Talk 11: Iwasawa theory

§1 Kato's Euler system

Notation: fix $k \geq 2$ weight, $N \geq 1$ level,

$f = \sum_{n\geq 1} a_n q^n \in S_k(X_1(N)) \otimes \mathbb{C}$ normalised newform

$F := \mathbb{Q}(a_n : n \geq 1)$ coefficient field (finite extn of $\mathbb{Q}$)

$\lambda | p$ a place of F

$r \in \mathbb{Z}$

$T := V_{\mathcal{O}_\lambda}(f)(k-r)$ ← Tate twist

cohomology on $Y_1(N)(\mathbb{C})$, namely

$V_{k,A}(Y_1(N)) := H^1(Y_1(N), \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1) \otimes_{\mathbb{Z}} A)$ where $\mathcal{H}^1 = R^1\lambda_* \mathbb{Z}$, $\lambda : E \to Y_1(N)$

Recall: $V_F(f) := V_{k,F}(Y_1(N)) / \langle T(n) \otimes 1 - 1 \otimes a_n : n \geq 1 \rangle_F$

$V_{F_\lambda}(f) := V_F(f) \otimes_F F_\lambda$

$V_{\mathcal{O}_\lambda}(f) :=$ $\mathcal{O}_\lambda$-submodule gen'd by the image of $V_{k,\mathbb{Z}_p}(Y_1(N))$

Fact: (8.3) $V_{\mathcal{O}_\lambda}(f)$ is a free $\mathcal{O}_\lambda$-module of rank 2, because $V_F(f)$ is an F-space of dim 2.

$G_\mathbb{Q} \curvearrowright V_{\mathbb{Q}_p}(Y_1(N))$ descends to an F_λ-linear action $G_\mathbb{Q} \curvearrowright V_{F_\lambda}(f)$ resp. an $\mathcal{O}_\lambda$-linear action $G_\mathbb{Q} \curvearrowright V_{\mathcal{O}_\lambda}(f)$.

Fact: this action is unramified at almost all primes (why?)

Fix $1 \leq j \leq k-1$ and $c, d \in \mathbb{Z} - \{0\}$

Let ξ be either a symbol $a(A)$, i.e. a tuple $(a, A) \in \mathbb{Z} \times \mathbb{Z}$ such that $(c, 6pA) = (d, 6pN) = 1$, or $\xi \in SL_2(\mathbb{Z})$, in which case we assume $(cd, 6pN) = 1$.

Let
$$\Sigma := \begin{cases} \mathrm{prime}(cdp\, AN) & \text{if } \xi = a(A) \\ \mathrm{prime}(cdp \cdot N) & \text{if } \xi \in SL_2(\mathbb{Z}) \end{cases}$$

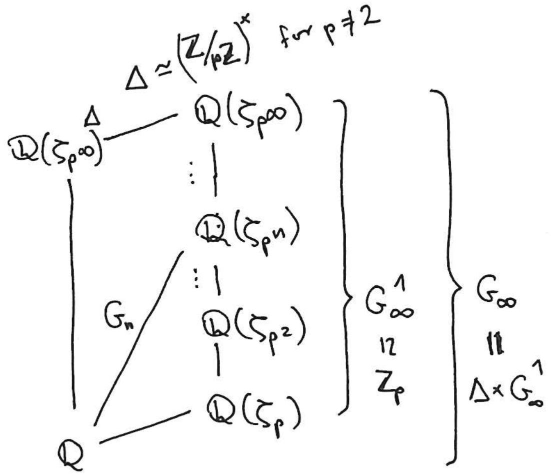
$\Xi := \{ m \geq 1 : \ell \in \Sigma - \{p\} \Rightarrow \ell \nmid m \}$ → indexing set for the ES

$$z_m := \begin{cases} {}_{c,d}z^{(p)}_m(f, r, j, \xi, \mathrm{prime}(mA)) & \text{if } \xi = a(A) \\ {}_{c,d}z^{(p)}_m(f, r, j, \xi, \mathrm{prime}(mN)) & \text{if } \xi \in SL_2(\mathbb{Z}) \end{cases}$$
$\in H^1(\mathbb{Z}[\zeta_m, \frac{1}{p}], T)$ for $m \in \Xi$

This is an Euler system for (T, F_λ, Σ) by Zhenghang's talk.

We will use the well-known Euler system machinery to gain information about the cohomology of T along the cyclotomic tower.

§2 Iwasawa cohomology



Let T be a fin gen $\mathbb{Z}_p$-module with a cts $G_\mathbb{Q} \curvearrowright T$ unramified almost everywhere

Let V be a fin dim $\mathbb{Q}_p$-v.s. with cts $G_\mathbb{Q} \curvearrowright V$ unram almost everywhere s.t. T is a $G_\mathbb{Q}$-stable $\mathbb{Z}_p$-lattice.

E.g. $T = V_{\mathcal{O}_\lambda}(f)$, $V = V_{F_\lambda}(f)$

We have the following Iwasawa-étale cohomology groups: recall $H^q(R,T) := \varprojlim_m H^q(R, T/p^m)$.

$$\mathbb{H}^q(T) := \varprojlim_n H^q_{ét}(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T) \qquad \mathbb{H}^q(V) := \mathbb{H}^q(T) \otimes \mathbb{Q}$$

$$\mathbb{H}^q_{loc}(T) := \varprojlim_n H^q_{ét}(\mathbb{Q}_p(\zeta_{p^n}), T) \qquad \mathbb{H}^q_{loc}(V) := \mathbb{H}^q_{loc}(T) \otimes \mathbb{Q}$$

Note: $\mathbb{H}^q(T) = 0$ for $q \neq 1,2$, and fin gen over $\mathbb{Z}_p[\![G_\infty]\!]$ for $q = 1,2$.

Leopoldt's conjecture: $\mathbb{H}^2(T)$ should be torsion over $\mathbb{Z}_p[\![G_\infty]\!]$.

3

<u>Euler system machinery</u>: <u>Theorem.</u> (Rubin et al.) Let $\Lambda := \mathcal{O}_\lambda[\![G_\infty]\!]$, and $Z := \langle (z_{p^n})_n : n \geq 0 \rangle_\Lambda \leq \mathbb{H}^1(T)$ submod, and $J := (h(Z) : h \in \mathrm{Hom}_\Lambda(\mathbb{H}^1(T), \Lambda)) \subseteq \Lambda$ ideal, and $\mathbb{H}^2(T)_0 := \ker(\mathbb{H}^2(T) \to \mathbb{H}^2_{loc}(T))$.

Assume that:
i) $\forall \mathfrak{q} \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(\mathfrak{q}) = 0$: $Z_\mathfrak{q} \neq 0$

T is free of rank 2 over $\mathcal{O}_\lambda$ $\longrightarrow$ ii) $\mathrm{rank}_{\mathcal{O}_\lambda}(T^+) = \mathrm{rank}_{\mathcal{O}_\lambda}(T^-) = 1$

Deligne $\longrightarrow$ iii) $\exists w \in \mathbb{Z}$ $\forall \ell \notin \Sigma$: all eigenvalues of Fr_ℓ^{-1} on $T \otimes_{\mathcal{O}_\lambda} \overline{F}_\lambda$ as an $\overline{F}_\lambda$-space ~~have~~ algebraic, and all cx conj have $|\cdot| = \ell^{w/2}$

Ribet $\longrightarrow$ iv) $T \otimes_{\mathcal{O}_\lambda} F_\lambda$ is an irreducible $G_\mathbb{Q}$-rep over F_λ

v) $\exists \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$: $\dim_{F_\lambda}\left(\ker(T \otimes_{\mathcal{O}_\lambda} F_\lambda \xrightarrow{1-\sigma} T \otimes_{\mathcal{O}_\lambda} F_\lambda)\right) = 1$.

Then:
1) $\mathbb{H}^2(T)$ is torsion over Λ.
2) If $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(\mathfrak{p}) = 1$, $p \notin \mathfrak{p}$, then $\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbb{H}^2(T)_{0,\mathfrak{p}}) \leq \mathrm{length}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/J_\mathfrak{p})$.
3) If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$ such that $\mathrm{coker}(T \xrightarrow{1-\sigma} T)$ is free over $\mathcal{O}_\lambda$ of rank 1, and if $T \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda/\mathfrak{m}_\lambda$ is an irreducible $G_\mathbb{Q}$-rep over $\mathcal{O}_\lambda/\mathfrak{m}_\lambda$, and if $p \neq 2$, then $\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbb{H}^2(T)_{0,\mathfrak{p}}) \leq \mathrm{length}_{\Lambda_\mathfrak{p}}(\Lambda_\mathfrak{p}/J_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(\mathfrak{p}) = 1$.

(So the existence of such a σ removes the $p \notin \mathfrak{p}$ condition.)

The theorem as stated here with conditions i – v) is the general form. For our specific ES (z_m), we need to check that these conditions actually hold.

To discuss condition v), we introduce the following:

<u>Def.</u> f has CM if $\exists K/\mathbb{Q}$ imag quadratic and $\exists \psi : \longrightarrow \mathbb{C}^\times$ Hecke char. (i.e. cts hom from the idèle class group) such that $L(f,s) = L(\psi, s)$, where $L(\psi, s) := \prod_v (1 - \psi(v) N(v)^{-s})^{-1}$, v ranges over finite unram primes.

<u>Theorem</u> (Ribet generalising Serre) If f has no CM, then for all finite places λ of F, $\exists T$ a $G_\mathbb{Q}$-stable O_λ-lattice of $V_{F_\lambda}(f)$ and an O_λ-basis of T such that the image of the action $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty})) \hookrightarrow G_\mathbb{Q} \to GL_2(O_\lambda)$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$ wrt this basis.

It follows that v) holds whenever f has no CM.
However, when f does have CM, Kato claims that v) fails, and the proof is substantially different, using Rubin's work on IMC for quadr. imag.
We <u>restrict our attention to the CM case.</u> $(\to$ §15 Kato$)$

§5 Verifying i)

It remains to verify condition i). (This will be independent of the non-CM assumption.)

We have:

$$V_{k,\mathbb{Q}}(Y(L)) \xrightarrow{\text{trace}} V_{k,\mathbb{Q}}(Y_1(N)) \xrightarrow{\text{quotient}} V_F(f)$$

$$\alpha^* \delta_{L,L}(k,j) \xmapsto{\text{def}} \delta_{1,N}(k,j,\alpha) \xmapsto{\text{def}} \delta(f,j,\alpha) \quad \text{for } \alpha \in SL_2(\mathbb{Z})$$

where $\delta_{L,L}(k,j)$ is defined as the image of a geodesic b/w cusps, and $N | L$.

Thm. (Ash–Stevens): $\langle \alpha^* \delta_{L,L}(k,j) \ : \ \alpha \in GL_2(\mathbb{Z}/L\mathbb{Z}),\ 1 \le j \le k-1 \rangle_\mathbb{Z} = V_{k,\mathbb{Z}}(Y(L))$, $L \geq 3$.

$\Rightarrow \exists \alpha \in SL_2(\mathbb{Z})\ \exists j :\ \delta(f,j,\alpha)^\pm \neq 0$ where $\pm = \begin{cases} + & \text{if } -\sigma_{-1} \text{ is } 1 \text{ in } \Lambda_q \\ - & \text{if } -\sigma_{-1} \text{ is } -1 \text{ in } \Lambda_q \end{cases}$

Let $c, d \in \mathbb{Z}$ s.t. $(cd, 6p) = 1$, $c \equiv d \equiv 1 \mod N$, $c^2 \neq 1 \neq d^2$.

<u>Thm. 6.6</u> (Kato)
$$\sum_{b \in (\mathbb{Z}/m)^\times} \chi(b)\, per_f\left(\sigma_b(z_m(f,r,r',\xi,S))\right)^\pm = L_S(f^*, \chi, r)(2\pi i)^{k-r-1}\, \delta(f,r',\xi)^\pm$$

$$\sum_{b \in (\mathbb{Z}/m)^\times} \chi(b)\, per_f\left(\sigma_b({}_{c,d}z_m(f,r,r',\xi,S))\right)^\pm = L_S(f^*,\chi,r)(2\pi i)^{k-r-1}\, \gamma^\pm$$

with γ given explicitly

$f^* = \sum_{n \geq 1} \overline{a_n} q^n$ dual modular form, $per_f : M_k(X_1(N)) \to V_{k,\mathbb{C}}(Y)$

Here $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ is a group homomorphism, r, r' satisfy some numerology, $L_S(f,\chi,s) = \sum_{(n,S)=1} \frac{a_n \chi(n)}{n^s} \Big(= \prod_{\ell \notin S} \frac{1}{1 - a_\ell \chi(\ell)\ell^{-s} + \varepsilon(\ell)\chi^2(\ell)\ell^{k-1-2s}}$

where $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, $\begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix}^* f = \varepsilon(d) f$. for $d \in (\mathbb{Z}/N)^\times$ and $\varepsilon(\ell) = 0$ for $\ell | N$. $\Big)$

Theorem 9.7 (Kato) $\exp_f^* : H^1\big(\mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p, V_{F_\lambda}(f)(k-r)\big) \longrightarrow S(f) \otimes_F F_\lambda \otimes \mathbb{Q}(\zeta_m)$

img of ${}_{c,d}z_m^{(p)}(f, r, r', \xi, S) \longmapsto {}_{c,d}z_m(f, r, r', \xi, S) \in S(f) \otimes \mathbb{Q}(\zeta_m$

Theorem (Jacquet – Shalika) $L(f,s) \neq 0$ for $\mathrm{Re}(s) \geq \frac{k+1}{2}$

Theorem. (Rohrlich) For $2 | k$: $\left\{ \chi \in \bigcup_{\substack{m \geq 1 \\ \mathrm{prime}(m) \subseteq S}} \mathrm{Hom}\big((\mathbb{Z}/m)^\times, \mathbb{C}^\times\big) : L_S(f, \chi, \tfrac{k}{2}) = 0 \right\}$ is finite.

Putting things together: consider the ~~sequence~~ composite map

$$) \quad H^1(V_{\mathcal{O}_\lambda}(f)) \simeq H^1(V_{\mathcal{O}_\lambda}(f)(1)) \longrightarrow H^1(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(1)) \xrightarrow{\exp^*} S(f) \otimes_F F_\lambda \xrightarrow{\chi} S(f) \otimes_F \bar{F}_\lambda$$

$$a \longmapsto \sum_{\sigma \in G_n} \sigma(a) \otimes \chi(\sigma)$$

where $\chi : G_\infty \to \bar{F}_\lambda^\times$ is of finite order, $\chi(-1) = \pm 1$, and $n \geq 0$ s.t. χ factors through G_n.

Kato + Jacquet – Shalika + Rohrlich $\Longrightarrow$ for almost all such χ, the image of $\big({}_{c,d}z_{p^n}^{(p)}(f, k, j, \alpha, \mathrm{prime}(pN))\big)_{n \geq 1} \in Z$ is not zero.

The map $(*)$ factors through $H^1(V_{F_\lambda}(f)) / \mathfrak{p} H^1(V_{F_\lambda}(f))$ for $\mathfrak{p} = \ker(\Lambda \to \bar{F}_\lambda)$

$G_\infty \ni \sigma \mapsto \chi_{cyc}(\sigma)^{-1}\chi(\sigma)^{-1}$

Therefore there are infinitely many primes $\mathfrak{p}$ of height 1 s.t. $\mathfrak{p} \supset \mathfrak{q}$ and s.t. the image of Z in $H^1(V_{F_\lambda}(f)) / \mathfrak{p}$ is not zero. Hence $Z_{\mathfrak{q}} \neq 0$.

ow we will use the ES machinery to prove some important theorems.

~~[illegible]~~

§ 6 Main theorem 1 $\quad \Lambda := \mathcal{O}_\lambda[[G_\infty]]$

Thm. For any $G_{\mathbb{Q}}$-stable $\mathcal{O}_\lambda$-lattice T of V_{F_λ}:

1) $H^2(T)$ is torsion over Λ
2) $\mathbb{H}^1(T)$ is torsion free over Λ, and $\mathbb{H}^1(T) \otimes \mathbb{Q} = \mathbb{H}^1(V_{F_\lambda}(f))$ is free of rank 1 over $\Lambda \otimes \mathbb{Q}$
3) If $p \neq 2$ and if $T/m_\lambda T$ is irreducible as a 2-dim $G_{\mathbb{Q}}$-rep ~~of~~ over $\mathcal{O}_\lambda/m_\lambda$, then $\mathbb{H}^1(T)$ is free of rank 1 over Λ.

The proof is not that interesting, and will be skipped.

Proof. 1 Is a consequence of ESM 1, reducing to the case $T = V_{\mathcal{O}_\lambda}(f)$, which can be done by Ribet's result in § 4.

2 The first assertion implies the rest. Indeed, assume torsion freeness of $\mathbb{H}^1(T)$. Tate and Perrin-Riou showed (in higher generality) that

$$\underbrace{\dim \mathbb{H}^1(T)_q}_{1} - \underbrace{\dim \mathbb{H}^2(T)_q}_{0 \text{ by } 1)} = \underbrace{\operatorname{rank}_{\mathbb{Z}_p}(T^-)}_{1 \text{ by ii) of ESM}}. \quad \forall q \in \operatorname{Spec}(\Lambda),\ \operatorname{ht}(q) = 0.$$

$1 \Leftarrow$

So it remains to show torsion freeness.

Let $x \in \Lambda$ be a non-zero-divisor. Wts $x : \mathbb{H}^1(T) \longrightarrow \mathbb{H}^1(T)$ injective.

$$\Lambda = \mathcal{O}_\lambda[[G_\infty]] \xrightarrow{\text{norm}} \mathcal{O}_\lambda[[G'_\infty]] \simeq \mathcal{O}_\lambda[[X]]$$
$$x \longmapsto p^n y \quad \text{for some } n \geq 1,\ y \text{ non-zero-divisor and } \Lambda/y\Lambda \text{ p-torsion-free.}$$

omit

Since $x \mid p^n y$, this reduces the proof to the two cases

•) $x = p$ and •) $\Lambda/x\Lambda$ is p-torsion-free.

If $x = p$: $\quad 0 \longrightarrow j_* T \xrightarrow{p} j_* T \longrightarrow j_*(T/pT) \longrightarrow 0$ exact sequence

where $j : \operatorname{Spec}(\mathbb{Z}[\frac{1}{p}]) - \operatorname{prime}(N) \hookrightarrow \operatorname{Spec}(\mathbb{Z}[\frac{1}{p}])$

By this sequence, p-multiplication is injective $\iff \varprojlim_n H^0(\mathbb{Q}(\zeta_{p^n}), T/p) = 0$

For $n \gg 0$: $H^0(\mathbb{Q}(\zeta_{p^n}), T/p) = H^0(\mathbb{Q}(\zeta_{p^\infty}), T/p)$. For such n, the norm map $H^0(\mathbb{Q}(\zeta_{p^{n+1}}), T/p) \longrightarrow H^0(\mathbb{Q}(\zeta_{p^n}), T/p)$ is multiplication by $[\mathbb{Q}(\zeta_{p^{n+1}}) : \mathbb{Q}(\zeta_{p^n})] = p$, hence 0, and thus so is the proj limit $\varprojlim_n H^0(\mathbb{Q}(\zeta_{p^n}), T/p) = 0$.

If $\Lambda/x\Lambda$ is p-torsion-free: we have for all $q \geq 0$:

$$H^q(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T) \simeq H^q(\mathbb{Z}[\tfrac{1}{p}], T \otimes \mathcal{O}_\lambda[G_n]) \qquad \text{(why?)}$$

where $G_\mathbb{Q} \curvearrowright T \otimes \mathcal{O}_\lambda[G_n]$ via $G_\mathbb{Q} \to G_\mathbb{Q} \otimes G_n$, $\sigma \mapsto \sigma \otimes \sigma_n^{-1}$ where $G_\mathbb{Q} \twoheadrightarrow G_n$, $\sigma \mapsto \sigma_n$ is the natural projection.

$$\Rightarrow \mathbb{H}^q(T) \overset{\text{def}}{=} \varprojlim_n H^q(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T) \simeq \varprojlim_n H^q(\mathbb{Z}[\tfrac{1}{p}], T \otimes \mathcal{O}_\lambda[G_n])$$

It follows that we have an exact sequence $H^0(\mathbb{Q}, T \otimes \Lambda/x\Lambda) \longrightarrow \mathbb{H}^1(T) \xrightarrow{x} \mathbb{H}^1(T)$.

Wts the leftmost term is 0.

$H^0(\mathbb{Q}, T \otimes \Lambda/x\Lambda) = \mathrm{Hom}_{\mathcal{O}_\lambda[G_\mathbb{Q}]}(\mathrm{Hom}_{\mathcal{O}_\lambda}(\Lambda/x\Lambda, \mathcal{O}_\lambda), T)$ with $G_\mathbb{Q} \curvearrowright \Lambda/x\Lambda$ via $\sigma \mapsto \sigma_\infty^{-1}$. We show that the right hand side is trivial.

Indeed, on the one hand, $G_\mathbb{Q} \curvearrowright \mathrm{Hom}(\Lambda/x\Lambda, \mathcal{O}_\lambda)$ factors through $G_\mathbb{Q}^{ab}$, but on the other hand, $V_{F_\lambda}(f)$ is irreducible of dimension 2 as a representation of $G_\mathbb{Q}$ and hence not abelian. So there is no non-trivial homomorphism here. So 2) is proved.

3) Assume $p \neq 2$, $T/m_\lambda T$ irreducible. Wts $\mathbb{H}^1(T)$ is free of rk 1 over Λ.

Recall that Λ is a 2-dimensional regular local ring.

Let $(x,y) \subseteq \Lambda$ be a maximal ideal. (For $\mathbb{Z}_p[[X]]$, one could pick $(x,y) = (p,T)$.) It's sufficient to show that x, y form a regular sequence, i.e. $x : \mathbb{H}^1(T) \longrightarrow \mathbb{H}^1(T)$ and $y : \mathbb{H}^1(T)/x \longrightarrow \mathbb{H}^1(T)/x$ are injective.

Indeed, this shows that the action of the maximal ideal is faithful, and all elements outside of (x,y) are units, hence also acting faithfully.

We have already seen that x-multiplication is injective. We have:

$$\mathbb{H}^1(T)/x\,\mathbb{H}^1(T) = \Big(\varprojlim_n H^1(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T)\Big)\Big/(x)$$

limits commute with limits

$$= \varprojlim_n \Big(H^1(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T)\Big/(x)\Big) \subseteq H^1(\mathbb{Z}[\tfrac{1}{p}], T \otimes_{\mathcal{O}_\lambda} \Lambda/x\Lambda)$$

We also have $0 \to j_*(T \otimes_{\mathcal{O}_\lambda} \Lambda/x\Lambda) \xrightarrow{y} j_*(T \otimes_{\mathcal{O}_\lambda} \Lambda/x\Lambda) \to j_*(T \otimes_{\mathcal{O}_\lambda} \Lambda/(x,y)\Lambda) \to 0$,

Long exact cohomology sequence: sts $H^0(\mathbb{Z}[\tfrac{1}{p}], T \otimes \Lambda/(x,y)) = 0$.

We have $\Lambda/(x,y) \simeq (\mathcal{O}_\lambda/m_\lambda)(r)$ because $G_\mathbb{Q}$ acts via $G_\mathbb{Q} \twoheadrightarrow G_\infty$.

Since T/m_λ is irred: $H^0(G_\mathbb{Q}, (T/m_\lambda T)(r)) = 0$,

so $H^0(\mathbb{Z}[\tfrac{1}{p}], T \otimes \Lambda/(x,y)) = 0$ as well. $\square$

§ 7 Definition of the elements $z_\gamma^{(p)}$

We will define elements $z_\gamma^{(p)} \in \mathbb{H}^1(V_{F_\lambda}(f)) \underset{\Lambda}{\otimes} Q(\Lambda)$. These will later be shown to interpolate the relevant L-function of f^*, and they will generate a submodule important in studying ~~lengths~~ the lengths of $\mathbb{H}^1$, $\mathbb{H}^2$.

Fix the following: $\alpha_1, \alpha_2 \in SL_2(\mathbb{Z})$

$j_1, j_2 \in \mathbb{Z}$ $\qquad 1 \leq j_i \leq k-1$

such that $\delta(f, j_1, \alpha_1)^+ \neq 0 \neq \delta(f, j_2, \alpha_2)^-$

Once again recall that these δs are in $V_{F_\lambda}(f)$ and come from geodesics between cusps. The point of fixing these elements is that we have an F_λ-basis for the plus resp minus parts $\xrightarrow{F_\lambda\text{-linearity}}$ for the whole $V_{F_\lambda}(f)$.

So if $\gamma \in V_{F_\lambda}(f)$, then $\exists\, b_1, b_2 \in F_\lambda$: $\quad \gamma = b_1\, \delta(f, j_1, \alpha_1)^+ + b_2\, \delta(f, j_2, \alpha_2)^-$.

Fix $c, d \in \mathbb{Z}$: $(cd, 6p) = 1$, $c \equiv d \equiv 1 \mod N$, $c^2 \neq 1 \neq d^2$.

<u>Def.</u>

$$z_\gamma^{(p)} := \left(\mu(c,d,j_1)^{-1} \cdot b_1 \cdot \left({}_{c,d}z_{p^n}^{(p)}(f, k, j_1, \alpha_1, \mathrm{prime}(pN))\right)_{n \geq 1}\right)^- +$$

$$+ \left(\mu(c,d,j_2)^{-1} \cdot b_2 \cdot \left({}_{c,d}z_{p^n}^{(p)}(f, k, j_2, \alpha_2, \mathrm{prime}(pN))\right)_{n \geq 1}\right)^+$$

where $\mu(c,d,j) := (c^2 - c^{k+1-j} \cdot \sigma_c)(d^2 - d^{j+1}\sigma_d) \prod_{\substack{\ell \mid N \\ \ell \neq p}} (1 - \bar{a}_\ell \ell^{-k} \sigma_\ell^{-1}) \in \Lambda$

We have: $\mu(c,d,j)$ non-zero-divisor $\Rightarrow \mu(c,d,j)^{-1} \in Q(\Lambda)$ total ring of quotients

$\Rightarrow z_\gamma^{(p)} \in \mathbb{H}^1(V_{F_\lambda}(f)) \underset{\Lambda}{\otimes} Q(\Lambda)$. $\qquad$ Later: $z_\gamma^{(p)} \in \mathbb{H}^1(V_{F_\lambda}(f))$.

Here $\sigma_c: \zeta_m \mapsto \zeta_m^c$.

<u>Properties.</u> ① If $\gamma \in V_F(f) \subset V_{F_\lambda}(f)$, $\chi: G_\infty \to \overline{\mathbb{Q}}^\times$ char of finite order s.t. $c^2 - c^{k-j}\bar{\chi}(c) \neq 0 \neq d^2 - d^j \bar{\chi}(d)$ for $j = j_1, j_2$, and for all $p \neq \ell \mid N$, we have $1 - \bar{a}_\ell \ell^{1-k} \chi(\ell) \neq 0$, then

$$\begin{tikzcd} \mathbb{H}^1(V_{F_\lambda}(f)) \otimes_\Lambda \Lambda[\mu^{-1}] \arrow[r, "(*)"] & S(f) \otimes_F \overline{F}_\lambda & V_F(f) \otimes \mathbb{C} \\ & \cup & \\ z_\gamma^{(p)} \arrow[rr, mapsto, bend right] & S(f) \otimes_F \overline{\mathbb{Q}} \arrow[uur, "\mathrm{per}_f"] & L_S(f^*, \chi, k-1) \cdot \gamma^\pm \end{tikzcd}$$

where $\mu := \mu(c,d,j_1) \cdot \mu(c,d,j_2)$.

② $\mathfrak{z}_\gamma^{(p)}$ is independent of the choices of $\alpha_1, \alpha_2, j_1, j_2, c, d$.

③ $\mathfrak{z}_{\iota(\gamma)}^{(p)} = -\sigma_{-1}(\mathfrak{z}_\gamma^{(p)})$ where $\iota: V_{F_\lambda}(f) \to V_{F_\lambda}(f)$ is the action of cx conjugation.

④ The elements $\left({}_{c,d}z_{p^n}^{(p)}(f,k,j,a(A),\mathrm{prime}(pA))\right)_{n\geq 1}$ and $\left({}_{c,d}z_{p^n}^{(p)}(f,k,j,\alpha,\mathrm{prime}(pN))\right)_{n\geq 1}$ can be expressed in terms of elements $\mathfrak{z}_\gamma^{(p)}$:

give only the formula for α

$$\left({}_{c,d}z_{p^n}^{(p)}(f,k,j,a(A),\mathrm{prime}(pA))\right)_{n\geq 1} = \prod_{p\neq \ell | A} \left(1-\bar{a}_\ell \ell^{-k}\sigma_\ell^{-1} + \bar{\varepsilon}(\ell)\ell^{-k-1}\sigma_\ell^{-2}\right)\cdot$$
$$\cdot\left(c^2d^2\mathfrak{z}_{\gamma_1}^{(p)} - c^{k+1-j}d^2\mathfrak{z}_{\gamma_2}^{(p)} - c^2d^{j+1}\varepsilon(d)\sigma_d\mathfrak{z}_{\gamma_3}^{(p)} + c^{k+1-j}d^{j+1}\varepsilon(d)\sigma_{cd}\mathfrak{z}_{\gamma_4}^{(p)}\right)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are given explicitly as δ_{S_i};

$$\left({}_{c,d}z_{p^n}^{(p)}(f,k,j,\alpha,\mathrm{prime}(pN))\right)_{n\geq 1} = (c^2-c^{k+1-j}\sigma_c)\cdot(d^2-d^{j+1}\sigma_d)\cdot\prod_{p\neq \ell|N}(1-\bar{a}_\ell \ell^{-k}\sigma_\ell^{-1})\cdot\mathfrak{z}_\gamma^{(p)}$$

with $\gamma = \delta(f,j,\alpha)$.

Proof. ① follows from the definition by the same argument as in §5, where we discussed the map $(*)$.

② follows from part 2) of Main Thm. 1: that shows that $H^1(V_{F_\lambda}(f))$ is free of rank 1 over $\Lambda[\frac{1}{p}]$.

Knowing where $\mathfrak{z}_\gamma^{(p)}$ goes under the nonzero map $(*)$ pins it down.

③ Under $(*)$, the action of σ_{-1} on $H^1(V_{F_\lambda}(f))$ becomes the action of $-\chi(-1)$ on $S(f)\otimes_F F_\lambda$.

Now the rank 1 freeness statement from before shows the claim.

④ Same as ①.

$\square$

Remark. Soon we will come to our remaining ~~two~~ main theorems. MT2 is about the module generated by the elements in ④. MT3 is about the elements $\mathfrak{z}_\gamma^{(p)}$ and L-values. and their relationship to the elements $\mathfrak{z}_\gamma^{(p)}$.

Lemma. For $A\geq 1$ and $\nu: (\mathbb{Z}/A)^\times \to \bar{\mathbb{Q}}^\times$:

④' If $c,d\in\mathbb{Z}$ s.t. $(c, 6Ap) = 1 = (d, 6ANp)$, then

$$\sum_{a\in(\mathbb{Z}/A)^\times} \nu(a)\cdot\left({}_{c,d}z_{p^n}^{(p)}(f,k,k-1,a(A),\mathrm{prime}(pA))\right)_{n\geq 1} = \left(\sum_{a\in(\mathbb{Z}/A)^\times}\nu(a)\,\mathfrak{z}^{(p)}_{\delta(f,k-1,a(A))}\right)\cdot$$
$$\cdot\left(\prod_{p\neq\ell|A}\left(1-\bar{a}_\ell\ell^{-k}\sigma_\ell^{-1}+\bar{\varepsilon}(\ell)\ell^{-k-1}\sigma_\ell^{-2}\right)\right)\cdot(c^2-c^2\nu(c)^{-1}\sigma_c)\cdot(d^2-d^k\varepsilon(d)\nu(d)\sigma_d)$$
$$\in H^1(V_{F_\lambda}(f))\otimes_\Lambda Q(\Lambda)$$

⑤ $\sum_{a\in(\mathbb{Z}/A)^\times}\nu(a)\,\delta(f,k-1,a(A)) \neq 0 \in V_{F_\lambda}(f)\otimes\bar{\mathbb{Q}}$ if $L_{\mathrm{prime}(A)}(f^*,\nu^{-1},k-1)\neq 0$ and ν has conductor A.

Proof: (4') is just a reformulation of (4). Probably even the statement can be omitted.

(5) is based on studying pairings:

$$\langle \cdot , \cdot \rangle : \mathcal{H}^1 \times \mathcal{H}^1 \longrightarrow \mathbb{Z} \qquad \text{(integrating against each other)}$$

induces a pairing

$$\langle \cdot , \cdot \rangle : \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1) \times \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1) \longrightarrow \mathbb{Q}$$

$$(x_1, \dots, x_{k-2}),\ (y_1, \dots, y_{k-2}) \longmapsto \frac{1}{(k-2)!} \sum_{\sigma \in S_{k-2}} \prod_{j=1}^{k-2} \langle x_j, y_{\sigma j} \rangle$$

which induces a perfect pairing

$$\langle \cdot , \cdot \rangle : V_k(Y_1(N)) \times V_{k,c}(Y_1(N)) \longrightarrow \mathbb{Q}$$

by the definitions of $V_k(Y_1(N)) := H^1(Y_1(N)(\mathbb{C}), \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1)) \otimes_{\mathbb{Z}} \mathbb{Q}$

and $V_{k,c}(Y_1(N)) := H^1_c(Y_1(N)(\mathbb{C}), \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1)) \otimes_{\mathbb{Z}} \mathbb{Q}$

This in turn induces a ~~pairing~~ map $\langle \cdot , \mathrm{per}(f^*) \rangle : V_F(f) \longrightarrow \mathbb{C}$.
since $\mathrm{per}: S_k(X_1(N)) \to V_{k,\mathbb{C},c}(Y)$.

Computation:

$$\sum_{a \in (\mathbb{Z}/A)^\times} v(a) \langle \delta(f, k-1, a(A)), \mathrm{per}(f^*) \rangle \overset{\text{unraveling def of pairing}}{=} \sum_{a \in (\mathbb{Z}/A)^\times} v(a) \cdot (-2\pi)^{k-1} \cdot A^{k-2} \int_0^\infty f^*\left(yi + \frac{a}{A}\right) y^{k-2}\, dy$$

$$= (-1)^{k-1} A^{k-2} \cdot (k-2)! \cdot \underbrace{\left(\sum_{a \in (\mathbb{Z}/A)^\times} v(a) \cdot \zeta_A^a \right)}_{\text{Gauß sum}} \cdot L_{\mathrm{prime}(A)}(f^*, v^{-1}, k-1)$$

For a Gauß sum $G(v, \zeta_A)$ where v is primitive of order A — i.e. has conductor A —, we have $|G(v, \zeta_A)|_{\mathbb{C}} = \sqrt{A}. \neq 0$.

So pairing against $\mathrm{per}(f^*)$ is non-zero, and thus the original sum is also $\neq 0$ by perfectness of the pairing. ◻

§8 Main theorem 2

$$Z(f, T) := \langle z_\gamma^{(p)} : \gamma \in T \rangle_\Lambda \leq H^1(T) \otimes \mathbb{Q} \qquad \text{for } T := V_{\mathcal{O}_\lambda}(f).$$

$$Z := \langle \text{the elements in (4)} \rangle_\Lambda \leq H^1(V_{\mathcal{O}_\lambda}(f))$$

Main theorem 2: $Z \subseteq Z(f, T)$ and $Z(f,T)/Z$ is finite.

<u>Proof.</u> (4) shows that $Z \subseteq Z(f, T)$. Note that the latter may admit denominators: the point is showing that this is not really that bad.

First note that $z_\gamma^{(p)}$ is F_λ-linear in γ by construction.

Recall the second formula of (4). Moreover recall that Ash-Stevens showed that the δs generate $V_{k,\mathbb{Z}}(Y(L))$, so we can write any γ as a linear combination of such δs, and this lin comb is finite because the set in Ash-Stevens is finite.

Taking the explicit factors in the formula into consideration, we find: $\exists \mu \in \Lambda$ nzd: $\mu Z(f, T) \subseteq Z$.

<u>We will show:</u> $\forall \mathfrak{p} \in \operatorname{Spec}(\Lambda)$, $\operatorname{ht}(\mathfrak{p}) = 1$, $p \notin \mathfrak{p}$: $Z(f, T)_\mathfrak{p} = Z_\mathfrak{p}$.

This shows the claim about finite index.

Let $\mathfrak{p}$ be such a prime, $c, d \in \mathbb{Z}$ s.t. $(c, 6p) = 1 = (d, 6Np)$, $c^2 \neq 1 \neq d^2$.

Rohrlich: $\exists \nu : (\mathbb{Z}/A\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times \subset \overline{F}_\lambda^\times$ of $\operatorname{cond}(\nu) = A$:

(i) $L_{\{p\}}(f^*, \nu^{-1}, k-1) \neq 0$

(ii) $c^2 - c^2\nu(c)^{-1} \hat{\sigma}_c \neq 0$

(iii) $d^2 - d^k \varepsilon(d)\nu(d)\hat{\sigma}_d \neq 0$

(iv) $\nu(-1) = \pm := \begin{cases} 1 & \text{img of } \sigma_{-1} \text{ in } \Lambda/\mathfrak{p} \text{ is } 1 \\ -1 & \text{— " —} \quad -1 \end{cases}$

auxiliary conditions

where $\hat{\ } : \Lambda \twoheadrightarrow \Lambda/\mathfrak{p} \hookrightarrow \overline{F}_\lambda$

$$\mathfrak{p}' := \ker\left(\mathcal{O}_{(\Lambda/\mathfrak{p})(\nu)}[\![G_\infty]\!] \xrightarrow{\hat{\ }} (\Lambda/\mathfrak{p})(\nu)\right)$$

Then $z_\gamma^{(p)}$ and $z_{\gamma^\pm}^{(p)}$ have the same img under $H^1(V_{F_\lambda}(f)) \to H^1(V_{F_\lambda}(f)) \otimes Q(\Lambda_\mathfrak{p})$

By (4'): $\gamma^\pm = b \cdot \sum_{a \in (\mathbb{Z}/A)^\times} \nu(a)\, \delta(f, k-1, a(A)) \in V_F(f) \otimes_F \overline{\mathbb{Q}}$ for $\exists b \in \overline{\mathbb{Q}}$

explicit computation

$$\Rightarrow \quad z_\gamma^{(p)} = b \sum_a \nu(a)\, z^{(p)}_{\delta(f, k-1, a(A))} \in H^1(V_{F_\lambda}(f)) \otimes_\Lambda Q(\Lambda_\mathfrak{p}) \otimes_{F_\lambda} (\Lambda/\mathfrak{p})(\nu)$$

by F_λ-linearity of z. Hence

$$\sum_a \nu(a) \cdot \left({}_{c,d}z^{(p)}_{p^n}(f, k, k-1, a(A), \operatorname{prime}(pA))\right)_{n \geq 1} = \tau \cdot \sum_a \nu(a)\, z^{(p)}_{\delta(f, k-1, a(A))}$$

for some $\tau \in \left(\mathcal{O}_{(\Lambda/\mathfrak{p})(\nu)}[\![G_\infty]\!]_{\mathfrak{p}'}\right)^\times$.

$$\Rightarrow \quad \sum_a \nu(a)\, z^{(p)}_{\delta(f, k-1, a(A))} \in Z \otimes_\Lambda \mathcal{O}_{(\Lambda/\mathfrak{p})(\nu)}[\![G_\infty]\!]_{\mathfrak{p}'} \quad \text{in } H^1(V_{F_\lambda}(f)) \otimes Q(\mathcal{O}_{\dots}[\![G_\infty]\!]_{\mathfrak{p}'})$$

$$\Rightarrow \quad z_\gamma^{(p)} \in \text{— " —} \quad \text{in — " —}$$

$$\Rightarrow \quad z_\gamma^{(p)} \in Z_\mathfrak{p} \quad \Rightarrow \quad Z(f, T)_\mathfrak{p} = Z_\mathfrak{p}, \text{ as claimed.}$$

∎

§9 Main theorem 3

We collect the properties of $\mathfrak{z}_\gamma^{(p)}$ in the following:

<u>Main theorem 3</u>: 1) The ~~[illegible]~~ map $\mathfrak{z}: V_{F_\lambda}(f) \longrightarrow \mathbb{H}^1(V_{F_\lambda}(f)),\ \gamma \mapsto \mathfrak{z}_\gamma^{(p)}$ is the unique F_λ-linear map with the property that:

- its image under
$$\mathbb{H}^1(V_{F_\lambda}(f)) \xrightarrow{-\otimes(\zeta_{p^n})_{n\geq 1}^{\otimes(k-r)}} \mathbb{H}^1(V_{F_\lambda}(f)(k-r)) \longrightarrow H^1(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(k-r)) \xrightarrow{\exp^*} S(f) \otimes_F F_\lambda \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n})$$
is in $S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n})$

- the image of this under
$$S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n}) \longrightarrow V_{\mathbb{C}}(f)^{\pm}$$
$$x \otimes y \longmapsto \sum_{\sigma \in G_n} \chi(\sigma)\,\sigma(y)\, \mathrm{per}_f(x)^{\pm}$$

for $\chi: G_n \to \mathbb{C}^\times$ and $\pm = (-1)^{k-r-1}\chi(-1)$ is $(2\pi i)^{k-r-1} \cdot L_{\{p\}}(f^*, \chi, r) \cdot \gamma^{\pm}$

2) $\mathbb{H}^1(V_{F_\lambda}(f)) / Z(f)$ is a torsion $\Lambda \otimes \mathbb{Q}$-module where

$$Z(f) := \langle \mathfrak{z}_\gamma^{(p)} : \gamma \in V_{F_\lambda}(f) \rangle_{\Lambda \otimes \mathbb{Q}} \leq \mathbb{H}^1(V_{F_\lambda}(f))$$

3) $\mathrm{length}_{\Lambda_\mathfrak{p}}\left(\mathbb{H}^2(V_{F_\lambda}(f))_\mathfrak{p}\right) \leq \mathrm{length}_{\Lambda_\mathfrak{p}}\left(\dfrac{\mathbb{H}^1(V_{F_\lambda}(f))_\mathfrak{p}}{Z(f)_\mathfrak{p}}\right) + \mathrm{length}_{\Lambda_\mathfrak{p}}\left(\mathbb{H}^2_{loc}(V_{F_\lambda}(f))_\mathfrak{p}\right)$

for all $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(\mathfrak{p}) = 1$, $p \notin \mathfrak{p}$.

omit

3') Strengthening: $\mathbb{H}^2_{loc} \neq 0 \Rightarrow k=2$, f is not potentially of good reduction at p, then ~~[illegible]~~ $\mathfrak{p} = \ker(\Lambda \to F_\lambda)$ induced by $x^{-2}\chi: G_\infty \to F_\lambda^\times$ with $\exists \chi: G_\infty \to F_\lambda^\times$ of fin order, and $\mathrm{length}_{\Lambda_\mathfrak{p}}\left(\mathbb{H}^2_{loc}(V_{F_\lambda}(f))_\mathfrak{p}\right) = 1$.

(The condition here means that $\nexists K/\mathbb{Q}_p$ finite such that $\forall v \nmid p$ finite place of F, $G_K \curvearrowright V_{F_v}(f)$ unramified, and $\forall v \mid p$, $G_K \curvearrowright V_{F_v}(f)$ crystalline.
In particular, $p \nmid N \Rightarrow$ pot. of good red.)

4) For any $G_\mathbb{Q}$-stable $\mathcal{O}_\lambda$-lattice $T \subseteq V_{F_\lambda}(f)$: if $p \neq 2$ and if T has an $\mathcal{O}_\lambda$-basis s.t. $SL_2(\mathbb{Z}_p) \subseteq \mathrm{im}\left(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty})) \longrightarrow GL_{\mathcal{O}_\lambda}(T) \simeq GL_2(\mathcal{O}_\lambda)\right)$, then

$$Z(f,T) \subseteq \mathbb{H}^1(T) \subseteq \mathbb{H}^1(T) \otimes \mathbb{Q}$$

and $\mathrm{length}_{\Lambda_\mathfrak{p}}\left(\mathbb{H}^2(V_{F_\lambda}(f))_\mathfrak{p}\right) \leq \mathrm{length}_{\Lambda_\mathfrak{p}}\left(\mathbb{H}^1(V_{F_\lambda}(f))_\mathfrak{p} / Z(f)_\mathfrak{p}\right)$

$\forall \mathfrak{p} \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(\mathfrak{p}) = 1$, unless the consequence of 3') holds.

<u>Proof</u>: 1) Let $r \in \mathbb{Z}$, $1 \le r \le k-1$, $\chi : G_\infty \to \bar{\mathbb{Q}}^\times$ finite order. Uniqueness: MT 1.2, 1-dimensionality.

By the same argument as in the proof of MT2, we can describe the image of $z_\gamma^{(p)}$ explicitly, and the theorems of Kato about the image of zeta elements under per / exp* finish the proof, see as in i)

2) Recall from MT1.2 that $\mathbb{H}^1(T) \otimes \mathbb{Q}$ is free of rk 1 over $\Lambda \otimes \mathbb{Q}$. Since $Z_{\gamma}(f)_q \neq 0$ by i) for $\mathrm{ht}(q) = 0$, the quotient has no other choice than to be torsion.

3) Follows from ESM.2: $\mathrm{length}_{\Lambda_p}(\ker(\mathbb{H}^2(T) \to \mathbb{H}^2_{loc}(T))) \le \mathrm{length}_{\Lambda_p}(\Lambda_p / J_p)$ where $J := (h(Z) : h \in \mathrm{Hom}_\Lambda(\mathbb{H}^1(T), \Lambda))$.

3') we omit, one has to do p-adic Hodge theory.

4) One can show that $T = a \cdot V_{O_\lambda}(f)$ for some $a \in F_\lambda^\times$. This is something Katharina may prove (Lemma 14.7). So wma $T = V_{O_\lambda}(f)$. Since $Z(f,T)/Z$ is finite,

$$\left.\begin{array}{l} \text{we have } Z(f,T)_p \subseteq \mathbb{H}^1(T)_p \ \ \forall p \in \mathrm{Spec}(\Lambda),\ \mathrm{ht}(p) = 1. \\ \text{MT1.2} \Rightarrow \mathbb{H}^1(T) \text{ is torsion free over } \Lambda \end{array}\right\} \Rightarrow Z(f,T) \subseteq \mathbb{H}^1(T)$$

The inequality is ESM.3. □

§10 Main conjecture

Classical Iwasawa theory:

$$H^1(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], \mathbb{Z}_p(1)) \simeq \mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \ni (1-\zeta_{p^n})\cdot(1-\zeta_{p^n}^{-1})$$

$$H^2(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], \mathbb{Z}_p(1)) \simeq Cl(\mathbb{Q}(\zeta_{p^n})) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

$$Z := \underbrace{\langle ((1-\zeta_{p^n})(1-\zeta_{p^n}^{-1}))_{n \ge 1} \rangle}_{\text{Euler system}}{}_{\mathbb{Z}_p[\![G_\infty]\!]_+} \le \mathbb{H}^1(\mathbb{Z}_p(1))^+$$

<u>Mazur–Wiles</u>: (IMC) $\mathrm{length}_{\mathbb{Z}_p[\![G_\infty]\!]_{+,p}}(\mathbb{H}^2(\mathbb{Z}_p(1))^+_p) = \mathrm{length}_{\mathbb{Z}_p[\![G_\infty]\!]_{+,p}}(\mathbb{H}^1(\mathbb{Z}_p(1))^+_p / Z_p)$

The analogous IMC for modular forms: (comes from motivic version Kato, Perrin-Riou)

<u>Conjecture.</u> (IMC) For T a $G_\mathbb{Q}$-stable O_λ-lattice in $V_{F_\lambda}(f)$, and $p \in \mathrm{Spec}(\Lambda)$, $\mathrm{ht}(p) = 1$ (assuming $2 \notin p$ if $p = 2$):

we have $Z(f,T)_p \subseteq \mathbb{H}^1(T)_p$

and $\mathrm{length}_{\Lambda_p}(\mathbb{H}^2(T)_p) = \mathrm{length}_{\Lambda_p}(\mathbb{H}^1(T)_p / Z(f,T)_p)$.

<u>So</u>: Here we proved one inequality.