left upwards", so we get a free group ($\mathbb{Z}$ or $\mathbb{Z}^{2g}$) on the right in the s.e.s. and the s.e.s. splits.

So we get for the cohomology of X:

$$\chi = 0 \qquad\qquad\qquad\qquad \chi \neq 0$$

$$H^*(\mathbb{S}T(\Sigma_g)) \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \mathbb{Z} & \mathbb{Z}^{2g+1} & \mathbb{Z}^{2g+1} & \mathbb{Z} \end{array} \qquad\qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z}^{2g} \oplus \mathbb{Z}/\chi & \mathbb{Z} \end{array}$$

Multiplicative Spectral Sequences

We call a s.s. of cohomological type <u>multiplicative</u> if

by (2)

(1) Each E_r is a bigraded (differential) algebra over some ring R, i.e. there is a multiplication

$$E_r^{p,q} \otimes E_r^{s,t} \longrightarrow E_r^{p+s,q+t}$$

$$a \otimes b \longmapsto a \cdot b$$

(2) The differentials $d_r : E_r \longrightarrow E_r$ are <u>derivations</u>

$$d_r(a \cdot b) = d_r(a) \cdot b + (-1)^{|a|} a \cdot d_r(b)$$

(with $|a| = p+q$ if $a \in E_r^{p,q}$.)

(3) The multiplication on E_{r+1} matches with the induced multiplication on $H^*(E_r, d_r)$, i.e. the isomorphism

$$H^*(E_r, d_r) \cong E_{r+1}$$

must be multiplicative (iso of algebras)

<u>Examples</u> ① Multiplicative exact couple:

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j=\alpha"] \\ & E \arrow[ul, "k=\beta"] & \end{tikzcd}$$

- A and E are bigraded algebras over R.
- i and j are algebra morphisms.

Bödigheimer forgot the details ☺, but remembers that $d = \alpha \circ \beta$ should be a derivation

$\Longrightarrow$ The derived exact couple stays multiplicative and we get a multiplicative spectral sequence.

Example ② Let C° be a filtered multiplicative cochain complex. i.e. we have multiplication

$$C^s \otimes C^t \longrightarrow C^{s+t}$$

with

$$F^pC^s \otimes F^qC^t \longrightarrow F^{p+q}C^{s+t}.$$

(For example $C^\circ = S^\circ(X)$ filtered.)

From example 2 We get the following:
Consider a fibration $F \to X \to B$ with B a connected CW complex B simply connected.
We get a filtered cochain complex $C^\circ = S^\circ(X)$.
$\rightsquigarrow$ We get a multiplicative Serre spectral sequence

$$E_2^{pq} \cong H^p(B, H^q(F; R)) \Longrightarrow H^{p+q}(X; R)$$

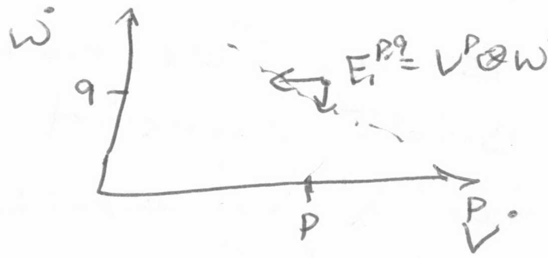
$$E_\infty^{pq} = \text{bigraded algebra on } R.$$

Example: Let V° and W° be DGA's over a field $\mathbb{F}$ ("differential graded algebra") with differentials ∂^V and ∂^W.
Then we get a double complex

$$?(a\otimes b) = |a|$$

$$E_1 := V^\circ \otimes W^\circ$$

with differential $\partial^V \otimes id_{W^\circ} + (-1)^? id_{V^\circ} \otimes \partial^W$



This d is a derivation (Leibniz rule) for the induced multiplication on E_1.

Assume $V^p = 0$ for $p < 0$, $W^q = 0$ for $q < 0$.
We get two filtrations of E, giving by "filtration on columns" or "filtration by rows".

For example

$$F_p(V^\circ \otimes W^\circ) = \Big(\bigoplus_{k=0}^{p} V^k\Big) \otimes W^\circ$$

(so this is the "filtration by columns.)
The filtration quotients are now

$$F_p(-)\big/ F_{p-1}(-) = V^p \otimes W^\circ = E_1^{p,\bullet}$$

The induced differential from $d = \partial^V \otimes id_{W^\circ} + \underbrace{(-1)^{?}\, id_{V^\circ} \otimes \partial^W}_{d_1}$
is $d_1 : E_1 \longrightarrow E_1$ given by

$$V^p \otimes W^\circ \xrightarrow{id_{V^\circ} \otimes \partial^W} V^p \otimes W^\circ.$$

Now, $\partial^V \otimes id_{W^\circ}$ is a chain map from

$$E_1^{p,\circ} \longrightarrow E_1^{p-1,\circ}$$

The map

$$\begin{tikzcd} d_2 : H(E_1) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H(E_1) \arrow[d, phantom, "\shortparallel"] \\ E_2 & E_2 \end{tikzcd}$$

is just the induced map $H(\partial^V \otimes id_{W^\circ})$.

Theorem: Let V°, W° be bounded DGAs over $\mathbb{F}$.
Let's assume we have

$$V^\circ = H^\circ(B, \mathbb{F})$$
$$W^\circ = H^\circ(F, \mathbb{F})$$

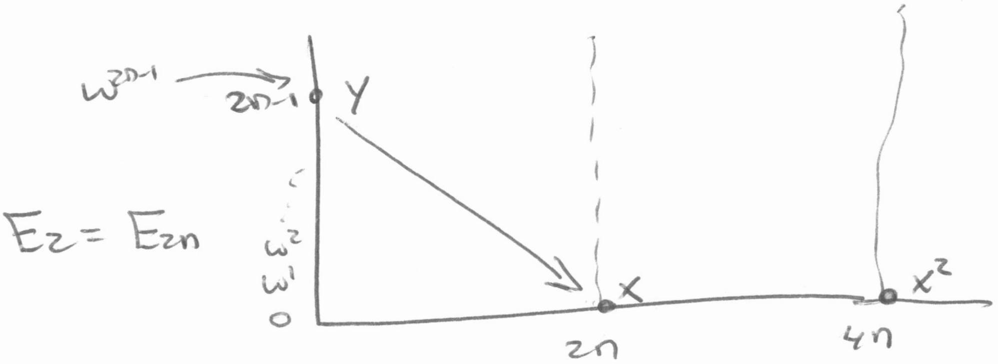
where $F \longrightarrow X \longrightarrow B$ is a fiber bundle with B simply connected and B and F connected (so $V^0 = W^0 = 0$). Assume that X is acyclic (for example if $X \simeq *$ is contractible.)
Then (see next page)

(i) If $\mathbb{F} = \mathbb{Q}$ and $V^\circ = \mathbb{Q}[x]$ with $|x| = 2u$, ~~then~~ then $W^\circ = \Lambda_{\mathbb{Q}}[y]$, $|y| = 2n-1$

(ii) If $V^\circ = \Lambda_{\mathbb{Q}}[x]$, $|x| = 2n+1$, then $W^\circ = \mathbb{Q}[y]$, $|y| = 2n$.

(Remark: when it's not over a field we have more subtle things like divided power algebras.)

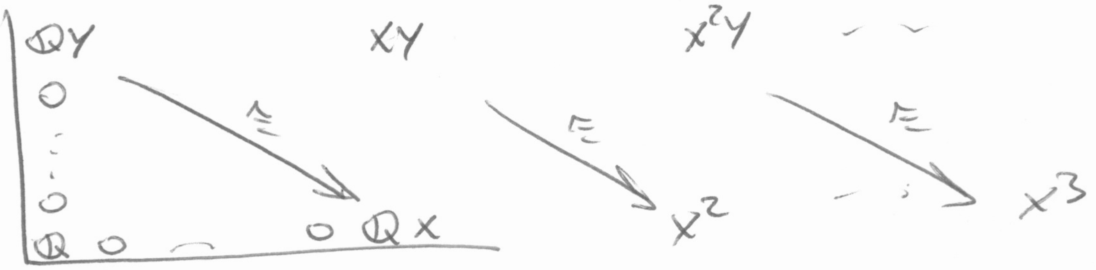
Proof (i) The E_2-page looks like



Since E_∞ should be zero (as $H^*(X) = 0$ by assumption) we see that $W^i = 0$ for $1 \le \overset{i \text{ (not } y)}{y} \le 2n-2$ (since there is no diff. that could kill them.)

But W^{2n-1} must have a $y \in W^{2n-1}$, since x must be killed by something, and d_{2n} is the only charge.

So we get



By the Leibniz rule, $d_{2n}(x^k y) = x^{k+1}$, so all d_{2n} are ~~differentials~~ isos.

This implies that there cannot be anything above line $q = 2n$, since it cannot be killed.

So in particular $y^2 = 0$ and $W^\circ \cong \Lambda_{\mathbb{Q}}[y]$.

Lecture 07-16-18

Consider the fibration $F \to X \to B$ over $R = \mathbb{Q}$, where F is connected, B simply-connected, X acyclic.

Then

H^*F exterior $\iff$ H^*B polynomial

H^*F polynomial $\iff$ H^*B exterior.

Example: $\Omega Y \longrightarrow PY \longrightarrow Y$

Example:

$$\begin{tikzcd} K(\mathbb{Z},1) \arrow[r] \arrow[d, phantom, "\shortparallel"] & PY \arrow[r] & Y = K(\mathbb{Z},2) \arrow[d, phantom, "\shortparallel"] \\ S^1 & & \mathbb{CP}^\infty \end{tikzcd}$$

Spectral sequence of a double complex

A double complex $D = (D_{pq})$ is a doubly indexed family of R-modules with two differentials:

vertical: $\partial' : D_{pq} \longrightarrow D_{p,q-1}$

horizontal: $\partial'' : D_{pq} \longrightarrow D_{p-1,q}$.

with $\partial'\partial'' = \partial''\partial'$

$$\begin{tikzcd} \cdot \arrow[d, "\partial'"'] & \cdot \arrow[l, "\partial''"'] \arrow[d, "\partial'"] \\ \cdot & \cdot \arrow[l, "\partial''"] \end{tikzcd}$$

In other words, we can regard this as a chain complex of chain complex, but in two ways!

Example: Let $(A_\cdot, \partial_A)$, $(B_\cdot, \partial_B)$ be chain complexes of R-modules. Set

$$D_{\cdot\cdot} = A_\cdot \otimes_R B_\cdot, \quad \partial' = id \otimes \partial_B, \quad \partial'' = \partial_A \otimes id_B.$$

We have the total complex

$$T_\cdot = \mathrm{Tot}(D_{\cdot\cdot}) \quad (\text{i.e. } T_n = \bigoplus_{p+q=n} D_{pq})$$

$\left(d(z) = \partial'(z) + (-1)^p \partial''(z) \text{ for } z \in D_{pq} \right)$

There are two filtrations:

$$F'_p T_n = \bigoplus_{\substack{r+s=n \\ r \le p}} D_{rs}$$

$$F''_q T_n = \bigoplus_{\substack{r+s=n \\ s \le q}} D_{rs}.$$

We get two spectral sequences:

$${}'E^1_{pq} := H_{p+q}(F'_p T_\bullet / F'_{p-1} T_\bullet ; d) \cong H_q(D_{p\bullet} ; \partial')$$

$${}''E^1_{pq} := H_{p+q}(F''_p T_\bullet / F''_{p-1} T_\bullet ; d) \cong H_q(D_{\bullet p} ; \partial'').$$

For the d_1-differentials

- ▷ $d_1' : {}'E^1_{pq} \longrightarrow {}'E^1_{p-1,q}$ induced by ∂'' (as chain map)
- ▷ $d_1'' : {}''E^1_{pq} \longrightarrow {}''E^1_{p-1,q}$ induced by ∂' (as chain map)

<u>Theorem</u>: For a ~~bo~~ double complex $D_{\bullet\bullet}$, there are two spectral seq. with

$${}'E^2_{pq} \cong H_p(H_q(D_{p\bullet} ; \partial') ; \partial''_*)$$

$${}''E^2_{pq} \cong H_p(H_q(D_{\bullet q} ; \partial'') ; \partial'_*)$$

If D is bounded on the right

$(\exists n \ \forall p,q \quad p > n \Rightarrow D_{pq} = 0)$

or bounded below

$(\exists n \ \forall p,q \quad q < n \Rightarrow D_{pq} = 0)$

then both s. sequences converge, and both to $H_{p+q}(\mathrm{Tot}(D), d)$.

<u>Example</u>: Kunneth Spectral sequence.

R a commutative ring, $A^\bullet$ a DGM over R, $\partial_A : A_n \to A_{n+1}$,

$B^\bullet$ a DGM over R, $\partial_B : B^n \to B^{n+1}$.

We get a spectral sequence

$$E_2^{pq} = \bigoplus_{s+t=q} \mathrm{Tor}^p_R(H^s(A^\bullet), H^t(B^\bullet)) \Rightarrow H^{p+q}(A^\bullet \otimes B^\bullet)$$

<u>The spectral sequence of a covering</u>

Let $F \longrightarrow X \xrightarrow{\xi} B$ be a covering, X and B both connected. Assume ξ is regular (= normal = Galois = the deck transformations act transitively on each fiber F = image of $\pi_1(X)$ in $\pi_1(B)$ is normal.)

Letting $\pi := \pi_1(B)$, this implies that

$$B \cong X/\pi.$$

Note that $\pi = \pi_1(B)$ also acts on the singular chain complex $S_\cdot(X)$ of X.

We have
$$S_\cdot(X) \otimes_{\mathbb{Z}[\pi]} \underset{\uparrow}{\mathbb{Z}} \cong S_\cdot(X)/_\pi \cong S_\cdot(B)$$
trivial π-action.

(3) Recall the ~~def~~ definition of $H_n(G, M)$ group homology with coefficients in a π-module M.

This is the n-th left derived functor of
$$M \longmapsto M/_{\langle m - gm \mid m \in M, g \in \pi \rangle}$$
(the "co-invariants").

So ~~[illegible]~~
$$H_n(G, M) := \mathrm{Tor}^n_{\mathbb{Z}[\pi]}(\mathbb{Z}, M).$$

To compute this, we choose a projective $\mathbb{Z}[\pi]$ resolution $P_\cdot \longrightarrow \mathbb{Z}$ of $\mathbb{Z}$. We get
$$\mathrm{Tor}^n_{\mathbb{Z}[\pi]}(\mathbb{Z}, M) \cong H_n(P_\cdot \otimes_{\mathbb{Z}[\pi]} M)$$

A possible resolution is the cellular chain complex of the Milnor construction EG, which is the <u>bar resolution</u>.
$$P_\cdot = \mathrm{Bar}_\cdot(\pi).$$

<u>Theorem (Cartan-Leray Spectral Sequence)</u>

There is a first quadrant spectral sequence
$$E^2_{p,q} \cong H_p(\pi ; \underbrace{H_q(X; A)}_{H_q(M_q)})) \Longrightarrow H_{p+q}(B; A),$$
where A is any abelian group.

<u>Proof</u>: M_q has π-action induced by the Deck-transformations acting on X.

Let
$$O_{p,q} := (\overbrace{A \otimes_{\mathbb{Z}} S_q(X)}^{M_q}) \otimes_{\mathbb{Z}[\pi]} \mathrm{Bar}_p(\pi)$$
be a double complex. We get two ~~sp~~ spectral sequences. For the first we have

$$'E^1_{pq} = H_q((\underbrace{\mathbb{A} \otimes_{\mathbb{Z}} S_p(X)}_{M_p}) \otimes_{\mathbb{Z}[\pi]} Bar_\cdot(\pi))$$

$$\cong H_q(\pi : \mathbb{A} \otimes_{\mathbb{Z}} S_p(X))$$

$$\cong H_q(\pi : \mathbb{M}_p)$$

So $'E^1_{p,0} = H_0(\pi : \mathbb{M}_p) = \mathbb{M}_p = \mathbb{A} \otimes_{\mathbb{Z}} S_p(X)$

$'E^1_{p,q} = H_q(\pi : \underset{\substack{\uparrow \\ \text{free } \pi\text{-module}}}{\mathbb{M}_p}) = 0$ for $q > 0$.

Thus

$'E^2_{p,0} = H_p(\mathbb{A} \otimes_{\mathbb{Z}} S_0(B)) = H_p(B : \mathbb{A})$

$'E^2_{pq} = \cdots = 0$ for $q > 0$

Thus $'E$ collapses and

$$H_n(Tot(D)) = H_n(B : \mathbb{A})$$

From the <u>second</u> filtration, we get

$$''E^1_{pq} = H_q(D_{p,\cdot}) \cong \neq$$

$$= H_q((\mathbb{A} \otimes_{\mathbb{Z}} S_\cdot(X)) \otimes_{\mathbb{Z}[\pi]} Bar_p(\pi))$$

Since for this homology p is fixed and $Bar_p(\pi)$ is a free $\mathbb{Z}[\pi]$-module, the tensor functor $- \otimes_{\mathbb{Z}[\pi]} Bar_p(\pi)$ is exact and thus preserves homology:

So

$$''E^1_{pq} \cong H_q(X : \mathbb{A}) \otimes_{\mathbb{Z}[\pi]} Bar_p(\pi).$$

Thus

$$''E^2_{pq} \cong H_p(H_q(X : \mathbb{A}) \otimes Bar_\cdot(\pi))$$

$$\cong H_p(\pi : H_q(X : \mathbb{A})).$$

Finite type theorem

Let $F \longrightarrow X \xrightarrow{\xi} B$ be a fibration, B a CW-complex, connected, ξ simple.

Let $R = \mathbb{Z}$ (or any Noetherian ring)

If any two of F, X, B have f.g. cohomology, then so has the third.

Proof: Case 1: Assume B and F are of finite type.

With the filtration on $H^*(X)$, we have

$$\bigoplus_p F_p H^n / F_{p-1} H^n \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

Since we have finitely many filtration modules

$$0 \subseteq F_0 H^n \subseteq F_1 H^n \subseteq \dots \subseteq F_n H^n = H^n,$$

it is enough to show that the E_∞^{pq} are f.g.

But $E_\infty^{pq} \cong E_r^{pq}$ for $r >> p+q$, and E_r^{pq} is an iterated subquotient of $E_2^{pq} = H^p(B, H^q(E))$. And this is a f.g. module, using the UCT.

Case 2: Assume X and B are of finite type.

Do induction on n. We have

$$H^0(F) = E_\infty^{0,0} = H^0(E)$$

so $H^n(F)$ is f.g. for $n=0$.

Let $n \geq 1$ be the smallest degree with $H^n(F)$ not f.g.

Then $E_2^{0,n} = H^n(F)$ is not f.g.

But $E_2^{2,n-1} = H^2(B, H^{n-1}(F))$ is f.g. by UCT, and the choice of n, so

$$E_3^{0,n} = \ker(d_2 : E_2^{0,n} \longrightarrow E_2^{2,n-1})$$

is not f.g.

Again as $E_3^{3,n-1}$ is a subquotient of the f.g. $H^3(B, H^{n-2}(F))$, so also f.g., so

$$E_4^{0,n} = \ker(d_3 : E_3^{0,n} \longrightarrow E_3^{3,n-2})$$ is <u>not f.g.</u>

And so on. We get that $E_\infty^{0,n}$ is not f.g.
But it is a subquotient of $H^n(X)$, which is f.g.!
Contradiction ↯

Case 3 If X and F are of finite type, we can do something similar.

Serre Classes

Let S = Serre class of R-modules (so closed under submodules quotients and extensions.)
S is called a Serre ring: if in addition we have

$$A, B \in S \Longrightarrow \mathrm{Tor}_R^i(A,B) \in S \quad \text{for all } i \geq 0.$$

In the exercise, we have seen the concepts of

- ▷ S-monomorphism
- ▷ S-epimorphism
- ▷ S-~~epi~~isomorphism (i.e. S-mono + S-epi)
- ▷ S-acyclic space if

$$H_k(X; \mathbb{Z}) \xrightarrow{\cong S} H_k(pt; \mathbb{Z})$$

is an S-iso for all k, in other words, if
$\tilde{H}_k(X; \mathbb{Z}) \in S$.

Theorem: Let S be a Serre ring and let $F \xrightarrow{\iota} X \xrightarrow{\xi} B$ be a fibration, with F connected, B connected + simply connected. If any two of the F, X, B are S-acyclic, then so is the third.

Corollary 1: S = Serre ring, Y simply connected.

$$Y \text{ S-acyclic} \Longleftrightarrow \Omega Y \text{ S-acyclic}.$$

Corollary 2: S = Serre ring, $A \in S$, and assume that $K(A,1)$ is S-acyclic. Then this is true for all $K(A,n)$ an $n \geq 1$, i.e. $K(A,n)$ is acyclic.

Theorem (Hurewicz mod S)

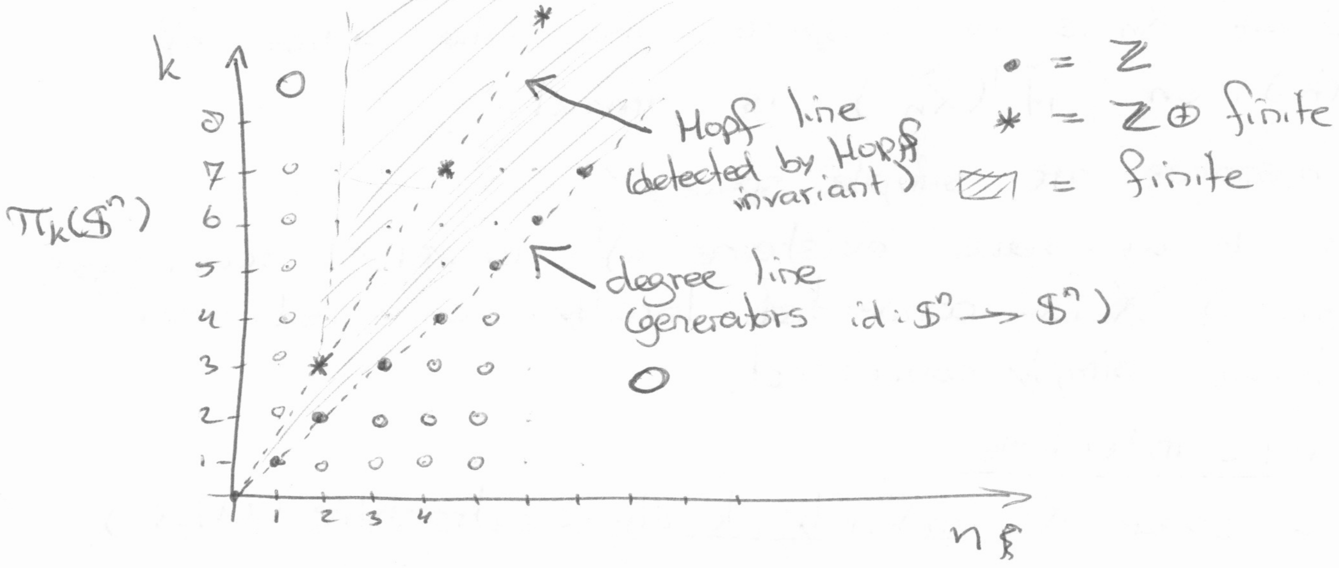
Let S be a Serre ring and assume that for all $A \in S$, $K(A,1)$ is S-acyclic.
Let X be connected and simply-connected; assume that for some $n \geq 2$, $\pi_k(X) \in S$ for $0 < k < n$.
Then (i) $H_k(X;\mathbb{Z}) \in S$ for $0 < k < n$
(ii) $hur : \pi_n(X) \longrightarrow H_n(X;\mathbb{Z})$ is an S-iso. ∎

Lecture 18-07-18

The theorem of Serre on the finiteness of homotopy groups of spheres. (1951)

Theorem I: $\pi_k(S^{2n-1})$ is finite for $k > 2n-1$

Theorem II: $\pi_k(S^{2n})$ is finite for $k > 2n$, $n > 1$, except $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus$ finite.



Corollary: The stable homotopy groups $\pi_m^{stab}(S^0)$ of spheres is finite for $m > 0$ and is $\cong \mathbb{Z}$ for $m = 0$.

Plan for today:
- ▷ 3 parts for proof of Theorem I, several lemmata
- ▷ Sketch for Theorem II
- ▷ Convert homotopy questions into homology questions.

Proof of Theorem I

Part I Let X be a connected, simply connected locally finite CW-complex. (These assumptions might be used later again without repeating.)

Define the following sequence of spaces:

$$\left.\begin{array}{l} X_0 := X. \\ X_0 \longleftarrow T_1 := \widetilde{X_0} = \text{universal covering of } X_0 \\ X_1 := \cancel{\Phi}\,\Omega T_1 \\ X_1 \longleftarrow T_2 := \widetilde{X_1} \\ X_2 := \Omega T_2 \\ \vdots \\ X_{n-1} \longleftarrow T_n := \widetilde{X_{n-1}} \\ X_n := \Omega T_n \end{array}\right\} \text{All spaces here are h-spaces.}$$

(Remark: using covering theory, one can show that a covering of an h-space is an h-space.)

Lemma 1: If X_n is an h-space, then the action of $\pi_1(X_n)$ on $\tilde{H}^q(\widehat{X_n})$ is trivial.

So "h-spaces are simple spaces".

Question: do we have existence of universal coverings?

This exists if X is connected, locally connected and semi-locally simply-connected.

Set-theoretic interlude

We call a space X uniformly locally contractible (ULC), if there is a neighborhood U of the diagonal in $X \times X$ and a homotopy $G : U \times [0,1]$ with

$$G(x,x,t) = x \quad \forall x \in X \ \forall t$$

$$\left.\begin{array}{l} G(x,y,0) = x \\ G(x,y,1) = y \end{array}\right\} \forall (x,y) \in U.$$

Think about $G(x,y,-)$ as a geodesic from differential geometry.

Examples for ULC's:

1) Absolute neighborhood retracts
2) Loc. finite CW-complex
3) X finite polyhedron
4) Triangulated manifold.

Lemma 2: Assume that X is a connected ULC-space. Then

(i) The universal covering $\hat{X}$ exists.

$$\left.\begin{array}{l}\text{(ii) } \hat{X} \text{ is ULC}\\ \text{(iii) } \Omega X \text{ is ULC}\end{array}\right\} \text{ so } \Omega\tilde{X} \text{ is ULC}$$

(iv) "Homotopy fibers are again ULC": (?)

If X, Y ULC, $f: X \to Y$ a map then $hfib(f)$ is ULC.

Remark Part (iv) is not stated in the reference for this lecture, McCleary's book about spectral sequences, but Bödigheimer thinks there is an argument missing that needs (iv).

So back to our construction of our sequence of spaces.

Since X is a CW-complex of finite type, it is ULC and thus all spaces X_i and T_i exist and are ULC by lemma 2.

We know

- ▷ $T_1, T_2, \dots$ are 1-connected
- ▷ $\pi_k(T_n) \cong \pi_k(X_{n-1})$ for $k \geq 2$
- ▷ $\pi_k(X_n) \cong \pi_k(\Omega T_n) \cong \pi_{k+1}(T_n) \cong \pi_{k+1}(X_{n-1})$

$$\cong \dots \cong \pi_{k+n}(X_0) = \pi_{k+n}(X)$$

Lemma 3: If Y has finite type, then ΩY has finite type.

Proof: As in the Finite Type Theorem, using the path-space fibration

$$\Omega Y \longrightarrow PY \longrightarrow Y$$

Corollary: All X_i are of finite type.

Then: $H_1(X_i) \cong \pi_1(X_i) \cong \pi_{i+1}(X)$ ~~[illegible]~~ is finitely generated.

Lemma 4: Let Y be connected, simply-connected and ULC. If $H_i(Y)$ is finitely generated for all $i \geq 0$, then the $\pi_i(Y)$ are fin. gen. for all $i \geq 0$.

Proof: Use the Cartan Leray spectral sequence for the universal covering $\hat{Y} \longrightarrow Y$:

$$E^2_{pq} = H_p(\pi_1(Y) ; H_q(\hat{Y} ; A)) \Rightarrow H_{p+q}(Y ; A)$$

where $H_q(\hat{Y} ; A)$ is an $\pi_1(Y)$-module and A an abelian group.

(You're confused why we're talking about $\hat{Y}$ if $\pi_1(Y) = 0$? Yes, we here too ...) (Probably this assumption was a mistake.)

We now proceed by induction on the sequence X_i as defined above:

▷ For $X = X_0$, we have $T_1 = \hat{X} = X$, since X is 1-connected. Thus $X_1 = \Omega T_1 = \Omega X$ is of finite type by lemma 3.

▷ By induction, assume that X_{n-1} is of finite type. Consider $T_n = \tilde{X}_{n-1}$. Since X_{n-1} is an H-space, the action of $\pi_1(X_{n-1})$ on $H_*(T_n)$ is trivial (lemma 1). Thus the E^2-page of the CL-spec. sequence of $T_n \longrightarrow X_{n-1}$ simplifies to

$$\begin{aligned} E^2_{pq} &= H_p(\pi_1(X_{n-1}) ; H_q(T_n)) \\ &\overset{UCT}{\cong} H_p(\pi_1(X_{n-1})) \otimes_{\mathbb{Z}} H_q(T_n) \\ &\quad \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(\pi_1(X_{n-1})), H_q(T_n)) \end{aligned}$$

By induction, $\pi_1(X_{n-1}) \cong H_1(X_{n-1})$ (recall X_{n-1} is an h-space) is fin. gen. Also $H_q(T_n)$ is fin. gen. (Bödigheimer wasn't sure why McCleary stated this and suspected we need some more arguments. Tobias Fleckenstein proposed to use that $\pi_q(T_n) \cong \pi_q(X_{n-1})$ is fin. gen. and then use the Hurewicz theorem rel. $S = \{\text{fin. gen. groups}\}$.

So $H_{p-1}(\pi_1(X_{n-1}))$ is fin. gen. (as the Bar resolution has fin. gen. abelian groups.) and this imply that the Tor - terms in E^2 are fin. gen.

So E^2 is fin. gen. and thus the convergence groups $H_{p+q}(X_{n-1})$ are finitely generated. by induction.

As in the proof of Case II or Case III in F.T. Theorem, we can argue that $H_q(T_n)$ is fin. gen. so T_n is of finite type.

By lemma 3. $X_n = \Omega T_n$ is of finite type. $\square$

Lemma 5: A fin. ~~ab~~ gen. abelian group A is finite if and only if $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ $\square$

Lemma (Rational Hurewicz Theorem)

Assume that X is connected 1-connected ULC, and of finite type, $n \geq 2$.

If $H_i(X; \mathbb{Q}) = 0$ for $0 < i < n$ then

(i) $\pi_i(X) \otimes \mathbb{Q} = 0$ for $0 < i < n$, and

(ii) $\pi_n(X) \otimes \mathbb{Q} \cong H_n(X; \mathbb{Q})$.

Proof: We use again our resolution $X = X_0, X_1, X_2, \ldots$

Claims:

① $H_i(X_j; \mathbb{Q}) = 0$ for $i > 0$, $i+j < n$

② $H_i(X_j, \mathbb{Q}) \cong H_n(X; \mathbb{Q})$ if $i+j = n$.

We prove the claims by induction on j.

For $j = 0$, the claims are clear by assumption, since $X_0 = X$.

Let $j \geq 1$ and assume ① and ② hold for all lower j.

For $j \geq 2$, X_{j-1} is a loop space. Thus $\pi_1(X_{j-1})$ acts trivially on $H_*(T_j; \mathbb{Q})$. The same also holds for $j = 1$ as $X_{1-1} = X_0 = X$ is simply-connected.

So by the Universal Coefficient Theorem:

$$E^2_{pq} \cong H_p(\pi_1(X_{j-1})) \otimes H_q(T_J : \mathbb{Q}) \underbrace{\oplus \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{p-1}(\pi_1(X_{j-1})), \underbrace{H_q(T_J : \mathbb{Q})}_{\mathbb{Q}\text{-module}}\big)}_{0}$$

$$\cong H_p(\pi_1(X_{j-1})) \otimes H_q(T_J : \mathbb{Q}).$$

In the FTth, we looked at the exact sequence

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \longrightarrow E^2_{0,1} \longrightarrow \underbrace{H_1(X_{j-1} : \mathbb{Q})}_{0} \to E^\infty_{1,0} \to 0$$

By induction, we have that $H_1(X_{j-1} : \mathbb{Q}) = 0$ for $j<n$.

This implies $0 = E^\infty_{1,0} = E^2_{1,0} = \underbrace{H_1(\pi_1(X_{j-1}))}_{= \cancel{H_1} \pi_1(X_{j-1}),\ \text{since } X_{j-1} \text{ is h-space}} \otimes_{\mathbb{Z}} \underbrace{H_0(T_J, \mathbb{Q})}_{= \mathbb{Q},\ \text{since } T_J \text{ connected}}$

Since $\pi_1(X_{j-1})$ is abelian we get $H_1(X_{j-1}) = \pi_1(X_{j-1})$.

So Alltogether we get that

$$\pi_1(X_{j-1}) \otimes \mathbb{Q} = 0$$

Since it is fin. gen., it must be finite by lemma 5.

Again with introspection onto the bar resolution we see that

$H_p(\pi_1(X_{j-1}))$ is finite for all $p>0$.

Thus when tensored ~~on~~ with any rational vector space

$H_q(T_j, \mathbb{Q})$, we get

$$E^2_{pq} = 0 \quad \text{for } p>0,\ q \geq 0.$$

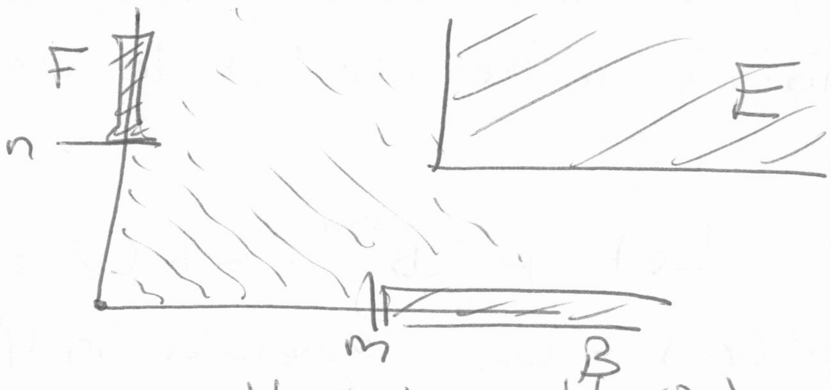
Thus the E^2-page consists only of the leftmost column $E^2_{0,q}$. So $E^\infty_{0,q} = E^2_{0q}$, $E^\infty_{p,q} = 0$ for $p>0$.

Thus $H_i(T_J : \mathbb{Q}) = H_i(X_{j-1} : \mathbb{Q})$ for $i>0$.

Now we can finish: for $i < 2(n-j+1) - 2$, we have

$$H_i(X_j ; \mathbb{Q}) \cong H_i(\Omega T_j ; \mathbb{Q})$$
$$\cong H_{i+1}(T_j ; \mathbb{Q})$$

By Serre's exact sequence not treated in the lecture)



$$H_t(F) \to H_t(E) \to H_t(B) \to H_{t-1}(F) \to H_{t-1}(E) \to \_\_ \qquad \text{for } t \leq m+n$$

So $H_i(X_j ; \mathbb{Q}) \cong H_{i+1}(T_j ; \mathbb{Q}) \cong H_{i+1}(X_{j-1} ; \mathbb{Q})$.

By the induction hypothesis, $H_i(X_j ; \mathbb{Q}) = 0$ for $i > 0$, $i+j < n$, and

$$H_{n-1}(X_j ; \mathbb{Q}) \cong H_n(X ; \mathbb{Q}) . \qquad \square$$

This proves claims ① and ②.

Now let's finish the proof of the lemma.

We have $H_1(X_j) \cong \pi_1(X_j) \cong \pi_{j+1}(X)$

▷ For $j < n-1$,
$$\begin{array}{c} H_1(X_j ; \mathbb{Q}) = 0 \\ \parallel \\ H_1(X_j) \otimes \mathbb{Q} \\ \parallel \\ \pi_{j+1}(X) \otimes \mathbb{Q} . \end{array}$$

▷ For $j = n-1$, $\pi_n(X) \otimes \mathbb{Q} \cong \pi_1(X_{n-1}) \otimes \mathbb{Q}$

$\cong H_1(X_{n-1}) \otimes \mathbb{Q} \cong$ ~~$H_1(X_{n-1} ; \otimes \mathbb{Q})$~~ $H_1(X_{n-1} ; \mathbb{Q}) \cong H_n(X ; \mathbb{Q})$.

This finishes the proof of the lemma.

$\boxed{\text{Part 2}}$

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) \cong \begin{cases} \Lambda_{\mathbb{Q}}(x_n) & n \text{ odd} \\ \mathbb{Q}[x_n] & n \text{ even} \end{cases}$$

$\boxed{\text{Part 3}}$ (We go quick since we're running out of time. It is 10:03 in the very last lecture of the semester)

Put $\underline{\underline{X = \mathbb{S}^{2n-1}}}$; Let $p : \mathbb{S}^{2n-1} \longrightarrow K(\mathbb{Z}, 2n-1)$

classify via $p^*(\tau_n) = \omega_{n-1} = \text{generator in } \underset{\mathbb{Z}}{\underset{\cong}{H^{2n-1}(\mathbb{S}^{2n-1};\mathbb{Z})}}$

We call $F_{2n-1} := \mathrm{hfib}(p)$.

We have

$$F_{2n-1} \longrightarrow \mathbb{S}^{2n-1} \longrightarrow K(\mathbb{Z}, 2n-1).$$

▷ $\pi_i(F_{2n-1}) = 0$ for $i \leq 2n-1$

▷ $\pi_i(F_{2n-1}) = \pi_i(\mathbb{S}^{2n-1})$ for $i > 2n-1$

▷ $\pi_{2n-1}(F_{2n-1}) = 0$

We will show: $\pi_i(F_{2n-1}) \otimes \mathbb{Q} = 0$ for $i > 2n-1$

Bödigheimer: "I have an examination in 10 minutes. I cannot be late. What should I do...? It is only a few more pages...

Well, I'll just be late..."

"Mr. Kramhold can you call #some name? Can you tell him that I'll be late. You can say I had a bicycle accident or someting".

XD

Consider

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & F_{2n-1} \arrow[d] \\
S^{2n-1} \arrow[r, equal] \arrow[d, "="] & S^{2n-1} \arrow[d, "p"] \\
S^{2n-1} \arrow[r, "p"] & K(\mathbb{Z}, 2n-1)
\end{tikzcd}$$

This gives

$$E \xleftarrow{p^*} \overline{E}$$

$$H^*(S^{2n+1} : \mathbb{Q}) \otimes = E_2^{*,0} \xleftarrow{p^*} \overline{E}_2^{*,0} = H^*(K : \mathbb{Q}),$$

is an isomorphism

At E^∞-page, we have an isomorphism of bigraded modules. because ι_{2n+1} is in the bottom the only possible ~~charts~~ element is dim $2n-1$ ~~vanishes~~ which may survive.

By ZCTh, we have

$$H^*(F_{2n-1} ; \mathbb{Q}) = 0$$

Lemma, we get $\pi_i(F_{2n-1}) \otimes \mathbb{Q}) = 0$. for $i > 2n-1$

We need

F_{2n-1} is uCC.

But F_{2n-1} is of fin. type since S^{2n-1} and K are. □