Interactions between analytic number theory and arithmetic geometry: 11. 4. 2018
counting rational points on algebraic varieties, lecture 1

kdestagnol@mpim-bonn.de

- Quantitative arithmetic of projective varieties (Browning)
- Intro to analytic and probabilistic number theory (Tenenbaum)
- Analytic N. T. (Kowalski, Iwaniec)
- Algebraic Geometry (Hartshorne)
- Diophantine Equations (Hindry, Silverman)

Time change: Tuesday 16 – 18

Introduction

$f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$ homogeneous polynomials

$$S_{\underline{f}} := \{ \underline{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{0\} \mid f_1(x) = \dots = f_r = 0 \}$$

Questions: 1) $S_{\underline{f}} = \emptyset$?

If $S_{\underline{f}} = \emptyset$, what are the obstructions?

2) If $S_{\underline{f}} \neq \emptyset$: finite or infinite?

3) If $S_{\underline{f}}$ finite: can we find all solutions?

If not, can we bound their number or even count them?

4) If $S_{\underline{f}}$ is infinite, how can we describe the complexity of $S_{\underline{f}}$?

We want some kind of quantification.

Example. Waring's problem

Lagrange: every integer can be written as a sum of 4 squares.
(but not as a sum of 3 squares)

Waring: the same question with k^{th} powers.

Involves counting points on $x_1^k + \dots + x_s^k = x_{s+1}^k + \dots + x_{2s}^k$

Our main point of interest is Question 4). (The others can be treated with anal. n. t. too.)

We will measure the size by using a norm $\|.\|$ on $\mathbb{R}^n$,

and we will try to understand the quantity

$$N_f(B) := \#\{\underline{x} \in S_f \mid \|\underline{x}\| \leq B\}$$

This is well-defined since $\{\underline{x} \in \mathbb{Z}^n \mid \|\underline{x}\| \leq B\}$ is compact, hence finite.

What is the behaviour of this as $B \longrightarrow +\infty$?

The most common choice is $\|\underline{x}\| := \max_i |x_i|$. Since all norms are equivalent, changing the norm does not change the behaviour, but it of course changes the values of $N_f(B)$.

A naive heuristic. $f \in \mathbb{Z}[x_1, \dots, x_n]$ homogeneous of degree $d \geq 1$

$$f = \sum_{i_1+\dots+i_n=d} a_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n}$$

Put $C := \sum_{i_1+\dots+i_n=d} |a_{i_1,\dots,i_n}|$

If we let $\|x\| \leq B$ with $\|x\| = \max_i |x_i|$, then $|f(x)| \leq CB^d$, so roughly $f(\underline{x})$ is an integer of size B^d.

The probability of $f(\underline{x}) = 0$ is then roughly B^{-d}.

$$N_f(B) \approx \sum_{\|\underline{x}\| \leq B} \mathbb{P}(f(x)=0) \approx B^{-d} \sum_{\|\underline{x}\| \leq B} 1$$

$$\#\{n \in \mathbb{Z} \mid |n| \leq B\} = 2B+1 \Rightarrow \sum_{\|x\| \leq B} 1 = (2B+1)^n \approx B^n$$

This means that one can expect $N_f(B) \asymp B^{n-d}$

where $f(x) \asymp g(x)$ if $\exists C_1, C_2 > 0$ s.t. $C_1 f(n) \leq g(n) \leq C_2 f(n)$.

I.e. we expect $N_f(B) \underset{B \to \infty}{\sim} c \cdot B^{n-d}$ for some $c \in \mathbb{R}$.

If $n > d$: $N_f(B) \xrightarrow[B\to\infty]{} \infty$

If $n = d$: finitely many solutions

If $n < d$: no solutions.

Of course, this was merely a heuristic derivation.

<u>What can go wrong?</u>

- $x_1^{2d} + \dots + x_n^{2d} = 0$ has no nontrivial real solutions (hence no integer solutions) $\to$ obstruction: too few solutions
- $x^2 + 3y^2 + 4z^2 = 0$ has no nontrivial solution (modulo 9):

  { let (x,y,z) be a nontrivial solution in $\mathbb{Z}^3$, wma $\gcd(x,y,z) = 1$.

  mod 3 we get $x^2 + 4z^2 \equiv 0$ (3), the squares mod 3 are 0 and 1,

  hence the only solution is $x \equiv z \equiv 0$ (3)

  $\Rightarrow x^2 \equiv z^2 \equiv 0$ (9) $\Rightarrow 3y^2 \equiv 0$ (9) $\Rightarrow y \equiv 0$ (3) $\Rightarrow 3 \mid \gcd(x,y,z)$. }

$\longrightarrow$ obstruction modulo some number

We say that the <u>Hasse principle</u> is satisfied if

$S_f \neq \emptyset \iff f$ has non-trivial solution over $\mathbb{R}$ and $\mathbb{Z}/M\mathbb{Z}$ $\forall M \geq 2$

The above two obstructions are called <u>local obstructions</u>.

Let $n > 2d$ and take $L_1, \dots, L_{2d}$ linearly independent linear forms in $\mathbb{Z}[x_1, \dots, x_n]$

Using $f_1(\underline{x}) = x_1^{2d} + \dots + x_n^{2d}$, define $f_3(\underline{x}) := f_1(L_1(\underline{x}), \dots, L_{2d}(\underline{x})) = 0 \iff$

$\iff L_1(\underline{x}) = \dots = L_{2d}(\underline{x}) = 0$

$$\#\{\underline{x} \mid L_1(\underline{x}) = \dots = L_{2d}(\underline{x}) = 0,\ \max|x_i| \leq B\} \approx B^{n-2d}$$

This agrees with what the heuristic would predict, but this should have been a counterexample (an actual counterexample will be presented later).

The problem can be too many solutions too:

$f_4(\underline{x}) = x_1^d - x_2\left(x_3^{d-1} + \dots + x_n^{d-1}\right)$

$\longrightarrow$ the points $(0, 0, x_3, \dots, x_n)$ are all solutions

$\#\{(0,0,x_3,\dots,x_n) \mid \max |x_i| \leq B\} \approx B^{n-2}$

$\Rightarrow N_f(B) \gg B^{n-2}$ $\quad$ ($f(x) \gg g(x)$ if $\exists C>0$ s.t. $g(x) \leq C f(x)$)

For $d \geq 3$, the heuristic fails

$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$ $\longrightarrow$ $(x_1, -x_1, x_2, -x_2)$ are solutions

$\longrightarrow$ roughly B^2 solutions.

This is a projective line.

Thm. (Birch, 1962) Let $f \in \mathbb{Z}[x_1, \dots, x_n]$ be a homog. polynomial of deg d, non-singular such that $n > (d-1)\cdot 2^{d-1}$.

Then if f has nontrivial solutions over $\mathbb{R}$ and $\mathbb{Z}/_{M\mathbb{Z}}$ $\forall M \geq 2$ then $\exists c_f$: $$N_f(B) \underset{B\to\infty}{\sim} c_f B^{n-d}$$

This supports the suspicion that the heuristic result is true for nice enough f.

In the projective space: if f is homogeneous and $f(x_1,\dots,x_n)=0$, then $\forall \lambda \in \mathbb{Q}$ $\lambda^d f(x_1,\dots,x_n) = f(\lambda x_1, \dots, \lambda x_n) = 0$

We say that $\underline{x} \sim \underline{y}$ over $\mathbb{Q}^n$ if $\underline{x} = \lambda \underline{y}$ for $\lambda \in \mathbb{Q}$,

and thus define $\mathbb{P}^{n-1} := \mathbb{Q}^n/_\sim$

We will be interested in $N_f(B) = \#\{[\underline{x}] \in \mathbb{P}^{n-1} \mid f(\underline{x}) = 0, H(\underline{x}) \leq B\}$

where $H(\underline{x}) = \|(x_1, \dots, x_n)\|$ for $(x_1,\dots,x_n)$ coprime integers. (There are two such representatives, they yield the same $\|\cdot\|$, H is well-def'd.)

H is called a height function.

Interactions ..., lecture 1

$$N_f(B) = \frac{1}{2} \# \left\{ (x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{0\} \;\middle|\; \gcd(x_1, \dots, x_n) = 1,\ f(\underline{x}) = 0,\ \|(x_1, \dots, x_n)\| \leq B \right\}$$

Möbius inversion $\longrightarrow$ a way to remove the gcd = 1 condition

Def. $\mu : \mathbb{N}_{>0} \longrightarrow \{0, \pm 1\}$

$$n \longmapsto \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 \mid n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ where the } p_i \text{ are distinct primes} \end{cases}$$

Prop: $\displaystyle\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$

Pf: $n = 1$ trivi

$n = p_1 \cdots p_r \Rightarrow \sum \mu(d) = \sum \binom{r}{i} (-1)^i = (-1+1)^r = 0$ $\square$

Removing the coprimality condition

$$N_f(B) = \frac{1}{2} \sum_{\substack{(x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{0\} \\ f(\underline{x}) = 0 \\ \|\underline{x}\| \leq B}} \underbrace{\mathbb{1}_{\gcd(x_1, \dots, x_n) = 1}}_{\sum_{d \mid \gcd(x_1, \dots, x_n)} \mu(d)} =$$

$$= \frac{1}{2} \sum_{d \leq B} \mu(d) \sum_{\substack{\underline{x} \in \mathbb{Z} \setminus \{0\} \\ f(\underline{x}) = 0 \\ \|\underline{x}\| \leq B \\ d \mid x_1, \dots, x_n}} 1 \qquad y_i = x_i / d$$

$$= \frac{1}{2} \sum_{d \leq B} \mu(d) \sum_{\substack{\underline{y} \in \mathbb{Z} \setminus \{0\} \\ f(\underline{y}) = 0,\ \|\underline{y}\| \leq \frac{B}{d}}} 1$$

$\longrightarrow$ this is the kind of counting function we studied earlier

So the coprimality condition does not really change anything.

$$N_f(B) = \#\{[\underline{x}] \in \mathbb{P}^{n-1} \mid f(\underline{x}) = 0,\ \max|x_i| \le B\}$$

Case of $\mathbb{P}^{n-1}$ (i.e. $f = 0$)

$$N_{\mathbb{P}^{n-1}}(B) = \#\{[x] \in \mathbb{P}^{n-1} \mid \max|x_i| \le B\} \approx B^n$$

Thm. (Schanuel, 1979) Let $n \geq 2$. Then

$$N_{\mathbb{P}^{n-1}}(B) = \frac{2^{n-1}}{\zeta(n)} B^n + O_n\left(B^{n-1} (\log B)^{b_n}\right) \quad \text{with} \quad b_n = \begin{cases} 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

the implied constant may depend on n

Pf: By Möbius inversion:

$$N_{\mathbb{P}^{n-1}}(B) = \frac{1}{2} \sum_{d \le B} \mu(d)\, \#\left\{(y_1, \dots, y_n) \in \mathbb{Z}^n \setminus \{0\} \;\middle|\; \max|y_i| \le \frac{B}{d}\right\}$$

$$= \frac{1}{2} \sum_{d \le B} \mu(d) \left(\left\lfloor \frac{2B}{d} + 1 \right\rfloor^n - 1\right)$$

Now use $\lfloor x \rfloor = x + O(1)$, i.e. $x - \lfloor x \rfloor$ is bounded:

$$= \frac{1}{2} \sum_{d \le B} \mu(d) \left(\left(\left(\frac{2B}{d} + 1\right)^{\cancel{n}} + O(n)\right)^n - 1\right)$$

binom. expansion

$$= \frac{1}{2} \sum_{d \le B} \mu(d) \left(\frac{2^n B^n}{d^n} + O_n\left(\frac{B^{n-1}}{d^{n-1}}\right)\right)$$

$$= 2^{n-1} B^n \sum_{d \le B} \frac{\mu(d)}{d^n} + O_n\left(B^{n-1} \sum_{d \le B} \frac{1}{d^{n-1}}\right)$$

$$\left| B^{n-1} \sum_{d \le B} O_n\left(\frac{1}{d^{n-1}}\right) \mu(d) \right| \le c B^{n-1} \sum_{d \le B} \frac{1}{d^n}.$$

Error term: $\displaystyle\sum_{d \le B} \frac{1}{d^{n-1}} = \begin{cases} O(\log B) & \text{if } n = 2 \\ O(1) & \text{if } n > 2 \end{cases} \implies$

$$\implies O_n\left(B^{n-1} \sum_{d \le B} \frac{1}{d^{n-1}}\right) = \begin{cases} O(B \log B) & \text{if } n = 2 \\ O(B^{n-1}) & \text{if } n > 2 \end{cases}$$

Interactions ..., lecture 1-2

Main term: $\left| \sum_{d \le B} \frac{\mu(d)}{d^{n-1}} \right| \le \sum_{d \le B} \frac{1}{d^n}$

$$\left| 2^{n-1} B^n \sum_{d \le B} \frac{\mu(d)}{d^n} - 2^{n-1} B^n \sum_{d} \frac{\mu(d)}{d^n} \right| =$$

$$= \left| 2^{n-1} B^n \sum_{d > B} \frac{\mu(d)}{d^n} \right| \le 2^{n-1} B^n \sum_{d > B} \frac{1}{d^n} \le$$

$$\le 2^{n-1} B^n \int_B^{+\infty} \frac{dt}{t^n} = 2^{n-1} B \, O(B^{n-1})$$

$$N_{\mathbb{P}^{n-1}}(B) = 2^{n-1} B^n \sum_{d \ge 1} \frac{\mu(d)}{d^n} + O_n\left(B^{n-1} (\log B)^{b_n} \right)$$

$$\sum_{d \ge 1} \frac{\mu(d)}{d^n} \times \sum_{d \ge 1} \frac{1}{d^n} = \sum_{d \ge 1} \frac{1}{d^n} \sum_{i | d} \mu(i) = 1$$

Cauchy product

$$\Rightarrow \sum_{d \ge 1} \frac{\mu(d)}{d^n} = \frac{1}{\zeta(n)}$$

$\square$

Next question (HW). $f = x_1 x_2 - x_3 x_4$

$$N_f(B) = \# \{ [\underline{x}] \in \mathbb{P}^3 \mid x_1 x_2 - x_3 x_4 = 0, \max |x_i| \le B \} = \underbrace{c B^2 \log B + o(B^2 \log B)}$$

$\forall \varepsilon > 0 \quad \log B \ll B^{\varepsilon} \Rightarrow$

$$\ll c B^{2+\varepsilon} + o(B^2 \log B),$$

so we are missing the conjecture but not by much.

Example. With too few rational points. 17.4.2018

Let $n > k > 2d$ and $f_1(x_1, \dots, x_k) := x_1^{2d} + \dots + x_k^{2d}$.

Let us choose $L_1, \dots L_k \in \mathbb{Z}[x_1, \dots, x_n]$ lin indep lin forms, e.g. $x_1, \dots, x_k$.

Consider $f_2(x_1, \dots, x_n) := f_1(L_1(x_1, \dots, x_n), \dots, L_k(x_1, \dots, x_n))$, this is a homog polynomial of degree $2d$ in $\mathbb{Z}[x_1, \dots, x_n]$.

Conjecture prediction: B^{n-2d}

$f_3 = 0 \iff L_1(\underline{x}) = 0, \ldots, L_2(\underline{x}) = 0$

This yields $\approx B^{n-2}$ solutions with $\max |x_i| \leq B$.

The "trick" of this counterexample is that the polynomial f_1 has no nontrivi zeroes.

Recall Schanuel's Thm: $n \geq 2$

$$N_{\mathbb{P}^{n-1}}(B) = 2^{n-1} \frac{B^n}{\zeta(n)} + O_n\left(B^{n-1} (\log B)^{b_n}\right) \quad \text{with } b_n = \begin{cases} 1 & n=2 \\ 0 & \text{else} \end{cases}$$

Rmk. on the constant: $n \geq 2$, $\zeta(n) = \prod_p (1-p^{-n})^{-1}$

$$\mathbb{P}^{n-1}(\mathbb{F}_p) = \mathbb{F}_p^n \setminus \{0\} \big/ \sim \qquad \frac{1}{\zeta(n)} = \prod_p (1-p^{-n})$$

$$\Downarrow$$

$$\# \mathbb{P}^{n-1}(\mathbb{F}_p) = \frac{p^{n}-1}{p-1} = p^{n-1} + \ldots + 1$$

$$\Rightarrow \frac{1}{\zeta(n)} = \prod_p \underbrace{\left(1 - \frac{1}{p}\right) \frac{\# \mathbb{P}^{n-1}(\mathbb{F}_p)}{\# \mathbb{F}_p^{n-1}}}$$

$$\left(1 - \frac{1}{p}\right) \cdot \frac{p^{n-1} + \ldots + 1}{p^{n-1}} = \frac{(p-1)}{p} \cdot \frac{p^n - 1}{(p-1)\, p^{n-1}} = 1 - \frac{1}{p^n}$$

Recall the 2nd example: $V := xy - zt = 0$

$$N_V(B) = \# \left\{ [x, y, z, t] \in \mathbb{P}^3 \;\middle|\; xy - zt = 0,\; H([x,y,z,t]) \leq B \right\}$$

1st step. Parametrisation.

$u_1 := \gcd(x, z) \longrightarrow x = u_1 v_1, \quad z = u_1 v_2, \quad \gcd(v_1, v_2) = 1$

$\Rightarrow v_1 y = v_2 t$ and $\gcd(v_1, v_2) = 1$ implies $v_1 \mid t$, $v_2 \mid y$.

$t = v_1 u_2$, $y = v_2 u_2'$ with $u_2 = u_2'$

In the end, $x = u_1 v_1$, $y = u_2 v_2$,

$z = u_1 v_2$, $t = u_2 v_1$ with $\gcd(v_1, v_2) = 1$.

$$\rightarrow N_V(B) = \frac{1}{2} \# \left\{ (x,y,z,t) \in \mathbb{Z}^4 \;\middle|\; \gcd(x,y,z,t) = 1,\; xy - zt = 0,\; \max |x|,|y|,|z|,|t| \leq B \right\}$$

Actually $\gcd(x,y,z,t)=1 \iff \gcd(v_1,v_2)=1$

In the end, $N_V(B) = \frac{1}{4}\,\#\Big\{(u_1,u_2),(v_1,v_2)\in\mathbb{Z}^2\times\mathbb{Z}^2 \;\Big|\; \gcd(u_1,u_2)=\gcd(v_1,v_2)=1,$
$\max(|u_1|,|u_2|)\cdot\max(|v_1|,|v_2|)\le B\Big\}$

$$\begin{array}{ccc} \mathbb{Z}^2\times\mathbb{Z}^2 & \longrightarrow & \{(x,y,z,t)\in\mathbb{Z}^4 \mid \gcd(x,y,z,t)=1,\; xy=zt\} \\ (u_1,u_2),\ (v_1,v_2) & & (u_1v_1,\ u_2v_2,\ u_1v_2,\ u_2v_1) \end{array}$$

with $\gcd(u_1,u_2)=1$,
$\gcd(v_1,v_2)=1$

This shows $V\simeq\mathbb{P}^1\times\mathbb{P}^1$.

$$N_V(B) = \frac{1}{4}\sum_{\substack{|u_1|,|u_2|\le B\\ \gcd(u_1,u_2)=1}} \underbrace{\sum_{\substack{|v_1|,|v_2|\le \frac{B}{\max(|u_1|,|u_2|)}\\ \gcd(v_1,v_2)=1}} 1}$$

$$\#\Big\{(v_1,v_2)\in\mathbb{Z}^2 \;\Big|\; \gcd(v_1,v_2)=1,\ \max(|v_1|,|v_2|)\le\frac{B}{\max(|u_1|,|u_2|)}\Big\}$$

Recall: $N_{\mathbb{P}^1}(T) = \frac{1}{2}\,\#\{(x_1,x_2)\in\mathbb{Z}^2 \mid \gcd(x_1,x_2)=1,\ |x_1|,|x_2|\le T\}$

$$\Rightarrow\quad 2\cdot N_{\mathbb{P}^1}\left(\frac{B}{\max(|u_1|,|u_2|)}\right) =$$

By Schanuel: $N_{\mathbb{P}^1}(T) = \frac{2\cdot 6}{\pi^2}T^2 + O(T\log T)$

$$N_V(B) = \frac{6}{\pi^2}B\cdot\sum_{\substack{|u_1|,|u_2|\le B\\ \gcd(u_1,u_2)=1}}\frac{1}{\max(|u_1|,|u_2|)^2} + O\left(\sum_{\substack{|v_1|,|v_2|\le B\\ \gcd(v_1,v_2)=1}}\frac{B}{\max(|v_1|,|v_2|)}\log B\right)$$

Main term: need to estimate

$$\sum_{\substack{|u_1|,|u_2|\le B\\ \gcd(u_1,u_2)=1}}\frac{1}{\max(|u_1|,|u_2|)^2} = \sum_{d\le B}\frac{1}{d^2}\underbrace{\sum_{\substack{u_1,u_2\\ \gcd(u_1,u_2)=1\\ \max(|u_1|,|u_2|)=d}} 1}_{\text{need to estimate this}}$$

$$\sum_{\substack{\gcd(u_1,u_2)=1 \\ \max(|u_1|,|u_2|)=d}} 1 = \sum_{\substack{u_1,u_2 \\ \max(|u_1|,|u_2|)=d}} \sum_{k|u_1,u_2} \mu(k) \quad \text{Möbius}$$

$$= \sum_{k|d} \mu(k) \underbrace{\sum_{\substack{u_1,u_2 \\ \max(|u_1|,|u_2|)=d \\ k|u_1,u_2}} 1}_{\text{need to compute this}}$$

Counting the number of such (u_1, u_2):

if $\max(|u_1|, |u_2|) = |u_1| \Rightarrow u_1 = \pm d$, this is automatically a k-multiple

u_2 has to be a k-multiple between $-d$ and d

$\longrightarrow$ $2 \cdot \frac{d}{k} + 1$ choices for u_2, 2 choices for u_1

$\longrightarrow$ the total is $8 \cdot \frac{d}{k} + \underbrace{O(1)}_{\substack{\text{will go into} \\ \text{the error term}}}$ (we counted (d,d), $(d,-d)$, $(-d,d)$, $(-d,-d)$ multiple times)

Main term $= \frac{6}{\pi^2} B^2 \sum_{d \leq B} \frac{1}{d^2} \sum_{k|d} \mu(k) \cdot 8 \frac{d}{k} = \frac{48}{\pi^2} B^2 \sum_{d \leq B} \frac{1}{d} \sum_{k|d} \frac{\mu(k)}{k} =$

$$= \frac{48}{\pi^2} B^2 \sum_{k \leq B} \frac{\mu(k)}{k} \sum_{\substack{d \leq B \\ d \equiv 0\ (k)}} \frac{1}{d} \qquad d = k \cdot d'$$

$$= \frac{48}{\pi^2} B^2 \sum_{k \leq B} \frac{\mu(k)}{k} \sum_{d' \leq B/k} \frac{1}{kd'}$$

$$= \frac{48}{\pi^2} B^2 \sum_{k \leq B} \frac{\mu(k)}{k^2} \underbrace{\sum_{d' \leq B/k} \frac{1}{d'}}_{\log(B/k) + O(1) = \log B - \log k + O(1)}$$

$$\text{Main term} = \underbrace{\frac{48}{\pi^2} B^2 \log B \underbrace{\sum_{k \leq B} \frac{\mu(k)}{k^2}}_{\zeta(2)}}_{\frac{48 \cdot 6}{\pi^4} B^2 \log B \,=\, c \cdot B^2 \log B} + O(B^2 \log B)$$

One can (and should) check that the error terms we neglected along the way will add up to $O(B^2 \log B)$

Note $\log B \ll_{\varepsilon} B^{\varepsilon} \quad \forall \varepsilon > 0$

$$c = c' \cdot \frac{1}{\zeta(2)^2} \quad \text{and} \quad \frac{1}{\zeta(2)^2} = \prod_p \left(1-\frac{1}{p}\right)^2 \frac{\# V(\mathbb{F}_p)}{\# \mathbb{F}_p^2}$$

In every known example: $N_V(B) \sim c B^a \log B^b$

We have a prediction for a; now we want one for b.

Ex. $V \subseteq \mathbb{P}^4$ $\quad x_0 x_1 = x_2^2$ and $x_3^2 + x_4^2 = x_0 x_1 + x_4 x_2$ intersected $\rightarrow$ of dim 2

Roughly: dimension $= n -$ number of equations

Def. V proj alg var. defined in $\mathbb{P}^n$ by $f_1, \dots, f_r$.

If $(f_1, \dots, f_r)$ has exactly r generators, then we say that this is the complete intersection of $f_1, \dots, f_r$ and $\dim V = d - r$.

Def. A complete intersection V defined by $f_1, \dots, f_r$ is smooth if

$\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}\right)$ is maximal

Def. A projective variety is irreducible if it is not the union of two proper subvarieties.

Ex. $V \subseteq \mathbb{P}^1$ def'd by $xy = 0$ not irreducable: $V = \{x=0\} \cup \{y=0\}$

Ex. Over $\mathbb{P}^1$: $f = \frac{g}{h}$ with g, h homogeneous

For f to be well-def'd, one needs $\deg g = \deg h$ since

$$f(x) = f(\lambda x) = \frac{g(\lambda x)}{h(\lambda x)} = \frac{\lambda^{\deg g}}{\lambda^{\deg h}} \cdot \underbrace{\frac{g(x)}{h(x)}}_{f(x)}$$

Def. V a smooth complete intersection of $f_1, \dots, f_r$ in $\mathbb{P}^n$. We define the set of rational functions on V to be

$$K(V) = \left(\mathbb{Q}[x_0, \dots, x_n] \big/ (f_1, \dots, f_r)\right)_{((0))} = \left\{ \frac{f}{g} \;\middle|\; f, g \text{ homog. of the same degree in } \mathbb{Q}[x_0, \dots, x_n] \right\}$$

We have $\pi: \mathbb{Q}[x_0,\dots,x_n] \twoheadrightarrow \mathbb{Q}[x_0,\dots,x_n]/(f_1,\dots,f_r)$ projection

$[f] \in \mathbb{Q}[x_0,\dots,x_n]/(f_1,\dots,f_r)$ is homog. of degree d if $\exists f \in \mathbb{Q}[x_0,\dots,x_n]$

s.t. $\pi(f) = [f]$.

Ex. $V: x^2+y^2-z^2=0$ in $\mathbb{P}^2$

A basis of the homog. polynomials of degree 1 in $\mathbb{Q}[x,y,z]/(x^2+y^2-z^2)$

is x, y, z.

The same for degree 2: x^2, xy, y^2, xz, zy

Def. V a smooth complete intersection in $\mathbb{P}^n$. A prime divisor on V is an irreducible closed subvariety of codimension 1 in V.

Def. A Weil divisor is an elt of the free abelian group generated by the prime divisors. $= \mathrm{Div}(V)$

$$\mathrm{Div}(V) = \left\{ \sum n_Y Y \;\middle|\; Y \text{ prime divisor, } n_Y \in \mathbb{Z}, \text{ only fin. many are } \neq 0 \right\}$$

Ex. $V = \mathbb{P}^n$ A codimension 1 subvariety is $\{F=0\}$ where $F \in \mathbb{Z}[x_0,\dots,x_n]$ homogeneous.

V is irreducible $\iff$ F is irreducible ("$\Rightarrow$" is trivial)

24.04.2018.

Def. The support of $D = \sum_{\text{finite}} n_Y Y$ is $\mathrm{Supp}(D) := \bigcup Y$.

Note that $\mathrm{Supp}(D)$ is closed.

Ex. $V = \mathbb{P}^n$. A rational function on V is of the form $f = \frac{g}{h}$ with $\deg h = \deg g$ homog., $f, g \in \mathbb{Q}[x_0,\dots,x_n]$.

We can write $g = \prod_{j=1}^{s} g_j^{m_j}$, $h = \prod_{i=1}^{r} h_i^{n_i}$ $\quad n_i, m_j \in \mathbb{N}_{>0}$, g_j, h_i irreducible

$H_i = \{h_i = 0\}$, $G_j = \{g_j = 0\}$ prime divisors

We will define the Weil divisor associated to f by

$$\mathrm{div}(f) := \sum_{j=1}^{s} m_j G_j - \sum_{i=1}^{r} n_i H_i$$

G_j and H_i are prime divisors

f vanishes on G_j

f has a pole on H_i

Def. V a smooth complete intersection in $\mathbb{P}^n$, f a rational function on V.

The Weil divisor associated to f is

$$\operatorname{div}(f) := \sum_{Y \text{ prime div.}} \operatorname{ord}_Y(f)\, Y$$

where $\operatorname{ord}_Y(f) := \begin{cases} \text{order of the zero of } f \text{ along } Y \text{ if } f \text{ vanishes on } Y \\ \text{order of the pole of } f \text{ along } Y \text{ if } f \text{ has a pole on } Y \\ 0 \end{cases}$

Def. A divisor of the form $\operatorname{div}(f)$ is called a principal divisor.
Clearly this generalises the previous example.

A more precise def. of ord can be looked up in Hartshorne.

Notation. $\operatorname{PDiv}(V)$ = principal divisors of V

Since $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$ and $\operatorname{div}\left(\frac{1}{f}\right) = -\operatorname{div}(f)$, $\operatorname{PDiv}(V)$ is a subgroup of $\operatorname{Div}(V)$.

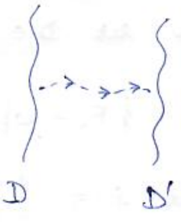
Def. Picard group: $\operatorname{Pic} V = \operatorname{Div}(V) / \operatorname{PDiv}(V)$

Rmk. $D, D' \in \operatorname{Div}(V)$, $[D] = [D']$ in $\operatorname{Pic}(V)$. Suppose $V = \mathbb{P}^n$.

$\Rightarrow$ $D - D' = \operatorname{div}(f)$ for a rat. function $f = \frac{g}{h}$

We get
$$\begin{aligned} V = \mathbb{P}^n &\longrightarrow \mathbb{P}^1 \\ [x_0, \dots, x_n] &\longmapsto [h(x_0, \dots, x_n), g(x_0, \dots, x_n)] \end{aligned}$$

such that $\varphi^{-1}([0,1]) = D'$, $\varphi^{-1}([1,0]) = D$.



Rmk. Analogue: class group of a number field.

It is also true that $\operatorname{Pic}(V) = H^1(V, \mathcal{O}_V)$.

Prop: $\mathrm{Pic}(\mathbb{P}^{n-1}) \cong \mathbb{Z} \quad \forall n \geq 2$.

Pf: A prime divisor is given by $F = 0$, F homog. irreducible.

We can define $\deg : \{\text{prime divisors}\} \longrightarrow \mathbb{Z}$

$$\{F=0\} \longmapsto \deg F$$

By additivity, this can be extended to a map $\deg : \mathrm{Div}(V) \longrightarrow \mathbb{Z}$:

$$\deg\left(\sum_i n_{Y_i} Y_i\right) = \sum_i n_{Y_i} \deg F_i \quad \text{where } Y_i = \{F_i = 0\},\ F_i \text{ irred. homog.}$$

Clearly $\deg$ is surjective: let $n \in \mathbb{Z}$ and F be an irreducible polynomial of degree 1 (such an F exists) in $\mathbb{Q}[x_1, \dots, x_n]$. Then $\deg(nY) = n \cdot \deg F = n$ where $Y = \{F=0\}$.

It remains to show that $\ker \deg = \mathrm{PDiv}(\mathbb{P}^{n-1})$.

Let $D = \mathrm{div}(f) \in \mathrm{PDiv}(\mathbb{P}^{n-1})$. For some $f = \frac{g}{h}$, $\deg g = \deg h$ write

$$g = \prod_j g_j^{m_j},\quad h = \prod_i h_i^{n_i} \quad \text{with } G_j = \{g_j = 0\},\ H_i = \{h_i = 0\}.$$

$$\Rightarrow \mathrm{div}(f) = \sum_j m_j G_j - \sum_i n_i H_i$$

$$\Rightarrow \deg(\mathrm{div}(f)) = \underbrace{\sum_j m_j \deg(g_j)}_{\deg g} - \underbrace{\sum_i n_i \deg(h_i)}_{\deg h} = 0$$

$\Rightarrow \ker \deg \supseteq \mathrm{PDiv}(\mathbb{P}^{n-1})$.

Now let $D \in \ker \deg$. $\Rightarrow D = \sum_i n_i Y_i$, $n_i \in \mathbb{Z}$, Y_i prime divisor, $Y_i = \{F_i = 0\}$.

Put $f := \prod_{n_i>0} F_i^{n_i} \Big/ \prod_{n_i<0} F_i^{-n_i}$.

$$\deg \prod_{n_i>0} F^{n_i} = \sum_{n_i>0} n_i \deg F_i$$

$$\deg \prod_{n_i<0} F^{-n_i} = -\sum_{n_i<0} n_i \deg F_i$$

Since $\deg D = 0 \Rightarrow \sum n_i \deg F_i = 0$

$\Big\}$ deg numerator = deg denominator, i.e. $f \in K(\mathbb{P}^{n-1})$.

So $\mathrm{div}\, f = D$. $\Rightarrow \ker \deg \subseteq \mathrm{PDiv}(\mathbb{P}^{n-1})$. Hence the result. $\square$

In general, the Picard group is not easy to compute.

<u>Prop.</u> Let $V := (xy - zt = 0) \subseteq \mathbb{P}^3$. Then $\mathrm{Pic}(V) \simeq \mathbb{Z}^2$.

PROOF 1. (SKETCH): We have shown that $V \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

One can show that the Picard groups of isomorphic varieties are isomorphic.

$\Rightarrow \mathrm{Pic}(V) \simeq \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$.

A prime divisor in $\mathbb{P}^1 \times \mathbb{P}^1$?

Let $F \in \mathbb{Q}[u_1, u_2, v_1, v_2]$. irreducible

If $F(u_1, u_2, v_1, v_2) = 0$ then $F(\lambda u_1, \lambda u_2, \mu v_1, \mu v_2) = 0$ should hold,

so we need $F(\lambda u_1, \lambda u_2, \mu u_1, \mu u_2) = \lambda^{d_1} \mu^{d_2} F(u_1, u_2, v_1, v_2)$ bihomogeneous.

(e.g. $u_1 v_2^2 + u_2 v_1 v_2$ is bihomogeneous.)

Now define $\deg : \{\text{prime divisor}\} \longrightarrow \mathbb{Z}^2$

$$\underset{F \text{ bihomog.}}{Y = \{F = 0\}} \longmapsto \left(\deg_{(u_1,u_2)} F, \deg_{(v_1,v_2)} F\right)$$

Extend deg, this is easily shown to give rise to an isomorphism

$$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2.$$

$\square$

PROOF 2.: Define $Z_1 := \{z = x = 0\}$, $Z_2 := \{z = y = 0\}$. lines.

Note that $Z_1, Z_2 \subseteq V$. These are irreducible closed subvarieties i.e. prime divisors of V.

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathrm{Pic}(V) \xrightarrow{\varphi} \mathrm{Pic}\left(V \setminus (Z_1 \cup Z_2)\right) \longrightarrow 0 \quad \text{exact}$$

$$D = \sum n_Y Y \longmapsto \sum n_Y \left(Y \cap V \setminus (Z_1 \cup Z_2)\right)$$

$D \in \ker \varphi$ implies $\mathrm{Supp}(D) \subseteq Z_1 \cup Z_2$. By the def of Supp this implies

$Y = Z_1$ or $Y = Z_2$ and $D = n_1 Z_1 + n_2 Z_2$

$\ker \varphi \simeq \mathbb{Z}^2$ if $[Z_1] \neq [Z_2]$. in Pic V.

<u>Claim.</u> $[Z_1] \neq [Z_2] \in \mathrm{Pic}\, V$.

Recall: $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} V$

$$([u_1, u_2], [v_1, v_2]) \longmapsto (u_1 v_1, u_2 v_2, u_1 v_2, u_2 v_1)$$

Now this isomorphism gives rise to

$$K(V) \longrightarrow K(\mathbb{P}^1 \times \mathbb{P}^1)$$

$$f = \frac{g(x,y,z,t)}{h(x,y,z,t)} \longmapsto \frac{g(u_1v_1, u_2v_2, u_1v_2, u_2v_1)}{h(u_1v_1, u_2v_2, u_1v_2, u_2v_1)}$$

Moreover: $\quad V \qquad\qquad\qquad \mathbb{P}^1 \times \mathbb{P}^1$

$$Z_1 = \{z = x = 0\} \longmapsto \{u_1 = 0\}$$

$$Z_2 = \{z = y = 0\} \longmapsto \{v_2 = 0\}$$

If $[Z_1] = [Z_2] \Rightarrow \{u_1 = 0\}$ is equivalent to $\{v_2 = 0\}$.

This implies that $\frac{u_1}{v_2}$ is rational in $\mathbb{P}^1 \times \mathbb{P}^1$. Not true. $\lightning$

$\rightarrow \ker \varphi \simeq \mathbb{Z}^2$.

$$V \setminus (Z_1 \cup Z_2) = \{z = 0\}$$

$V \setminus (Z_1 \cup Z_2) \xrightarrow{\sim} \mathbb{A}^2 = \{(x_1, x_2) \in \mathbb{Q}^2\}$ isomorphism

$$[x,y,z,t] \longmapsto \left(\frac{x}{z}, \frac{y}{z}\right)$$

$$[x, y, 1, xy] \longleftarrow (x, y)$$

Lemma. $\mathrm{Pic}(\mathbb{A}^n) = \{0\} \quad \forall n \geq 1$

PF: EXERCISE. Hint: $\mathbb{P}^{n+1} \setminus \{x_{n+1} = 0\} = \mathbb{A}^n$.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \underbrace{\mathrm{Pic}(\mathbb{P}^{n+1})}_{\mathbb{Z}} \longrightarrow \underbrace{\mathrm{Pic}(\mathbb{P}^{n+1} \setminus \{x_{n+1} = 0\})}_{0} \longrightarrow 0$$

To be proven as above. $\square$

Hence φ is an iso, and the statement follows. $\square$

So far: $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}, \quad \mathrm{Pic}(xy - zt) = \mathbb{Z}^2$

$$N_{\mathbb{P}^n}(B) = c B^{n+1} (\log B)^{\mathrm{rk}(\mathrm{Pic}\, \mathbb{P}^n) - 1} + O(\cdots)$$

$$N_V(B) = c B^2 (\log B)^{\mathrm{rk}(\mathrm{Pic}\, V) - 1} + O(\cdots)$$

Interactions..., lecture 3–4

Conjecture. (Manin, 1989) Let $V \subseteq \mathbb{P}^n$ be a smooth complete intersection of dim 2 given by $f_1, \dots, f_r$ of respective degrees $d_1, \dots, d_r$.

If $n > d_1 + \dots + d_r$ and there is at least one solution,

then there is an open subset U of V and $c > 0$ such that

$$N_V(B) = c\, B^{n-d_1-\dots-d_r} (\log B)^{\mathrm{rk}(\mathrm{Pic}\, V)-1} (1+o(1)).$$

For arbitrary varieties, there are known counterexamples.

For surfaces, no such counterexamples are known and the conjecture is believed to be true.

Back to $N_V(B)$ for $V = \{xy - zt = 0\}$.

$$N_V(B) = \underset{B^2 \log B}{\text{main term}} + O\left(\underbrace{B \sum_{\substack{u_1, u_2 \le B \\ \gcd(u_1,u_2)=1}} \frac{\log\left(\frac{B}{\max(|u_1|,|u_2|)}\right)}{\max(|u_1|,|u_2|)}}\right)$$

$$O\left(B \sum_{u_1 \le u_2 \le B} \frac{\log \frac{B}{u_2}}{u_2}\right) \quad \text{by symmetry of } u_1 \text{ and } u_2$$

$$\log \frac{B}{u_2} < \log B, \qquad \sum_{u_1 \le u_2 \le B} \frac{1}{u_2} = \sum_{u_1 \le B} \sum_{u_1 \le u_2 \le B} \frac{1}{u_2} = \sum_{u_1 \le B} \log\left(\frac{B}{u_1}\right)$$

This would be too big!

Instead: $$B \log B \sum_{u_1 \le u_2 \le B} \frac{1}{u_2} - B \sum_{u_1 \le u_2 \le B} \frac{\log u_2}{u_2}$$

Comparison with integrals $\longrightarrow (B \log B)^2$

$$\sum_{d \le B} \frac{1}{d^2} \sum_{k|d} \mu(k) \left(\frac{d}{k} + O(1)\right)$$

$$\downarrow$$

$$O\Big(\sum_{d \le B} \frac{1}{d^2} \sum_{k|d} 1\Big) = O\Big(\sum_{d \le B} \frac{\tau(d)}{d^2}\Big) \quad \text{where } \tau(d) = \sum_{k|d} 1$$

τ is multiplicative

Prop. One has $\tau(n) = O_\varepsilon(n^\varepsilon)$ $\forall \varepsilon > 0$

Thm. f multiplicative. If $f(p^\nu) \longrightarrow 0$ if $p^\nu \longrightarrow \infty$ where p prime, $\nu \in \mathbb{N}_{>0}$,

then $f(n) \longrightarrow 0$ if $n \longrightarrow \infty$.

(Proof in Tennenbaum.)

08.05.2018

Recall the conjecture by Manin:

$V \subseteq \mathbb{P}^1$ smooth complete intersection (of dimension 2) given by $f_1, \dots, f_r$ of resp. degrees $d_1, \dots, d_r$,

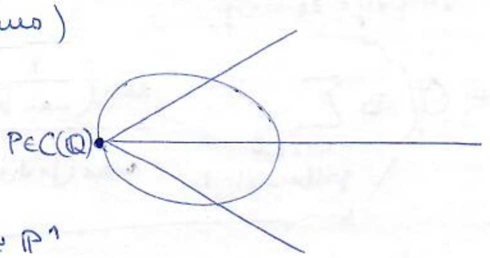
s.t. $n > d_1 + \dots + d_r$. If $f_1(\underline{x}) = 0, \dots, f_r(\underline{x}) = 0$ has a rational solution

then there is a closed subvariety F of V and $c > 0$ s.t.

$$\#\{[x_1, \dots, x_n] \in V \setminus F(\mathbb{Q}) \mid H([x_1, \dots, x_n]) \leq B\} = c \cdot B^{n-d_1-\dots-d_r} (\log B)^{\mathrm{rk}(\mathrm{Pic}\, V) - 1} (1 + o(1))$$

Situation for curves: (g = genus)

- $g = 0$, conic C in $\mathbb{P}^2$



$n = 3,\ d = 2$

$\longrightarrow n - d > 0$

if $C(\mathbb{Q}) = \emptyset$, $C \simeq \mathbb{P}^1$

- $g = 1$ elliptic curves E of deg 3 in $\mathbb{P}^2$, $y^2 = f(x)$ $\quad n = d = 3 \longrightarrow n - d = 0$

$$\#\{(y,x) \in E(\mathbb{Q}) \mid H(x,y) \leq B\} \sim c(\log B)^{\frac{1}{2}\mathrm{rk}(E)}$$

- $g = 2$ hyperelliptic curves C, $y^2 = f(x)$, $\deg f > 4$, $\quad n - d < 0$

Faltings: $C(\mathbb{Q})$ is finite.

$\dim V = n - 1 - r = 2 \longrightarrow n = r + 3 > 3$

Assume that $d_r \geqslant d_{r-1} \geqslant \dots \geqslant d_1 \geqslant 2$

$\Rightarrow n > d_1 + \dots + d_r \geqslant 2r = 2(n-3)$

$\Rightarrow n > 2n - 6 \Rightarrow n < 6$

Under these assumptions, we have $n \in \{3, 4, 5\}$.

- $n = 3 \longrightarrow r = 0$, $V = \mathbb{P}^2$, we have already dealt with this one.

- $\underset{\sim\sim}{n=4}$ $\longrightarrow r=1$ in $\mathbb{P}^3$, one equation f of $\deg < 4$, i.e.

$$d=2 \quad (\text{quadric in } \mathbb{P}^3)$$
$$\text{or } d=3 \quad (\text{cubic in } \mathbb{P}^3)$$

- $\underset{\sim\sim}{n=5}$ $\longrightarrow r=2$ in $\mathbb{P}^4$, two equations f_1, f_2 of $d_1+d_2<5$

$\Rightarrow d_1 = d_2 = 2$, intersection of two quadrics in $\mathbb{P}^4$.

Del Pezzo surfaces: Manin, Cubic Forms

Browning, Quantitative Arithmetic of Projective Varieties

Hartshorne, Algebraic Geometry

Dan Loughran's PhD thesis (on his website)

Def. V_1, V_2 projective varieties. We say that V_1 is birational to V_2 if $\exists\, U_1 \subseteq V_1$ open, $U_2 \subseteq V_2$ open s.t. $U_1 \simeq U_2$

Since closed subsets are relatively thin, this means that V_1 and V_2 are almost the same.

Def. S is a Del Pezzo surface if S is smooth and $\exists\, K/\mathbb{Q}$ finite Galois extension s.t. S is birational to $\mathbb{P}^2$ over K.

Ex. $x_1^2 + x_2^2 = 0$ and $x_1^2 - x_2^2 = 0$ are not birational over $\mathbb{Q}$ but over $\mathbb{Q}(i)$

Ex. $xy - zt = 0$ in $\mathbb{P}^3$. During the computation of $\operatorname{Pic} V$, we showed that

$$V \setminus \{z=x=0,\ z=y=0\} \simeq \mathbb{A}^2$$

Def. X a projective variety, $Z \subseteq X$ a closed subvariety (typically a pt). Then the blowup of X at Z is a variety $\tilde{X}$ together with $\pi: \tilde{X} \longrightarrow X$ s.t. $\pi^{-1}(Z)$ is a Weil divisor and such that the following universal property is satisfied: if $\varphi: \tilde{Y} \to X$, $\varphi^{-1}(Z)$ is a Weil divisor then φ factors through π:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ \tilde{Y} \arrow[ur, dashed] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Hartshorne: $\exists \tilde{X}$. and $X \setminus Z \simeq \tilde{X} \setminus \pi^{-1}(Z)$. Notation: $Bl_Z(X) = \tilde{X}$.

Ex. Blowup of A^n at $(0, \dots, 0) = O$

Hartshorne: $Bl_O(A^n)$ is the subvariety of $A^n \times \mathbb{P}^{n-1}$ given by the equations $x_i y_j = x_j y_i \quad \forall i, j = 1, \dots, n$ $\qquad (*)$

where $(x_1, \dots, x_n), (y_1, \dots, y_n) \in A^n \times \mathbb{P}^{n-1}$.

Indeed, we have $\pi : Bl_0(A^n) \longrightarrow A^n$ as the projection $A^n \times \mathbb{P}^{n-1} \twoheadrightarrow A^n$.

$\pi^{-1}(0) = \{ \forall x_i = 0, \ \forall y_i \text{ arbitrary} \} = \mathbb{P}^{n-1}$, $\dim = n-1$.

Note that we don't need all the equations in $(*)$:

$$\left.\begin{array}{l} x_1 y_2 = x_2 y_1 \\ x_1 y_3 = x_3 y_1 \end{array}\right\} \Rightarrow x_2 y_3 = x_3 y_2$$

The $n-1$ equations $x_1 y_i = y_1 x_i \quad (i = 2, \dots, n)$ are necessary and sufficient.

$\Rightarrow \pi^{-1}(0)$ is a codim 1 closed subvariety, hence a Weil divisor.

We won't check the universal property.

Let $P \in A^n \setminus \{0\}$, $P = (a_1, \dots, a_n)$.

$$\pi^{-1}(P) = \{ (y_1, \dots, y_n) \in \mathbb{P}^{n-1} \mid a_i y_j = a_j y_i \Rightarrow \underbrace{y_j = \left(\frac{a_j}{a_i}\right) y_i}_{\text{this fixes } (y_0, \dots, y_n)} \} = \text{one point}$$

Ex. $y^2 = x^2(x+1)$ which passes through $O = (0,0)$, denoted by C.

Def. Let Y be a closed subvariety of A^n passing through O. The blowup of Y at O is given by $\tilde{Y} = \overline{\pi^{-1}(Y \setminus \{0\})}$ where $\pi : Bl_0(A^n) \longrightarrow A^n$ is the blowup of A^n at O.

Ex. (cont.) $\pi : Bl_0(A^2) \longrightarrow A^2$ where $Bl_0(A^2)$ is given by $xu = yt$ for $(x, u), (y, t) \in A^2 \times \mathbb{P}^1$

Now $\pi^{-1}(Y)$ is given by $\begin{cases} xu = yt \\ y^2 = x^2(x+1) \end{cases}$ in $A^2 \times \mathbb{P}^1$

$\|$

$\pi^{-1}(Y \setminus \{0\}) \cup \pi^{-1}(0) \simeq Bl_0(C)$

$\mathbb{P}^1$ is covered by the charts $t \neq 0$ and $u \neq 0$

Suppose $t \neq 0$. When $t=1$, use u as an affine parameter $(\mathbb{P}^1 \setminus \{t=0\} \simeq \mathbb{A}^1)$

Now $\pi^{-1}(Y)$ is given by $\begin{cases} xu = y \\ y^2 = x^2(x+1) \end{cases}$

$$\Leftrightarrow \begin{cases} xu = y \\ 0 = x^2u^2 - x^2(x+1) = x^2\left(u^2 - (x+1)\right) \end{cases}$$

$$\Leftrightarrow x = y = 0 \quad \text{OR} \quad \left\{ \begin{matrix} u^2 = x+1 \\ xu = y \end{matrix} \right\} = \tilde{Y} = Bl_0(Y)$$

Thm. (Manin 24.4) S a del Pezzo surface over $\overline{\mathbb{Q}}$.

Then either $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$

or S is the blowup of $\mathbb{P}^2$ at r points in general position

where $0 \leq r \leq 8$.

General position: no 3 of them lie on a line,

no 6 of them lie on a conic,

no 8 of them lie on a cubic.

Prop. V smooth, $P \in V$ point.

$$\Rightarrow \operatorname{Pic}\left(Bl_P(V)\right) \simeq \operatorname{Pic}(V) \oplus \mathbb{Z}$$

PF (SKETCH): Let $\pi : \tilde{V} = Bl_P(V) \longrightarrow V$

$$\Rightarrow \pi : \tilde{V} \setminus \pi^{-1}(P) \xrightarrow{\sim} V \setminus P$$

In lec 3 we have seen

$$0 \to \operatorname{Pic}(V) \xrightarrow{\sim} \operatorname{Pic}(V \setminus P) \to 0$$
$$\sum n_Y Y \longmapsto \sum n_Y (Y \cap P)$$

On the other hand:

$$\underbrace{0 \longrightarrow \mathbb{Z}}_{\text{uses intersection thy}} \longrightarrow \operatorname{Pic} \tilde{V} \longrightarrow \underbrace{\operatorname{Pic}(\tilde{V} \setminus \pi^{-1}(P))}_{= \operatorname{Pic} V} \longrightarrow 0$$

Using π^*: $\operatorname{Pic} V \longrightarrow \operatorname{Pic} \tilde{V}$

$$\sum n_Y Y \longmapsto \sum n_Y \pi^{-1}(Y)$$

we split this s.e.s.

$$\Rightarrow \operatorname{Pic} \tilde{V} \cong \operatorname{Pic} V \oplus \mathbb{Z}. \qquad \square$$

<u>Prop.</u> S a del Pezzo surface $\overset{\text{over } \overline{\mathbb{Q}}}{,}$ blowup of r pts in general position.

$\Rightarrow \operatorname{Pic}(S) \cong \mathbb{Z}^{1+r}$ (Manin even gives an explicit basis)

<u>Ex.</u> $[1,0,0], [1,i,0], [1,-i,0]$

<u>Thm.</u> S a del Pezzo surface, blowup of $P_1 = [x_{11}, \dots, x_{1n}], \dots, P_r = [x_{r1}, \dots, x_{rn}]$.

Let K be the field obtained by adjoining all the coordinates of $P_1, \dots, P_r$.

Then $\operatorname{Pic}(S) \cong \left(\operatorname{Pic}_{\overline{\mathbb{Q}}}(S)\right)^{\operatorname{Gal}(K/\mathbb{Q})}$

Blowup of r pts, $r = 0, 1, \dots, 8$. Manin's conjecture is proven for $r \leq 3$

(Batyrev, Tschinkel, Chambert-Loir, ...)

For $r = 0$, we just have $\mathbb{P}^2$.

$r = 4$: 2 examples: La Bretèche – Fouvry, Batyrev – Popov

for which the conj is proven

$[1,0,0], [0,1,0]$,	$[1,0,0], [0,1,0]$,
$[0,0,1], [1,1,1]$	$[1,1,i], [1,1,-i]$

$r = 5$: only 1 example La Bretèche – Browning (2011), two quadrics in $\mathbb{P}^4$

$r = 6, 7, 8$: no example is known.

Generalised del Pezzo surf: d.P. surf + isolated singularities.

Resolution of singularities: $\pi: \tilde{X} \to X$, $\tilde{X}$ smooth, birational.

<u>Thm.</u> S gen. del Pezzo surface. $\Rightarrow \exists$ minimal resolution of singularities for S, i.e. every other resolution factors through it.

$$\begin{tikzcd} & \tilde{S} \arrow[d, "\pi"] \\ X \arrow[ur, dashed, "\exists!"] \arrow[r, "\varphi"'] & S \end{tikzcd}$$

Interactions..., lecture 4–5

$\rightarrow$ we obtain more examples for $r = 4, 5, 6$, still nothing for $r = 7, 8$.

$\pi: \tilde{S} \to S$ $\quad \# \{ x \in U(\mathbb{Q}) \mid H(x) \le B \} = \# \{ y \in \tilde{U}(\mathbb{Q}) \mid H \circ \pi(y) \le B \}$

$\tilde{U} \simeq U$

09.05.2018

Let S be the surface in $\underset{[x_1, y_1]\ [x_2, y_2]\ [x_3, y_3]}{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$ given by $x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_2 = 0$

This is a generalised del Pezzo surface with $r = 3$.

Singularities: the Jacobian is the following

$$\begin{array}{cccccc} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ \end{array}$$
$$\begin{pmatrix} y_2 y_3 & x_2 y_3 + x_3 y_2 & y_1 y_3 & x_1 y_3 + x_3 y_1 & y_1 y_2 & x_1 y_2 + x_2 y_1 \end{pmatrix}$$

Singularity $\Leftrightarrow \nexists$ 1x1 nonzero det $\Leftrightarrow$ every entry is 0.

$$\begin{cases} y_2 y_3 = y_2 y_1 = y_1 y_3 = 0 \quad \Leftrightarrow \text{ 2 of the 3 are } 0, \text{ wlog } y_1 = y_2 = 0 \\ \underbrace{x_2 y_3 + x_3 y_2}_{0} = \underbrace{x_1 y_3 + y_3 y_1}_{0} = \underbrace{x_1 y_2 + x_2 y_1}_{0} = 0 \end{cases}$$

$x_1, x_2 \ne 0$ (since we are in $\mathbb{P}^1$) $\Rightarrow y_3 = 0$, the only singularity is $P = [1,0], [1,0], [1,0]$.

Computation of $\mathrm{Pic}(S)$: $\{y_1 = y_2 = 0\}$ is a line in S, explicitely this is $[1,0] \times [1,0] \times \mathbb{P}^1 = L_1$

Similarly $[1,0] \times \mathbb{P}^1 \times [1,0] = L_2$ and $L_3 = \mathbb{P}^1 \times [1,0] \times [1,0]$ are lines in S as well.

$$0 \to \underset{\text{divisor}}{\{D = n_1 L_1 + n_2 L_2 + n_3 L_3\}} \longrightarrow \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(S \setminus L_1 \cup L_2 \cup L_3) \longrightarrow 0$$

since $\mathrm{supp}\, D \subseteq L_1 \cup L_2 \cup L_3$

Using intersection thy, $\{D = n_1 L_1 + n_2 L_2 + n_3 L_3\} \simeq \mathbb{Z}^3$

L_1, L_2, L_3 indep.

Now we compute $\mathrm{Pic}(S \setminus L_1 \cup L_2 \cup L_3)$.

In $S \setminus L_1 \cup L_2 \cup L_3$ there are $i \neq j$ s.t. $y_i \neq 0 \neq y_j$

Wlog: $i=1$, $j=2$.

Then $y_3 \neq 0$. ∄ Otherwise $0 + 0 + \underbrace{x_3}_{\neq 0} \underbrace{y_1}_{\neq 0} \underbrace{y_2}_{\neq 0} = 0$. ↯

$\Rightarrow$ $S \setminus L_1 \cup L_2 \cup L_3$ is the open subset of S given by $y_1 y_2 y_3 \neq 0$

Rewrite the equation: $\frac{x_1}{y_1} = - \frac{x_2}{y_2} - \frac{x_3}{y_3}$

The map

$$S \setminus L_1 \cup L_2 \cup L_3 \xrightarrow{\sim} \mathbb{A}^2 \quad \text{is thus an iso.}$$
$$[x_1, y_1], [x_2, y_2], [x_3, y_3] \longmapsto \left(\frac{x_2}{y_2}, \frac{x_3}{y_3} \right)$$

But $\mathrm{Pic}(\mathbb{A}^2) = 0$ $\Rightarrow$ $\mathrm{Pic}(S \setminus L_1 \cup L_2 \cup L_3) = 0$ $\Rightarrow$ $\mathrm{Pic}(S) \simeq \mathbb{Z}^3$.

$\Rightarrow$ $\mathrm{Pic}(\tilde{S}) \simeq \mathbb{Z}^4$ if $\tilde{S} \to S$ is the minimal resolution of S.

Thus we expect $(\log B)^3$ in Manin's conjecture.

Height

Segre embedding: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \overset{\varphi}{\hookrightarrow} \mathbb{P}^7$

$$[x_1, y_1] \; [x_2, y_2] \; [x_3, y_3] \longmapsto [x_1 x_2 x_3, \; x_1 x_2 y_3, \; x_1 y_2 x_3, \; x_1 y_2 y_3, \; y_1 x_2 x_3, \; y_1 x_2 y_3, \; y_1 y_2 x_3, \; y_1 y_2 y_3]$$

The img of S, $\varphi(S)$ is given by the intersection of 9 quartics and one linear equation.

$H\left([x_1, y_1], [x_2, y_2], [x_3, y_3]\right) :=$ max of abs values of the coordinates of $\varphi(S)$.

The number of $x_1, y_1, x_2, y_2, x_3, y_3 \in \mathbb{Z}^6$ satisfying $\prod_{i=1}^{3} \max(|x_i|, |y_i|) \leq B$ is roughly

$$\sum_{|x_1|, |y_1| \leq B} \; \sum_{|x_2|, |y_2| \leq \frac{B}{\max(|x_1|, |y_1|)}} \; \sum_{|x_3|, |y_3| \leq \frac{B}{\max(|x_1|, |y_1|, |x_2|, |y_2|)}} 1 \approx B^2 \sum_{x_1 \leq y_1 \leq B} \frac{1}{y_1^2} \sum_{x_2 \leq y_2 \leq \frac{B}{y_1}} \frac{1}{y_2^2}$$

$$\sum_{y_2 \leq \frac{B}{y_1}} \frac{1}{y_2^2} \sum_{x_2 \leq y_2} 1 = \sum_{y_2 \leq \frac{B}{y_1}} \frac{1}{y_1} \approx \log B$$

$$B^2 (\log B)^2 \approx B^{2+\varepsilon}$$

Interactions ..., lecture 5

$x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 \approx B$

$\Rightarrow \mathbb{P}(\text{---}\ ''\ \text{---} = 0) \approx B^{-1}$

Heuristically we get $\frac{B^{2+\varepsilon}}{B} = B^{1+\varepsilon}$. Neglecting the logs, we get B.

Thm. Let $U = S \setminus L_1 \cup L_2 \cup L_3$ and $N_0(B) = \# \{ (\underline{x}, \underline{y}) \in U(\mathbb{Q}) \mid H(\underline{x}, \underline{y}) \le B$
$x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 = 0 \}$

Then there exist $c_1, c_2 \ge 0$ s.t. $N_0(B) = c_1 B(\log B)^3 + c_2 B (\log B)^2 + O(B \log B)$.

where $c_1 = \frac{c_2}{144} \prod_p \left(1 - \frac{1}{p}\right)^4 \cdot \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)$.

This will take a LONG time to prove.

Previous approach: $(x_1, \dots, x_{n+1}) \in \mathbb{Z}^{n+1}$ with $\gcd(x_1, \dots, x_{n+1}) = 1$

$\mathbb{P}^n(\mathbb{Q})$

These parametrisations are called torsors.

(Literature: Skorobogatov: Torsors and rational points).

Parametrisation:

for $[x_i : y_i]$ there are 2 representatives with $\gcd(x_i, y_i) = 1$

$N_0(B) = \frac{1}{8} \# \{ (\underline{x}, \underline{y}) \in \mathbb{Z}^6 \mid \gcd(x_i, y_i) = 1, \prod_{i=1}^{3} \max(|x_i|, |y_i|) \le B, y_1y_2y_3 \neq 0, x_1y_2y_3 + \dots = 0 \}$

On $y_1y_2y_3 \neq 0$ the equation becomes $\frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} = 0$

If $y_1 < 0$: $\frac{x_1}{y_1} = \frac{-x_1}{-y_1}$. $\Rightarrow$ Up to multiplication by 8, we can ensure that $y_1, y_2, y_3 > 0$.

$\Rightarrow N_0(B) = \# \{ \underline{x} \in \mathbb{Z}^3, \underline{y} \in \mathbb{Z}^3_{>0} \mid \gcd(x_i, y_i) = 1, \prod_{i=1}^{3} \max(|x_i|, |y_i|) \le B, \frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} = 0 \}$

Contribution of $x_1x_2x_3 = 0$

If $x_3 = 0, y_3 = 1 \longrightarrow x_2y_1 + x_1y_2 = 0$

$\Leftrightarrow x_2y_1 = -x_1y_2$ & $\gcd(x_2, y_2) = \gcd(x_1, y_1) = 1$

$\Rightarrow x_2 \mid x_1, y_2 \mid y_1 \Rightarrow y_1 = y_2, x_1 = -x_2$

$$H(\underline{x}, \underline{y}) = \max(|x_2|, |y_2|)^2 \leq B$$

$$\begin{pmatrix}\text{Contribution} \\ \text{of } x_1x_2x_3=0\end{pmatrix} = 3 \sum_{\substack{|x_2|, y_2 \leq B^{1/2} \\ \gcd(x_2, y_2) = 1}} 1 = 3 \sum_{|x_2|, y_2 \leq B^{1/2}} \sum_{k | x_2, y_2} \mu(k) \qquad \text{Möbius inversion}$$

$$= 3 \sum_{k \leq B^{1/2}} \mu(k) \underbrace{\sum_{\substack{|x_2|, y_2 \leq B^{1/2} \\ k | x_2, y_2}} 1}_{} =$$

$$= \sum_{|x_2'|, y_2' \leq \frac{B^{1/2}}{k}} 1 = \frac{2B}{k^2} + O(1) \qquad \text{since } \lfloor x \rfloor = x + O(1)$$

$$= 6B \sum_{k \leq B^{1/2}} \frac{\mu(k)}{k^2} + O(B^{1/2})$$

$$= 6 \cdot B \underbrace{\sum_{k} \frac{\mu(k)}{k^2}}_{6/\pi^2} + O(B^{1/2}) = O(B) = \theta(B \log B)$$

Since we expect a $B \log B$-order sum, the above can be neglected. I.e. we may neglect the case $x_1 x_2 x_3 = 0$, i.e. when $x_1 x_2 x_3 \neq 0$.

General idea. $x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3 = 0$

If $p \mid y_1, y_2, y_3$ then p^2 divides each term. This enlarges the terms unnecessarily $\longrightarrow$ we want to get rid of it. by parametrisation.

We will have an equation $x_1 z_1 + x_2 z_2 + x_3 z_3 = 0$, the solutions of which can be counted by counting solutions to

$$x_1 z_1 + x_2 z_2 \equiv 0 \pmod{z_3}.$$

So this is the plan.

Parametrisation: Use a unique factorisation of (y_1, y_2, y_3)

Let $z_0 := \gcd(y_1, y_2, y_3)$,

$z_3 := \gcd\left(\frac{y_1}{z_0}, \frac{y_2}{z_0}\right), \quad z_2 := \gcd\left(\frac{y_1}{z_0}, \frac{y_3}{z_0}\right), \quad z_1 := \gcd\left(\frac{y_2}{z_0}, \frac{y_3}{z_0}\right),$

$z_3' := \frac{y_3}{z_0 z_1 z_2}, \quad z_2' := \frac{y_2}{z_0 z_1 z_3}, \quad z_1' := \frac{y_1}{z_0 z_2 z_3}$

common factors in all three

common factors of y_i, y_j

factors only in y_i

$$\Rightarrow \quad y_1 = z_0 z_1' z_2 z_3$$
$$y_2 = z_0 z_1 z_2' z_3$$
$$y_3 = z_0 z_1 z_2 z_3'$$

$$x_1 y_2 y_3 + \underbrace{y_1 x_2 y_3}_{z_1' \mid} + \underbrace{y_1 y_2 x_3}_{z_1' \mid} = 0$$

$\Rightarrow z_1' \mid x_1 y_2 y_3 \Rightarrow z_1' \mid x_1$. But $\gcd(x_1, y_1) = 1 \Rightarrow z_1' = 1$.

Similarly: $z_1' = z_2' = z_3' = 1$.

We have $\gcd(z_i, z_j) = 1 \ \forall 1 \leq i \neq j \leq 3$ by def.

Thus we have a 1:1-correspondence between (y_1, y_2, y_3) and (z_0, z_1, z_2, z_3) satisfying $\gcd(z_i, z_j) = 1 \ \forall 1 \leq i \neq j \leq 3$

$\longrightarrow$ Instead of counting $\underline{y}$ we will count $\underline{z}$.

One can check that the equation for $\underline{z}$ becomes

$$x_1 z_1 + x_2 z_2 + x_3 z_3 = 0$$

We have shown

$$M(B) = \#\left\{ x \in \mathbb{Z}^3_{\neq 0}, z \in \mathbb{Z}^4_{>0} \;\middle|\; \begin{array}{l} x_1 z_1 + x_2 z_2 + x_3 z_3 = 0 \\ \gcd(z_i, z_j) = 1 \quad \forall 1 \leq i \neq j \leq 3 \\ \gcd(x_i, z_0 z_j z_k) = 1 \quad \forall \{i,j,k\} = \{1,2,3\} \\ \prod \max(|x_i|, z_0 z_j z_k) \leq B \end{array} \right\}$$

Height condition

$$\max\Big\{ |s_0^2 s_1^2 s_2^2 s_3^2|, \ |x_1 x_2 x_3|, \ |x_1 x_2 s_0 s_1 s_2|, \ |x_1 x_3 s_0 s_1 s_3|, \ |x_2 x_3 s_0 s_2 s_3|,$$
$$|s_0^2 s_1^2 s_2 s_3 x_1|, \ |s_0^2 s_2^2 s_1 s_3 x_2|, \ |s_0^2 s_3^2 s_1 s_2 x_3| \Big\}$$

$$s_0 s_1^2 x_1^2 = x_1 y_2 y_3 \times \frac{x_1}{y_1}$$

$$|x_1 x_2 s_0 s_1 s_2| = \sqrt{|s_0 s_1^2 x_1^2|}\sqrt{|s_0 s_2^2 x_2^2|} \leq B$$

$$|s_0^2 s_1^2 s_2 s_3 x_1| = \sqrt{|s_0^2 s_1^2 s_2^2 s_3^2|}\sqrt{|s_0 x_1^2 s_1^2|}$$

Hence the height condition can be replaced by

$$\psi(x,s) := \max\Big\{ |s_0^2 s_1^2 s_2^2 s_3^2|, |x_1x_2x_3|, |s_0 s_1^2 x_1^2|, |s_0 s_2^2 x_2^2|, |s_0 s_3^2 x_3^2| \Big\} \leq B.$$

We will be interested in the following surface S. 05.06.2018

$$x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_3 = 0 \quad \text{in} \quad \underset{[x_1,y_1]}{\mathbb{P}^1} \times \underset{[x_2,y_2]}{\mathbb{P}^1} \times \underset{[x_3,y_3]}{\mathbb{P}^1}$$

We have seen: rk $\mathrm{Pic}(S) = 3$, S is a singular del Pezzo surface.

One singular point: $([1,0],[1,0],[1,0])$

If $\tilde{S}$ is the minimal desingularisation of S, rk $\mathrm{Pic}(\tilde{S}) = 4$.

To measure the size of a rational pt in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we use Segre's embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^7$$
$$([x_1,y_1][x_2,y_2][x_3,y_3]) \longmapsto [x_1x_2x_3, x_1y_2x_3, x_1x_2y_3, x_1y_2y_3,$$
$$y_1x_2x_3, y_1y_2x_3, y_1x_2y_3, y_1y_2y_3]$$

We will then use the following height:

$$H(\underline{x},y) = \prod_{i=1}^{3} \min(|x_i|, |y_i|) \quad \text{for } (x_i, y_i) \text{ coprime integers.}$$

3 lines in S: $y_1 = y_2 = 0$, $y_1 = y_3 = 0$, $y_2 = y_3 = 0$. $\longrightarrow$ contribute too much!

So we want to look at

$$N(B) = \#\left\{ ([x_1,y_1],[x_2,y_2],[x_3,y_3]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; \begin{array}{l} x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 = 0 \\ y_1y_2y_3 \neq 0 \text{ (exclude the lines)} \\ H(\underline{x},y) \leq B \end{array} \right\}$$

$$N(B) = \frac{1}{2^3} \# \left\{ (x_1,x_2,x_3,y_1,y_2,y_3) \in \mathbb{Z}^6 \,\middle|\, \begin{array}{l}\forall \\ \gcd(x_i,y_i)=1,\end{array} \ x_1y_2y_3+\dots=0,\ y_1y_2y_3\neq 0,\ H(\underline{x},\underline{y})\leq B \right\}$$

The heuristic predicts $N(B) \underset{B\to\infty}{\sim} B(\log B)^3$. The objective of the remaining lectures is to prove this. More precisely we wts:

Thm. $\exists c_1, c_2$ constants for which

$$N(B) = c_1 B (\log B)^3 + c_2 B(\log B)^2 + O(B \log B)$$

Rmk. There is a refined version of Manin's conjecture:

$$N_0(B) = B^{n+1-d} P(\log B) + O(B^{1-\delta})$$

where $0<\delta<1$ and P is a polynomial of degree $\operatorname{rk}(\operatorname{Pic} V)-1$.

We showed that

$$N(B) = 2\,\# \left\{ s_0,s_1,s_2,s_3,x_1 \in \mathbb{Z}^5_{>0},\ x_2,x_3\in\mathbb{Z}^2 \,\middle|\, \begin{array}{l} s_1x_1+s_2x_2+s_3x_3=0 \\ \gcd(x_i, s_0s_js_k)=1 \quad \forall\{i,j,k\}=\{1,2,3\} \\ \gcd(s_i,s_j)=1 \quad \forall i\neq j \in\{1,2,3\} \\ \max\big(\underbrace{|x_1x_2x_3|}_{①}, \underbrace{|s_0^3s_1^2s_2^2s_3^2|}_{②}, \underbrace{|s_0s_1^2x_1^2|}_{③}, \underbrace{|s_0s_2^2x_2^2|}_{④}, \underbrace{|s_0s_3^2x_3^2|}_{⑤}\big)\leq B \end{array}\right\} + O(B)$$

$$= 2\,T(B) + O(B)$$

Goal: $T(B) = B^{2/3} \sum_{n\leq B} \Delta(n)\cdot F\left(\frac{n}{B}\right)$ where Δ is multiplicative and F is a regular function. (cont. diff.)

This can be tackled using anal. n.t.: use partial summation, it is enough to study $\sum_{n\leq B} \Delta(n)$.

(Note: today's lecture is rather technical, the next one is supposed to contain more general principles.)

1st step. Rewrite the height condition.

$$X_0 := \left(\frac{s_0^3 s_1^2 s_2^2 s_3^2}{B}\right)^{1/3} \qquad \text{Then } ② \iff X_0 \leq 1$$

For $i = 1,2$ let $X_i := \left(\dfrac{B\, s_1 s_2 s_3}{s_i^3}\right)^{1/3}$. Then ③ $\Longleftrightarrow$ $f_1(x_1) = X_0 \left(\dfrac{x_1}{X_1}\right)^2 \leq 1$

④ $\Longleftrightarrow$ $f_2(x_2) = X_0 \left(\dfrac{x_2}{X_2}\right)^2 \leq 1$

Moreover ①' $\Longleftrightarrow$ $\left| x_1 x_2 \dfrac{s_1 x_1 + s_2 x_2}{x_3} \right| \leq B \Longleftrightarrow \dfrac{x_1 x_2}{X_1 X_2}\left(\dfrac{x_1}{X_1} + \dfrac{x_2}{X_2}\right) \leq 1$

⑤ $\Longleftrightarrow$ $\left| s_0 (s_1 x_1 + s_2 x_2)^2 \right| \leq B \Longleftrightarrow X_0 \left(\dfrac{x_1}{X_1} + \dfrac{x_2}{X_2}\right)^2 \leq 1$

① + ⑤ $\Longleftrightarrow$ $g(x_1, x_2) = \max\left(\dfrac{x_1 x_2}{X_1 X_2}\left(\dfrac{x_1}{X_1} + \dfrac{x_2}{X_2}\right),\ X_0 \left(\dfrac{x_1}{X_1} + \dfrac{x_2}{X_2}\right)^2 \right) \leq 1$

Putting $S := \left\{ (s_0, s) \in \mathbb{Z}^4_{>0} \;\middle|\; \gcd(s_i, s_j) = 1 \quad \forall i \neq j \in \{1,2,3\},\ X_0 \leq 1 \right\}$

we see that

$$T(B) = \sum_{(s_0, s) \in S} \# \left\{ \begin{matrix} x_1 \in \mathbb{Z}_{>0} \\ x_2, x_3 \in \mathbb{Z}_{\neq 0} \end{matrix} \;\middle|\; \begin{matrix} s_1 x_1 + s_2 x_2 + s_3 x_3 = 0, \quad \gcd(x_i, s_0 s_j s_k) = 1, \\ |f_1(x_1)| \leq 1, \quad |f_2(x_2)| \leq 1, \quad |g(x_1, x_2)| \leq 1 \end{matrix} \right\}$$

The main idea is to replace $s_1 x_1 + s_2 x_2 + s_3 x_3 = 0$ by $s_1 x_1 + s_2 x_2 \equiv 0 \ (s_3)$.

$\longrightarrow$ this automatically takes care of the summation over x_3.

The trouble is that there is a coprimality condition involving x_3,

but this may be removed by Möbius inversion and requiring $\gcd(x_3, s_0 s_1 s_2) = 1$.

Möbius:

$$T(B) = \sum_{(s_0, s) \in S} \sum_{x_3 \in \mathbb{Z}_{\neq 0}} \underbrace{\mathbb{1}_{\gcd(x_3,\, s_0 s_1 s_2) = 1}}_{\sum_{\substack{k_3 | x_3 \\ k_3 | s_0 s_1 s_2}} \mu(k_3)} \# \left\{ x_1, x_2 \neq 0 \;\middle|\; \begin{matrix} s_1 x_1 + s_2 x_2 + s_3 x_3 = 0 \\ \gcd(x_i, s_0 s_j s_k) = 1 \quad i = 1,2 \\ |f_i(x_i)| \leq 1, \quad |g(x_1, x_2)| \leq 1 \end{matrix} \right\}$$

$$= \sum_{(s_0, s) \in S} \sum_{k_3 | s_0 s_1 s_2} \mu(k_3)\, \# \left\{ \begin{matrix} x_1 \in \mathbb{Z}_{>0} \\ x_2, x_3 \in \mathbb{Z}_{\neq 0} \end{matrix} \;\middle|\; \begin{matrix} s_1 x_1 + s_2 x_2 + s_3 x_3 = 0 \\ \gcd(x_i, s_0 s_j s_k) = 1 \quad \forall i = 1,2 \\ |f_i(x_i)| \leq 1, \quad |g(x_1, x_2)| \leq 1, \quad \underbrace{x_3 \equiv 0 \ (k_3)}_{x_3 = k_3 x_3'} \end{matrix} \right\}$$

$$= \sum_{(s_0, s) \in S} \sum_{k_3 | s_0 s_1 s_2} \mu(k_3)\, \# \left\{ \begin{matrix} x_1 \in \mathbb{Z}_{>0} \\ x_2, x_3 \in \mathbb{Z}_{\neq 0} \end{matrix} \;\middle|\; \underbrace{\begin{matrix} x_1 s_1 + x_2 s_2 + s_3 k_3 x_3' = 0 \\ \gcd(x_i, s_0 s_j s_k) = 1 \quad \forall i = 1,2 \\ |f_i(x_i)| \leq 1, \quad |g(x_1, x_2)| \leq 1 \end{matrix}}_{(*)} \right\}$$

Counting the solutions of (#) is the same as counting

$$\#\left\{ \begin{matrix} x_1 \in \mathbb{Z}_{>0} \\ x_2 \in \mathbb{Z}_{\neq 0} \end{matrix} \;\middle|\; \begin{matrix} x_1 s_1 + x_2 s_2 \equiv 0 \pmod{k_3 s_3} \\ \gcd(x_i, s_0 s_j s_3) = 1 \quad \forall i = 1,2 \\ |f_i(x_i)| \leq 1, \; |g(x_1, x_2)| \leq 1 \end{matrix} \right\} =: S_{k_3}(B)$$

$$\Rightarrow \quad T(B) = \sum_{(s_0, s) \in S} \; \sum_{k_3 \mid s_0 s_1 s_2} \mu(k_3)\, S_{k_3}(B)$$

Rmk. We have $\gcd(k_3, s_1 s_2) = 1$:

↯ If $p \mid k_3$ and $p \mid s_1 \Rightarrow p \mid x_2 s_2$. But $p \nmid x_2$, $p \nmid s_2$. ↯

Same for $p \mid k_3$ and $p \mid s_2$.

Idea. If we rewrite the condition $x_1 \equiv * \pmod{k_3 s_3}$ we will be able to use the following:

Lemma. $\#\{ n \in]a, b] \cap \mathbb{Z} \mid n \equiv n_0 \pmod{q} \} = \frac{b-a}{q} + O(1)$

Pf: $\#\{ n \in]a,b] \cap \mathbb{Z} \mid n \equiv n_0 \pmod{q} \} = \#\{ n \in]a,b] \cap \mathbb{Z} \mid n = n_0 + kq \}$

$$= \#\{ k \in \mathbb{Z} \mid a < n_0 + kq \leq b \} = \#\left\{ k \in \mathbb{Z} \;\middle|\; \frac{a-n_0}{q} < k \leq \frac{b-n_0}{q} \right\}$$

$$= \left[\frac{b-n_0}{q}\right] - \left[\frac{a-n_0}{q}\right] = \frac{b-n_0}{q} + O(1) - \left(\frac{a-n_0}{q} + O(1)\right) = \frac{b-a}{q} + O(1)$$

□

$$x_1 s_1 + x_2 s_2 \equiv 0 \pmod{k_3 s_3} \iff x_1 \equiv -\overline{s_1}\, s_2 x_2 \pmod{k_3 s_3}$$

Möbius inversion to remove $\gcd(x_1, s_0 s_2 s_3) = 1$

$$S_{k_3}(B) = \sum_{k_1 \mid s_0 s_2 s_3} \mu(k_1)\, \#\left\{ \begin{matrix} x_1' \in \mathbb{Z}_{>0} \\ x_2 \in \mathbb{Z}_{\neq 0} \end{matrix} \;\middle|\; \begin{matrix} s_1 k_1 x_1' + s_2 x_2 \equiv 0 \pmod{k_3 s_3} \\ \gcd(x_2, s_0 s_1 s_3) = 1 \\ |f_2(x_2)| \leq 1, \; |f_1(k_1 x_1')| \leq 1, \; |g(k_1 x_1' x_2)| \leq 1 \end{matrix} \right\}$$

As before, one can check that $\gcd(k_1, k_3 s_3) = 1$.

$s_1 k_1$ has (unique) inverse in $\mathbb{Z}/k_3 s_3 \mathbb{Z}$ denoted $\overline{s_1 k_1}$.

$$S_{k_3}(B) = \sum_{\substack{k_1 \mid s_0 s_2 \\ \gcd(k_1, s_3 k_3)=1}} \mu(k_1)\, S_{k_1,k_3}(B) \quad \text{where}$$

$$S_{k_1,k_3}(B) = \# \left\{ \begin{array}{l} x_1' \in \mathbb{Z}_{>0} \\ x_2 \in \mathbb{Z}_{\neq 0} \end{array} \middle| \begin{array}{l} x_1' \equiv 0 \pmod{k_3 s_3}, \\ \gcd(x_2, s_0 s_1 s_3) = 1 \end{array} \quad \begin{array}{l} |f_2(x_2)| \le 1, \ |f_1(k_1 x_1')| \le 1, \\ |g(k_1 x_1', x_2)| \le 1 \end{array} \right\}$$

when x_2 is fixed, this is a disjoint union of intervals:

$$|f_1(x_1)| = \left| X_0 \left(\frac{k_1 x_1'}{X_1}\right)^2 \right| \le 1 \iff |x_1'| \le \frac{X_1}{k_1 X_0^{1/2}} \rightsquigarrow \text{interval}$$

$$|g(k_1 x_1', x_2)| = \max \left| \frac{k_1 x_1' \cdot x_2}{X_1 X_2} \left(\frac{k_1 x_1'}{X_1} + \frac{x_2}{X_2} \right),\ X_0 \cdot \left(\frac{k_1 x_1'}{X_1} + \frac{x_2}{X_2} \right) \right| \le 1 \rightsquigarrow \text{intervals}$$

The lemma yields $\# \{ n \in \mathcal{I} \cap \mathbb{Z} \mid n \equiv n_0 \pmod{q} \} = \frac{\mathrm{vol}(\mathcal{I})}{q} + O(1)$ when $\mathcal{I}$ is a finite union of disjoint intervals.

$$\Rightarrow S_{k_1,k_3}(B) = \sum_{\substack{x_2 \in \mathbb{Z}_{\neq 0} \\ \gcd(x_2, s_0 s_1 s_3) = 1 \\ |f_2(x_2)| \le 1}} \left(\frac{1}{k_3 s_3} \int_{\left\{ x_1' > 0 \,\middle|\, \begin{array}{l} |f_1(k_1 x_1')| \le 1 \\ |g(k_1 x_1', x_2)| \le 1 \end{array} \right\}} dx_1' + O(1) \right)$$

Putting $t = \frac{k_1 x_1'}{X_1}$ we get

$$S_{k_1,k_3}(B) = \underbrace{\sum_{\substack{x_2 \in \mathbb{Z}_{\neq 0} \\ \gcd(x_2, s_0 s_1 s_3) = 1 \\ |f_2(x_2)| \le 1}} \left(\frac{X_1}{k_1 k_3 s_3} F_1\left(X_0, \frac{x_2}{X_2}\right) + O(1) \right)}_{\text{this will be the main term, to be dealt with next week}}$$

where $F_1(u, v) = \displaystyle\int_{\{ t > 0 \mid |u t^2| \le 1,\ |t v (t + v)| \le 1,\ |u (t + v)| \le 1 \}} dt$

Interactions..., lecture 6-7

$$\text{Error term} \ll \sum_{(s_0, \underline{s}) \in \mathfrak{S}} \sum_{k_3 | s_0} \underbrace{|\mu(k_3)|}_{\leq 1} \sum_{k_1 | s_0 s_2} \underbrace{|\mu(k_1)|}_{\leq 1} \underbrace{\sum_{\substack{x_2 \in \mathbb{Z}_{\neq 0} \\ X_0 \left(\frac{x_2}{X_2}\right) \leq 1}} 1}_{\frac{X_2}{X_0^{1/2}}}$$

$$\ll \sum_{s_0, \underline{s}} \tau(s_0)\, \tau(s_0 s_2)\, \frac{X_2}{X_0^{1/2}} \qquad \tau(n) = \sum_{d | n} 1$$

$$\ll B^{1/2} \sum_{s_0^3 s_1^2 s_2^2 s_3^2 \leq B} \frac{\tau(s_0)\, \tau(s_0 s_2)}{s_0^{1/2}\, s_2}$$

$$\ll B^{1/2} \sum_{\substack{s_0 \leq B^{1/3} \\ s_1, s_2 \leq B^{1/2}}} \frac{\tau(s_0)\, \tau(s_0 s_2)}{s_0^{1/2}\, s_2} \underbrace{\sum_{s_3 \leq \frac{B^{1/2}}{s_0^{3/2} s_1 s_2}} 1}_{\frac{B^{1/2}}{s_0^{3/2} s_1 s_2}}$$

$$\ll B \sum_{\substack{s_0 \leq B^{1/3} \\ s_1, s_2 \leq B^{1/2}}} \frac{\tau(s_0)\, \tau(s_0 s_2)}{s_0^2\, s_1\, s_2^2} \underset{\text{sum over } s_1}{\ll} B \log B \sum_{\substack{s_0 \leq B^{1/3} \\ s_2 \leq B^{1/2}}} \frac{\tau(s_0)\, \tau(s_0 s_2)}{s_0^2\, s_2^2}$$

$\forall \varepsilon > 0 : \quad \tau(n) \ll_\varepsilon n^\varepsilon$ (Tenenbaum, Chapter maximal order) $\qquad \varepsilon := 1/2$

$$\ll \frac{1}{s_0^{3/2}\, s_2^3}$$

$$\ll B \log B$$

Want to estimate: 19.06.2018

$$\sum_{\substack{x_2 \in \mathbb{Z}_{\neq 0} \\ |x_2| \le X_2/\sqrt{X_0} \\ \gcd(x_2, \mathfrak{I}_0, \mathfrak{I}_1, \mathfrak{I}_3) = 1}} F_1\left(X_0, \frac{x_2}{X_2}\right)$$

<u>Lemma.</u> $A < B$, $A, B \in \mathbb{Z}$, $a \in \mathbb{Z}_{\geq 0}$, $f: \mathbb{R} \to \mathbb{R}$ cont diff

$$\Rightarrow \sum_{\substack{A \le n \le B \\ \gcd(n,a)=1}} f(n) = \varphi^*(a) \int_A^B f(t)\, dt + O\left(\tau(a) \sup_{t \in [A,B]} |f(t)|\right)$$

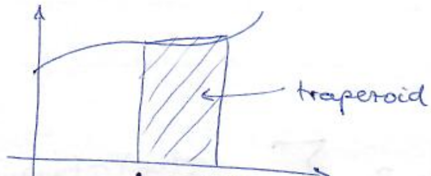
Note that φ^* is a multiplicative function where

$$\varphi^*(n) = \prod_{p | n} \left(1 - \frac{1}{p}\right)$$

Pf: Uses the Euler-Maclaurin formula in a special case.

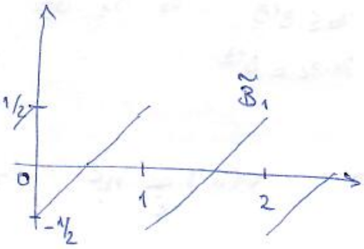
By Möbius inversion we have:

$$\sum_{\substack{A \le n \le B \\ \gcd(n,a)=1}} f(n) = \sum_{k|a} \mu(k) \sum_{\frac{A}{k} \le n \le \frac{B}{k}} f(kn)$$



Euler-Maclaurin:

$$\int_k^{k+1} f(t)\, dt = \frac{f(k) + f(k+1)}{2} + \text{error}$$



$\tilde{B}_1(x) := x - \lfloor x \rfloor - \frac{1}{2}$ (The notation comes from the fact that $B_1(x) = x - 1/2$ is the 1^{st} Bernoulli polynomial and the general EM-formula uses higher Bernoulli polynomials.)

$$\int_k^{k+1} f(t)\, dt = \Big[f(t)\, \tilde{B}_1(t) \Big]_k^{k+1} - \int_k^{k+1} \tilde{B}_1(t) f'(t)\, dt \qquad \text{int by parts}$$

$$= \frac{f(k) + f(k+1)}{2} - \int_k^{k+1} \tilde{B}_1(t) f'(t)\, dt$$

Interactions..., lecture 7

Summing over $k = A, \dots, B$:

$$\int_A^B f(t)\,dt = \sum_{A \le n \le B} f(n) - \frac{f(A)+f(B)}{2} - \int_A^B \widetilde{B}_1(t) f'(t)\,dt$$

$$\Rightarrow \sum_{A \le n \le B} f(n) = \int_A^B f(t)\,dt + \underbrace{\frac{f(A)+f(B)}{2}}_{\le \max_{[A,B]} |f(t)|} + \underbrace{\int_A^B \widetilde{B}_1(t) f'(t)\,dt}_{\le \frac{1}{2}\cdot 2 \cdot \max_{[A,B]} |f(t)|}$$

$$\Rightarrow \sum_{\frac{A}{k} \le n \le \frac{B}{k}} f(kn) = \underbrace{\int_{A/k}^{B/k} f(kt)\,dt}_{\frac{1}{k}\int_A^B f(t)\,dt} + \mathcal{O}\left(\max_{[A,B]} |f(t)|\right)$$

$$\sum_{k|a} \mu(k) \sum_{\frac{A}{k} \le n \le \frac{B}{k}} f(kt) = \int_A^B f(t)\,dt \cdot \underbrace{\sum_{k|a} \frac{\mu(k)}{k}}_{f(a):=} + \mathcal{O}\Big(\underbrace{\sum_{k|a} 1}_{\tau(a)} \cdot \max_{[A,B]} |f(t)|\Big)$$

multiplicative function of a because μ is multiplicative
(We like multiplicative functions.)

$$\varphi^*(a) = \prod_{p|a} \left(1 - \frac{1}{p}\right).$$ If $a = \prod_{p|a} p^{v_p(a)}$: $f(a) = \prod_{p|a} f\left(p^{v_p(a)}\right)$

$$f(p^v) = \sum_{k=0}^{v} \frac{\mu(p^k)}{p^k} = 1 - \frac{1}{p} \quad \Rightarrow \quad f(a) = \prod_{p|a}\left(1 - \frac{1}{p}\right).$$

This proves the lemma. $\square$

Applying the lemma yields:

$$\sum_{\substack{x_2 \in \mathbb{Z}_{\neq 0} \\ |x_2| \le X_2/X_0^{1/2} \\ \gcd(x_0, s_0, s_1, s_3) = 1}} F_1\left(X_0, \frac{x_2}{X_2}\right) = \varphi^*(s_0 s_1 s_3) \int_{|x_2| \le \frac{X_2}{X_0^{1/2}}} F_1\left(X_0, \frac{x_2}{X_2}\right) dx_2 + \mathcal{O}\left(\tau(s_0 s_1 s_3) \max_{|x_2| \le \frac{X_2}{X_0^{1/2}}} F_1\left(X_0, \frac{x_2}{X_2}\right)\right)$$

$v := x_2 / X_2$ gives:

$$X_2 \int_{v^2 X_0 \le 1} F_1(X_0, v)\,dv = X_2 \int_{\substack{t, v \in \mathbb{R},\ t > 0 \\ |v^2 X_0|,\ |t^2 X_0|,\ |tv(t+v)|,\ |X_0(t+v)| \le 1}} dt\,dv$$

$$= X_2 F_2(X_0)$$

Here $F_2(u) := \displaystyle\int_{\substack{t,\sigma\in\mathbb{R},\ t>0 \\ |u\sigma^2|,\ |ut^2|,\ |t\sigma(t+\sigma)|,\ |\sigma(t+\sigma)|\le 1}} dt\, d\sigma$

$$F_1(X_0,\sigma) = \int_{\substack{t>0 \\ |X_0t^2|,\ \dots,\ \le 1}} dt \le \int_{\substack{t>0 \\ t\le \frac{1}{X_0^{1/2}}}} dt = \frac{1}{X_0^{1/2}}$$

$$\sum_{(s_0,s)\in S}\ \sum_{\substack{k_3|s_0 \\ \gcd(k_3,s_1s_2)=1}} \mu(k_3) \sum_{\substack{k_1|s_0s_2 \\ \gcd(k_1,k_3,s_3)=1}} \mu(k_1)\cdot \frac{X_1}{k_1k_3s_3}\left(X_2\,\varphi^*(s_0s_1s_3)F_2(X_0) + O\left(\frac{\tau(s_0s_1s_3)}{X_0^{1/2}}\right)\right)$$

Error term:

$$\ll \sum_{(s_0,s)\in S} \tau(s_0)\tau(s_0s_2)\frac{X_1}{s_3}\frac{\tau(s_0s_1s_3)}{X_0^{1/2}} = B^{1/2}\sum_{(s_0,s)\in S}\frac{\tau(s_0)\tau(s_0s_2)\tau(s_0s_1s_3)}{s_3\cdot s_0^{1/2}s_1} \ll B$$

$$\sum_{s_0^3s_1^2s_2^2s_3^2\le B}\frac{1}{s_3s_1s_0^{1/2}} < \sum_{\substack{s_0\le B^{1/3} \\ s_1,s_3\le B^{1/2}}}\frac{1}{s_0^{\frac12}s_1s_3}\underbrace{\sum_{s_2\le \frac{B^{1/2}}{s_0^{1/2}s_1s_3}} 1}_{B^{1/2}/s_0^{\frac12}s_1s_3}$$

$$\ll B^{1/2}\sum_{\substack{s_0\le B^{1/3} \\ s_1,s_3\le B^{1/2}}}\frac{1}{s_0^2s_1^2s_3^2} \ll B^{1/2}$$

Main term:

$$\sum_{(s_0,s)\in S}\ \sum_{\substack{k_3|s_0 \\ \gcd(k_3,s_1s_2)=1}}\frac{\mu(k_3)}{k_3}\underbrace{\sum_{\substack{k_1|s_0s_2 \\ \gcd(k_1,k_3s_3)=1}}\frac{\mu(k_1)}{k_1}}_{\frac{\varphi^*(s_0s_2)}{\varphi^*(\gcd(s_0s_2,\ k_3s_3))}}\cdot\underbrace{\frac{X_1X_2}{s_3}}_{=\frac{B^{2/3}}{s_1^{1/3}s_2^{1/3}s_3^{1/3}}}\varphi^*(s_0s_1s_3)F_2\left(\left(\frac{s_0^3s_1^2s_2^2s_3^2}{B}\right)^{1/3}\right)$$

$$\sum_{k|a}\frac{\mu(k)}{k} = \varphi^*(a)$$

Interactions..., lecture 7

In the end, the main term becomes

$$\sum_{\substack{s_0,s_1,s_2,s_3\in\mathbb{Z}_{>0}^4\\ \gcd(s_i,s_j)=1\\ s_0^3 s_1^2 s_2^2 s_3^2\le B}} \frac{1}{(s_1s_2s_3)^{1/3}} \underbrace{\sum_{\substack{k_3\mid s_0\\ \gcd(k_3,s_1s_2)=1}} \frac{\mu(k_3)}{k_3}\cdot\frac{\varphi^*(s_0s_2)}{\varphi^*(\gcd(k_3s_3,s_0s_2))}\cdot\varphi^*(s_0s_1s_3)}$$

$=:\vartheta(s_0,\underline{s})$ if $\gcd(s_i,s_j)=1\ \ \forall i\neq j\in\{1,2,3\}$

and $\vartheta(s_0,\underline{s}):=0$ otherwise

$$=\sum_{\substack{s_0,s_1,s_2,s_3\in\mathbb{Z}_{>0}^4\\ s_0^3 s_1^2 s_2^2 s_3^2\le B}} \frac{\vartheta(s_0,\underline{s})}{(s_1s_2s_3)^{1/3}}\, F_2\left(\left(\frac{s_0^3 s_1^2 s_2^2 s_3^2}{B}\right)^{1/3}\right)$$

Good news: ϑ is multiplicative in the sense that

if $\gcd(s_0s_1s_2s_3,\ s_0's_1's_2's_3')=1$ then

$$\vartheta(s_0s_0',\ s_1s_1',\ s_2s_2',\ s_3s_3')=\vartheta(s_0,\underline{s})\cdot\vartheta(s_0',\underline{s}').\quad \text{(Painful computation.)}$$

$\Rightarrow$ Main term:

$$B^{2/3}\sum_{n\le B}\underbrace{\sum_{\substack{s_0,s_1,s_2,s_3\in\mathbb{Z}_{>0}^4\\ s_0^3s_1^2s_2^2s_3^2=n}}\frac{\vartheta(s_0,\underline{s})}{(s_1s_2s_3)^{1/3}}}_{=:\Delta(n)}\cdot F_2\left(\left(\frac{n}{B}\right)^{1/3}\right)$$

Thus we have shown:

$$N_0(B)=2B^{2/3}\sum_{n\le B}\Delta(n)\,F_2\left(\left(\frac{n}{B}\right)^{1/3}\right)+O(B\log B).$$

Perron's formula

$h : \mathbb{N} \to \mathbb{C}$ a multiplicative function

$$S_h(x) := \sum_{n \le x} h(n)$$

In order to understand $S_h(x)$ as $x \to \infty$, we will actually study the analytic properties of the associated Dirichlet series

$$H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} \quad \text{for } s \in \mathbb{C} \text{ where } \operatorname{Re} s \text{ is large enough}$$

Let $f \in \mathcal{H}(\Omega)$ where $\Omega \subseteq \mathbb{C}$ open, $\mathcal{H}$ is the class of holomorphic functions

Let γ be a path in $\mathbb{C}$ piecewise contdiff. E.g. a circle of radius r with center a, parametrised as $\gamma(t) = a + re^{it}$, $t \in (0, 2\pi]$

Another example is $[a,b]$ parametrised as $\gamma(t) = a + (b-a)t$, $t \in [0,1]$.

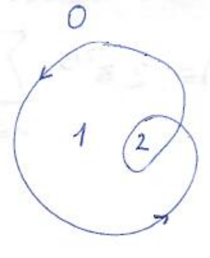
$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt \quad \text{if } \gamma : [a,b] \to \mathbb{C} \text{ contdif.}$$

<u>Cauchy's Theorem.</u> $\Omega \subseteq \mathbb{C}$ open, $f \in \mathcal{H}(\Omega \setminus \{pt\})$, f cont. on Ω.

If γ is a closed path then $\int_\gamma f(z)\, dz = 0$.

Special case of the <u>Residue Theorem</u>: γ closed path, $\Omega \subseteq \mathbb{C}$ convex open, $f \in \mathcal{H}(\Omega)$, $z \in \Omega$. $\Rightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{z - \xi}\, d\xi = f(z) \cdot \operatorname{Ind}_\gamma(z)$

where $\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int \frac{d\xi}{z - \xi}$ is the index.



These are the preliminaries for Perron's formula.

Goal: Estimate $\sum_{n\le B} \Delta(n)$

$$\Delta(n) = \sum_{\substack{s_3,s_0,s_1,s_2>0\\ n=s_0^3 s_1^2 s_2^2 s_3^2}} \frac{\vartheta(s_0,\underline{s})}{(s_1 s_2 s_3)^{1/3}}$$

$$\vartheta(s_0,\underline{s}) = \begin{cases} 0 & \text{else} \\ \dfrac{\varphi^*(s_0 s_2)\,\varphi^*(s_0 s_1 s_3)}{\varphi^*(\gcd(s_0,s_3))} \displaystyle\sum_{\substack{k_3 \mid s_0\\ \gcd(k_3,s_1s_2)=1}} \frac{\mu(k_3)}{k_3}\, \frac{\varphi^*(\gcd(k_3, s_0 s_3))}{\varphi^*(\gcd(k_3,s_0))} & \gcd(s_i,s_j)=1 \ \forall i,j=1,2,3 \end{cases}$$

Method: study the assoc Dirichlet series analytically

$$D(s) = \sum_{n=1}^{+\infty} \frac{\Delta(n)}{n^s} = \sum_{s_0,s_1,s_2,s_3=1}^{+\infty} \frac{\vartheta(s_0,\underline{s})}{s_0^{3s}\, s_1^{2s+1/3}\, s_2^{2s+1/3}\, s_3^{2s+1/3}} \qquad s \in \mathbb{C}$$

<u>Lemma</u> This is abs convergent if $Re(s) > 1/3$

Pf: $0 \le \vartheta(s_0,\underline{s}) \le 1$ implies this result. So now we prove this.

$$(\dots) = \prod_{\substack{p\mid s_0\\ p\nmid s_1,s_2,s_3}} \left(1-\frac{1}{p}\cdot\frac{1}{1-\frac{1}{p}}\right) \prod_{\substack{p\mid \gcd(s_0,s_3)\\ p\nmid s_1 s_2}} \left(1-\frac{1}{p}\cdot\frac{1-\frac{1}{p}}{1-\frac{1}{p}}\right)$$

$$= \prod_{\substack{p\mid s_0\\ p\nmid s_1s_2s_3}} \left(\frac{1-\frac{2}{p}}{1-\frac{1}{p}}\right) \prod_{\substack{p\mid \gcd(s_0,s_3)\\ p\nmid s_1 s_2}} \left(1-\frac{1}{p}\right)$$

Recall:

$$\varphi^*(a) = \prod_{p\mid a}\left(1-\frac{1}{p}\right)$$

$$\varphi^*(nm) = \frac{\varphi^*(n)\,\varphi^*(m)}{\varphi^*(\gcd(n,m))}$$

If $\gcd(s_i,s_j)=1$:

$$\vartheta(s_0,\underline{s}) = \frac{\varphi^*(s_0)\ \varphi^*(s_0)\,\varphi^*(s_2)\,\varphi^*(s_1s_3)}{\varphi^*(\gcd(s_0,s_2))\,\varphi^*(\gcd(s_0,s_1s_3))} \cdot \frac{1}{\prod_{p\mid s_0,s_3}\left(1-\frac{1}{p}\right)} \cdot \prod_{\substack{p\mid s_0,s_3\\ \underbrace{p\nmid s_1,s_2}_{\text{automatic}}}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid s_0\\ p\nmid s_1s_2s_3}} \frac{1-\frac{2}{p}}{1-\frac{1}{p}}$$

simplify

$$= \varphi^*(s_0)\,\varphi^*(s_1s_2s_3) \prod_{\substack{p\mid s_0\\ p\nmid s_1s_2s_3}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid s_0\\ p\nmid s_1s_2s_3}} \left(\frac{1-\frac{2}{p}}{1-\frac{1}{p}}\right)$$

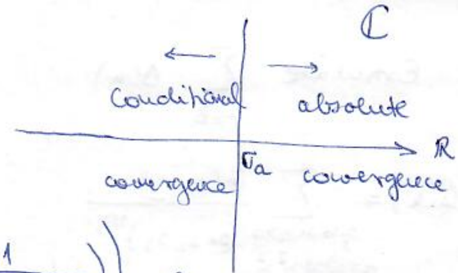
$$= \varphi^*(s_0)\,\varphi^*(s_1s_2s_3) \prod_{\substack{p\mid s_0\\ p\nmid s_1s_2s_3}} \left(1-\frac{2}{p}\right)$$

simplify

This is in $[0,1]$. $\square$

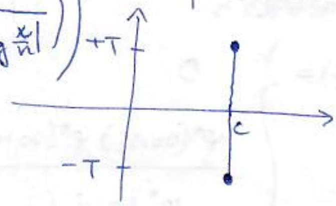
Perron's formula (see also p. 38)

Suppose H has abscissa of absolute convergence σ_a



Let $c > \sigma_a$, $x, T > 0$, $x \notin \mathbb{Z}$.

$$\Rightarrow \quad S_h(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} H(s)\frac{x^s}{s}\,ds + O\left(\frac{x^c}{T} + \sum_{\frac{x}{2}\le n\le\frac{3x}{2}} |h(n)| \min\left(1, \frac{1}{T\left|\log\frac{x}{n}\right|}\right)\right)$$



Main idea:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\,ds = \chi_{]1,+\infty[}(x) \qquad \delta_{x=0} = \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases}$$

Lemma. Let $\delta(x) = \begin{cases} 0 & \text{if } 0<x<1 \\ 1/2 & \text{if } x=1 \\ 1 & \text{if } x>1 \end{cases}$

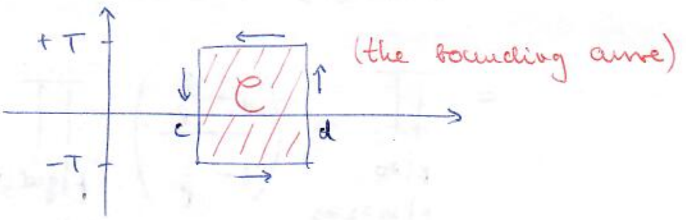
For $c, x, T > 0$ we have

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\,ds - \delta(x)\right| < \begin{cases} x^c \min\left(1, \dfrac{1}{T|\log x|}\right) & \text{if } x\neq 1 \\ \dfrac{c}{T} & \text{if } x=1 \end{cases}$$

PF OF LEMMA:

Case 1.: $0<x<1$: $\delta(x)=0$

Let $d > c$.



Cauchy: $\displaystyle \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{x^s}{s}\,ds = 0 = \delta(x)$

$$\| $$

$$\frac{1}{2\pi i}\left(-\int_{c-iT}^{c+iT}\frac{x^s}{s}\,ds + \underbrace{\int_{c-iT}^{d-iT}\frac{x^s}{s}\,ds}_{I_-} + \underbrace{\int_{d-iT}^{d+iT}\frac{x^s}{s}\,ds}_{\mathcal{J}} + \underbrace{\int_{d+iT}^{c+iT}\frac{x^s}{s}\,ds}_{I_+}\right)$$

$$\Rightarrow \left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{x^s}{s}\,ds - \delta(x)\right| \le |I_+| + |I_-| + |\mathcal{J}|.$$

$$|I_+| = \left|-\frac{1}{2\pi i}\int_c^d \frac{x^{\sigma+iT}}{\sigma+iT}\,d\sigma\right| \le \frac{1}{2\pi}\int_c^d \frac{x^\sigma}{T}\,d\sigma =$$

$$= \frac{1}{2\pi\, T\log x}\cdot(x^d - x^c) = \frac{x^c - x^d}{2\pi T|\log x|}.$$

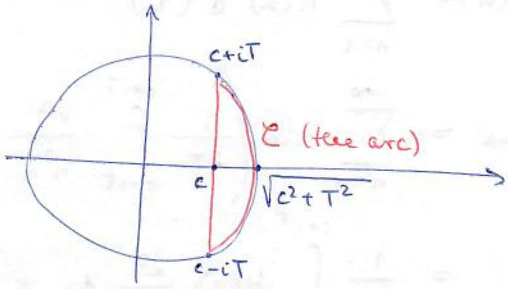
Interactions..., lecture 8

$d \to +\infty$ implies $|I^{\pm}| \le \dfrac{x^c}{T\,|\log x|}$

$$|\mathcal{J}| = \left| \frac{1}{2\pi} \int_{-T}^{+T} \frac{x^{d-it}}{d-it}\, dt \right| \le \frac{1}{2\pi} \int_{-T}^{+T} \frac{x^d}{d}\, dt = \frac{T}{\pi} \cdot \frac{x^d}{d} \xrightarrow[d \to +\infty]{} 0$$

$\Rightarrow$ get the bound $x^c \cdot \dfrac{1}{T\,|\log x|}$.

Use another contour of integration:



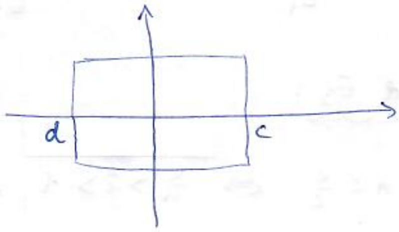
Cauchy:

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s}\, ds = -\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{x^s}{s}\, ds$$

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s}\, ds - \delta(x) \right| \le \frac{1}{2\pi} \frac{1}{\sqrt{c^2+T^2}}\, x^c\, \pi \sqrt{c^2+T^2} < x^c,$$

so we get the other bound x^c too.

Case 2. $x>1$: Let $d<0$.

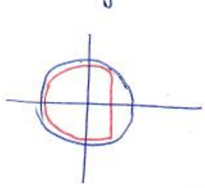


Residue thm. $\Rightarrow$ $\dfrac{1}{2\pi i} \displaystyle\int_{\mathcal{C}} \frac{x^s}{s}\, ds = x^0 \cdot 1 = 1 = \delta(x)$

$$\| $$

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s}\, ds - I_- - I_+ - \mathcal{J}$$

By the same argument as in Case 1 we get $\dfrac{x^c}{T\,|\log x|}$

Consider  $\longrightarrow$ get x^c with the residue thm.

Case 3. $x=1$

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{ds}{s} = \frac{1}{2\pi i} \int_{-T}^{+T} \frac{dt}{c+it} = \frac{c}{2\pi} \int_{-T}^{T} \frac{dt}{c^2+t^2} - \overbrace{\frac{i}{2\pi} \int_{-T}^{T} \frac{t\,dt}{c^2+t^2}}^{0}$$

$$= \frac{c}{\pi} \int_0^T \frac{dt}{c^2+t^2} = \frac{1}{\pi} \int_0^{T/2} \frac{d\sigma}{1+\sigma^2}$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{d\sigma}{1+\sigma^2} - \frac{1}{\pi} \int_{T/2}^{\infty} \frac{d\sigma}{1+\sigma^2} = \frac{1}{2} + \frac{1}{\pi} \int_{T/2}^{\infty} \frac{d\sigma}{1+\sigma^2}$$

$$\frac{1}{\pi}\int_{T/2}^{\infty}\frac{d\sigma}{1+\sigma^2} \le \frac{1}{\pi}\int_{T/2}^{\infty}\frac{d\sigma}{\sigma^2} = \frac{1}{\pi}\cdot\frac{c}{T} < \frac{c}{T}$$

WTS $S_h(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} H(s)\,\frac{x^s}{s}\,ds + O\left(\frac{x^c}{T} + \sum_{\frac{x}{2}\le n\le\frac{3x}{2}} |h(n)| \min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right)$ $\Big|$ $c>\sigma_a$, $x\in\mathbb{Z}$

Pf: $\sum_{n\le x} h(n) = \sum_{n=1}^{+\infty} h(n)\,\delta\left(\frac{x}{n}\right)$ $\qquad x\notin\mathbb{Z} \Rightarrow \frac{x}{n}\notin\mathbb{Z}\ \forall n\in\mathbb{N}$

use Lemma $= \sum_{n=1}^{+\infty} h(n)\left(\frac{1}{2\pi}\int_{c-iT}^{c+iT}\frac{x^s}{n^s s}\,ds + O\,\frac{x^c}{n^c}\min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right)$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} H(s)\,\frac{x^s}{s}\,ds + O\left(x^c\sum_{n=1}^{+\infty}\frac{|h(n)|}{n^c}\min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right)$$

$\log\frac{x}{n}$ is small if x and n are close. We split the summation range into 3 intervalls based on this:

$n\in[1,\frac{x}{2}[$, $\quad n\in]\frac{3x}{2},+\infty[$, $\quad n\in[\frac{x}{2},\frac{3x}{2}]$

$E_1 \qquad\qquad E_3 \qquad\qquad E_2$

E_1 and E_3:

E_1: $n<\frac{x}{2} \iff \frac{x}{n}>2>1 \Rightarrow \left|\log\frac{x}{n}\right| \ge \log 2 \gg 1$

E_3: $n>\frac{3x}{2} \iff \frac{x}{n}<\frac{3}{2}<1 \Rightarrow \left|\log\frac{x}{n}\right| \gg 1$

$$\Rightarrow \quad E_1, E_3 \ll \sum_{n=1}^{+\infty}\frac{|h(n)|}{n^c}\underbrace{\min\left(1,\frac{1}{T}\right)}_{\le 1/T} \ll \frac{1}{T}\sum_{n=1}^{+\infty}\frac{|h(n)|}{n^c} \ll \frac{1}{T}$$

$E_2 = \sum_{n=1}^{+\infty}\frac{|h(n)|}{n^c}\min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)$, $\quad n^c \ge \frac{x^c}{2^c} \gg x^c$

$$E_2 \ll x^{-c}\sum_{\frac{x}{2}\le n\le\frac{3n}{2}} |h(n)|\min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)$$

Apply Perron's formula for $h := \Delta$ recalled at the beginning of lec. 8 (p. 39)

$\Rightarrow 1/3 \geq \sigma_a$ (in fact, $1/3 = \sigma_a$ holds). Choose $c = \frac{1}{3} + \varepsilon$ with $\varepsilon > 0$.

Let $B \notin \mathbb{Z}$,

$$\sum_{n \leq B} \Delta(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} D(s) \frac{B^s}{s} \, ds + O\left(\frac{B^{\frac{1}{3}+\varepsilon}}{T} + \sum_{\frac{x}{2} \leq n \leq \frac{3x}{2}} |\Delta(n)| \min\left(1, \frac{1}{T|\log \frac{x}{n}|}\right) \right)$$

ERROR TERM.

$$(\ldots) = \sum_{\substack{s_0, s_1, s_2, s_3 \\ \frac{B}{2} \leq s_0^3 s_1^2 s_2^2 s_3^2 \leq \frac{3B}{2}}} \frac{\overbrace{\vartheta_1(s_0, s)}^{\leq 1}}{(s_1 s_2 s_3)^{1/3}} \min\left(1, \frac{1}{T\left|\log \frac{B}{s_0^3 s_1^2 s_2^2 s_3^2}\right|}\right)$$

$$\ll \sum_{s_1, s_2, s_3 \ll B^{1/2}} \frac{1}{(s_1 s_2 s_3)^{1/3}} \sum_{\frac{1}{2^{1/3}}\left(\frac{B}{s_1 s_2 s_3}\right)^{1/3} \leq s_0 \leq \left(\frac{3}{2}\right)^{1/3} \underbrace{\left(\frac{B}{s_1 s_2 s_3}\right)^{1/3}}_{y}} \min\left(1, \frac{1}{T\left|\log \frac{s_0}{y}\right|}\right)$$

i.e. $\frac{1}{2^{1/3}} y \leq s_0 \leq \left(\frac{3}{2}\right)^{1/3} y$

Put $N := \left\lfloor \frac{1}{2^{1/3}} y \right\rfloor$, $s_0 = N + m$ for $0 \leq m \leq \left(\frac{3}{2}\right)^{1/3} y = by$

$$\Rightarrow \sum_{0 \leq m \leq by} \min\left(1, \frac{1}{T\left|\log \frac{N+m}{y}\right|}\right) \ll \sum_{0 \leq m \leq by} \underbrace{\min\left(1, \frac{y}{Tm}\right)}_{= 1 \text{ if } m \leq \frac{y}{T}} \ll$$

$$\ll \sum_{0 \leq m \leq \frac{y}{T}} 1 + \sum_{0 \leq m \leq by} \frac{y}{Tm} \ll \frac{y}{T} + \frac{y}{T} \log y \ll \frac{B^{1/3}}{T} \cdot \frac{1}{(s_1 s_2 s_3)^{2/3}} + \frac{B^{1/3}}{T (s_1 s_2 s_3)^{2/3}} \log B$$

$$\ll_\varepsilon \frac{B^{1/3+\varepsilon}}{T} \cdot \frac{1}{(s_1 s_2 s_3)^{1/3}} \quad \text{for } \forall \varepsilon > 0 \quad (\text{since } \log B \ll B^\varepsilon)$$

$$\Rightarrow (\ldots) \ll \frac{B^{1/3+\varepsilon}}{T} \sum_{s_1, s_2, s_3 \ll B^{1/2}} \frac{1}{s_1 s_2 s_3} \ll B^{1/3+\varepsilon}$$

So we have proven

$$\sum_{n \leq B} \Delta(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} D(s) \frac{B^s}{s} \, ds + O_\varepsilon\left(\frac{B^{1/3+\varepsilon}}{T} \right)$$

MAIN TERM. To compute the main term, we need to study the integrand function.

Thm. [Tenenbaum] (ref.) $s \in \mathbb{C}$, f multiplicative function. and $\sum_p \sum_{\nu \geq 0} \left| \frac{f(p^\nu)}{p^{\nu s}} \right| < +\infty$

then $F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely and $F(s) = \prod_p \sum_{\nu \geq 0} \frac{f(p^\nu)}{p^{\nu s}}$.

$$D(s) = \sum_{s_0, s_1, s_2, s_3 = 1}^{+\infty} \frac{1}{(s_1 s_2 s_3)^{1/3}} \frac{\vartheta(s_0, \underline{s})}{s_0^3 s_1^2 s_2^2 s_3^2} \qquad Re(s) > 1/3$$

We will consider $D\left(s + \frac{1}{3}\right) = \sum_{s_0, s_1, s_2, s_3 = 1}^{+\infty} \frac{\vartheta(s_0, \underline{s})}{s_0^{3s+1} (s_1 s_2 s_3)^{2s+1}}$ $\qquad Re(s) > 0$

Multidimensional application of the Thm.:

$$D\left(s + \frac{1}{3}\right) = \prod_p \underbrace{\sum_{\nu_0, \nu_1, \nu_2, \nu_3 = 0}^{+\infty} \frac{\vartheta(p^{\nu_0}, p^{\nu_1}, p^{\nu_2}, p^{\nu_3})}{p^{(3s+1)\nu_0 + (2s+1)(\nu_1 + \nu_2 + \nu_3)}}}_{a_p(s)}$$

$a_p(s) = 0$ unless $\gcd(s_i, s_j) = 1 \quad \forall i,j = 1,2,3$

$$\Rightarrow a_p(s) = 1 + 3 \sum_{\nu_1 = 1}^{+\infty} \frac{\vartheta(1, p^{\nu_1}, 1, 1)}{p^{(2s+1)\nu_1}} + \sum_{\nu_0 = 1}^{+\infty} \frac{\vartheta(p^{\nu_0}, 1, 1, 1)}{p^{(3s+1)\nu_0}} + 3 \sum_{\nu_0, \nu_1 = 1}^{+\infty} \frac{\vartheta(p^{\nu_0}, p^{\nu_1}, 1, 1)}{p^{(3s+1)\nu_0 + (2s+1)\nu_1}}$$

$\nu_0 = 0,\ \nu_i \neq 0,\ \nu_j = 0, \nu_k = 0$ $\qquad$ $\nu_0 \neq 0$, $\nu_1 = \nu_2 = \nu_3 = 0$ $\qquad$ $\nu_0, \nu_i \neq 0$, $\nu_j, \nu_k = 0$

$$\vartheta(s_0, \underline{s}) = \varphi^*(s_0)\, \varphi^*(s_1 s_2 s_3) \prod_{\substack{p \mid s_0 \\ p \nmid s_1 s_2 s_3}} \left(1 - \frac{2}{p}\right)$$

$\vartheta(1, p^{\nu_1}, 1, 1) = 1 - \frac{1}{p}$; $\quad \vartheta(p^{\nu_0}, 1, 1, 1) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)$; $\quad \vartheta(p^{\nu_0}, p^{\nu_1}, 1, 1) = \left(1 - \frac{1}{p}\right)^2$

In the end:
$$a_p(s) = 1 + \underbrace{\frac{3\left(1 - \frac{1}{p}\right)}{p^{2s+1}\left(1 - \frac{1}{p^{2s+1}}\right)}}_{\text{looks like } \zeta(2s+1)^3} + \underbrace{\frac{\left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)}{p^{3s+1}\left(1 - \frac{1}{p^{3s+1}}\right)}}_{\zeta(3s+1)} + \underbrace{\frac{3\left(1 - \frac{1}{p}\right)^2}{p^{5s+2}\left(1 - \frac{1}{p^{2s+1}}\right)\left(1 - \frac{1}{p^{3s+1}}\right)}}_{\zeta(2s+1)^3 \zeta(3s+1)}$$

$$D\left(s+\tfrac{1}{3}\right) = \underbrace{\zeta(2s+1)^3\,\zeta(3s+1)}_{E_1(s)}\,E_2(s) \quad \text{where} \quad E_2(s) = \frac{D\left(s+\frac{1}{3}\right)}{E_1(s)}$$

Lemma. E_2 is holomorphic and bounded for $Re(s) > -\frac{1}{6}$

E_1 has a holomorphic continuation to $\mathbb{C}$ with a quadruple pole at $s=0$.

$\Rightarrow$ Get a meromorphic continuation for $D(s)$ for $Re(s) > -\frac{1}{6}$ with quadruple pole at $s = \frac{1}{3}$.

PF OF LEMMA:

$$E_2(s) = \prod_p \frac{a_p(s)}{\left(1-\frac{1}{p^{2s+1}}\right)^{-3}\left(1-\frac{1}{p^{3s+1}}\right)^{-1}} \quad \text{Expanding the denominator:}$$

$$\left(1-\frac{1}{p^{2s+1}}\right)^{-3}\left(1-\frac{1}{p^{3s+1}}\right)^{-1} = \left(1+\frac{3}{p^{2s+1}}+\frac{6}{p^{4s+2}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)\right)\cdot\left(1+\frac{1}{p^{3s+1}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)\right)$$

$$= 1+\frac{3}{p^{2s+1}}+\frac{1}{p^{3s+1}}+\frac{6}{p^{4s+2}}+\frac{3}{p^{5s+2}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)$$

$$a_p(s) = 1+3\left(\frac{1}{p^{2s+1}}-\frac{1}{p^{2s+2}}\right)\left(1+\frac{1}{p^{2s+1}}+\frac{1}{p^{4s+2}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)\right)+$$
$$+\left(\frac{1}{p^{3s+1}}-\frac{3}{p^{3s+2}}+\frac{2}{p^{3s+3}}\right)\left(1+\frac{1}{p^{3s+1}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)\right)+$$
$$+3\left(\frac{1}{p^{3s+2}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)\right)\left(1+\frac{1}{p^{2s+1}}+\frac{1}{p^{3s+1}}-\dots\right)$$
$$= \boxed{1+\frac{3}{p^{2s+1}}+\frac{1}{p^{3s+1}}+\frac{3}{p^{5s+2}}}+O\left(\frac{1}{p^{6Re\,s+2}}\right)$$

For $Re\,s > -\frac{1}{6}$ we have $O\left(\frac{1}{p^{6Re\,s+2}}\right) = O\left(\frac{1}{p^{1+\varepsilon}}\right)$

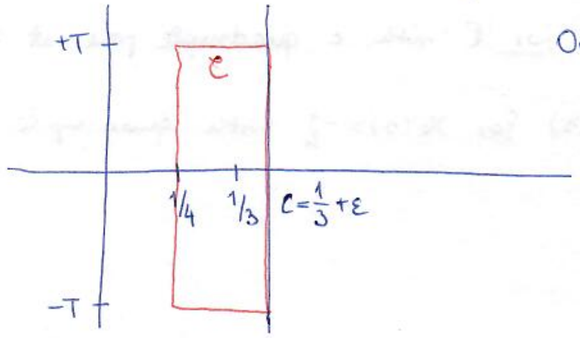
$$\Rightarrow E_2 = \prod_p \frac{1+\sim\sim+O\left(\frac{1}{p^{1+\varepsilon}}\right)}{1+\underset{\text{the same}}{\sim\sim}+O\left(\frac{1}{p^{1+\varepsilon}}\right)} = \underbrace{\prod_p\left(1+O\left(\frac{1}{p^{1+\varepsilon}}\right)\right)}_{\text{convergent because } \sum_p \frac{1}{p^{1+\varepsilon}} \text{ is convergent}}$$

$\square$

In the end we have proved that

$$\sum_{n\le B} \Delta(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} E_1\left(s-\frac{1}{3}\right) E_2\left(s-\frac{1}{3}\right)\frac{B^s}{s}\, ds + O_\varepsilon\left(\frac{B^{1/3+\varepsilon}}{T}\right)$$

Now we will use the following contour of integration:



On $\mathcal{C}$, $E_2\left(s-\frac{1}{3}\right)$ is holomorphic and bounded

Residue theorem. Let f be a meromorphic function on $\Omega\subset\mathbb{C}$ open with a single (not simple!) pole at $s=s_0$. Let $\mathcal{C}$ be a closed path in Ω s.t. s_0 is in the bounded connected component defined by $\mathcal{C}$. Then

$$\frac{1}{2\pi i}\int_{\mathcal{C}} f(z)\, dz = \operatorname*{Res}_{z=s_0} f = a_{-1}$$

where $f(z)=\sum_{n\geq -N}^{+\infty} a_n (z-s_0)^n$ for some $N\in\mathbb{Z}_{>0}$ (Laurent series).

Applying the residue thm yields:

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \underbrace{E_1\left(s-\frac{1}{3}\right) E_2\left(s-\frac{1}{3}\right)\frac{B^s}{s}}_{f(s)}\, ds = \operatorname*{Res}_{s-\frac{1}{3}} f$$

We need the Laurent series of f.

Lemma. [ref. Tenenbaum] $\zeta(s)=\frac{1}{s-1}+\underbrace{\gamma}_{\text{Euler – Mascheroni constant}}+\sum_{n=1}^{\infty}\frac{\gamma_n(-1)^n}{n!}(s-1)^n$ is the Laurent series of ζ.

$$\zeta\left(2\left(s-\frac{1}{3}\right)+1\right)^3\zeta\left(3\left(s-\frac{1}{3}\right)+1\right)=\zeta\left(2s+\frac{1}{3}\right)^3\zeta(3s)$$

$$=\left(\frac{1}{2\left(s-\frac{1}{3}\right)}+\sum_{n=0}^{\infty}\gamma_n'\left(s-\frac{1}{3}\right)^n\right)^3\left(\frac{1}{3\left(s-\frac{1}{3}\right)}+\sum_{n=0}^{\infty}\gamma_n'\left(s-\frac{1}{3}\right)^n\right)$$

$$=\frac{1}{2\cdot 3}\cdot\frac{1}{\left(s-\frac{1}{3}\right)^4}+\sum_{n=-3}^{\infty}a_n\left(s-\frac{1}{3}\right)^n$$

The higher order constants are more painful to compute (still doable, but pointless: as there are no conjectures involving them there is no way to check them.)

Interactions, lecture 9 ~ 10

$$E_2\left(s-\frac{1}{3}\right) = E_2(0) + \sum_{n=1}^{\infty} b_n \left(s-\frac{1}{3}\right)^n$$

$$\frac{B^s}{s} = 3B^{1/3}\left(1 + \left(s-\frac{1}{3}\right)\log B + \frac{\left(s-\frac{1}{3}\right)^2}{2}\log^2 B + \frac{\left(s-\frac{1}{3}\right)^3}{6}\log^3 B + \sum_{n=4}^{\infty}\frac{\left(s-\frac{1}{3}\right)^n}{n!}\log^n B\right)$$

$$\frac{a_{-1}}{s-\frac{1}{3}} = \frac{3\,B^{1/3}\,E_2(0)\,\log^3 B}{6} + \frac{3b_1}{2}B^{1/3}\log^2 B + \text{more terms but none have } \log^3 B$$

$$\Rightarrow \operatorname*{Res}_{\frac{1}{3}} f = \frac{B^{1/3}}{6\cdot 2^3\cdot 3}\cdot E_2(0)\cdot \underbrace{P(\log B)}_{\substack{\text{polynomial in } \log B,\\ \text{monic of degree } 3}}$$