Graduated orders in equivariant Iwasawa theory

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Preliminaries on orders

R noetherian integrally closed integral domain

$k = \mathrm{Frac}(R)$ quotient field, characteristic 0 (avoid separability questions)

A finite dimensional k-algebra

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Maximal orders exist, and they are well-behaved under direct sums, localisation, completion

Motivation: equivariant Iwasawa theory ($p \neq 2$)

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Equivariant Iwasawa main conjecture (very imprecise formulation)

$$\mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times \xleftarrow{\mathrm{nr}} K_1\big(\mathcal{Q}(\mathcal{G})\big) \xrightarrow{\partial} K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$$

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Goal: understand orders in $\mathcal{Q}(\mathcal{G})$, then show that ζ comes from an order

Outline

One-dimensional case: local fields

Skew fields over local fields

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Theorem (Hasse)

The valuation on k admits a unique extension v to D, i.e. $\forall a, b \in D$:

- $v(a) = \infty \Leftrightarrow a = 0$;
- $v(ab) = v(a) + v(b) = v(ba)$;
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This is the integral closure of R in D.

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Note: W is unique only up to conjugacy. There are infinitely many ω!

Explicit description of skew fields over local fields

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Theorem (Hasse)

There exists a uniformiser $\pi_D \in D$ and $1 \leq r \leq \operatorname{ind} D$ such that

$$\pi_D^{\operatorname{ind} D} = \pi, \quad \pi_D \omega \pi_D^{-1} = \omega^{q^r}, \quad \gcd(r, \operatorname{ind} D) = 1, \quad D = \bigoplus_{i=0}^{\operatorname{ind} D - 1} k(\omega)\pi_D^i.$$

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Conversely, for any given k and $1 \le r \le n$ coprime, there exists D with $n = \operatorname{ind} D$ as above.

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As part of the proof, one constructs an explicit splitting isomorphism:

$$D \hookrightarrow k(\omega) \otimes_k D \xrightarrow{\sim} M_{\operatorname{ind} D}(k(\omega))$$

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Proposition

The maximal R-orders in $M_n(D)$ are precisely the $uM_n(\Omega)u^{-1}$ with $u \in \mathrm{GL}_n(D)$.

Two-dimensional case: Iwasawa algebras

General theory

$\mathscr{R}$ complete regular local ring of dimension 2 — later: $\mathscr{R} = \mathbb{Z}_p[[T]]$
$\mathscr{D}$ skew field with centre $\mathfrak{z}(\mathscr{D}) = \boldsymbol{k} = \mathrm{Frac}(\mathscr{R})$

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Theorem (Ramras 1969)

- *Maximal $\mathscr{R}$-orders $\mathfrak{Y}$ in $\mathscr{D}$ are unique only up to conjugation.*
- *Maximal $\mathscr{R}$-orders in $M_n(\mathscr{D})$ are also unique up to conjugation.*
- *Any $\mathfrak{Y}$ is a local ring, i.e. it has a unique two-sided maximal ideal $\mathfrak{m}_{\mathfrak{Y}}$*

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$\mathscr{D} = \bigoplus_{i=0}^{[K:k]\,\mathrm{ind}(D)-1} \mathscr{W}\,(\pi_D X)^i$; $\pi_D X$ acts as rth power of Frobenius times τ

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A finite dimensional K-algebra

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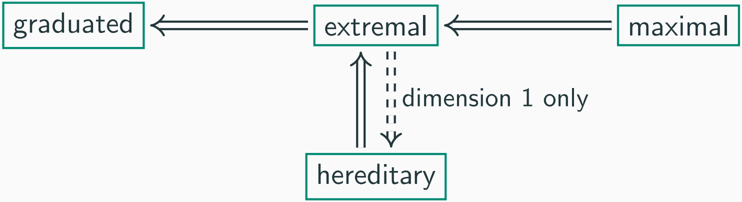

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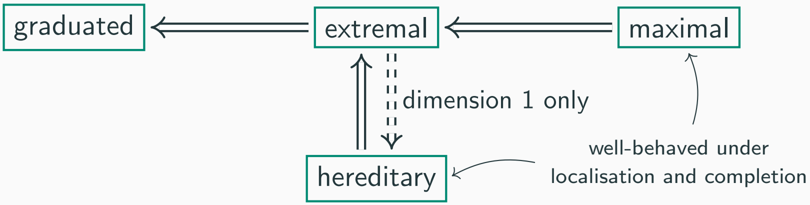

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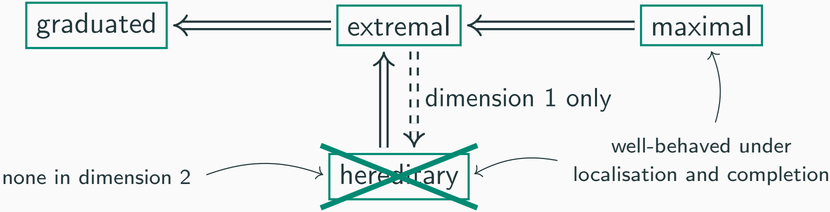

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Note: the data in the Theorem are not uniquely determined by $\mathfrak{N}$.

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$\mathcal{G} = H \rtimes \Gamma$ with H finite, $\Gamma \simeq \mathbb{Z}_p$, $\Lambda(\mathcal{G}) = \mathbb{Z}_p[\![\mathcal{G}]\!]$, $\mathcal{Q}(\mathcal{G}) = \mathrm{Quot}(\Lambda(\mathcal{G}))$

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$$\mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times \xleftarrow{\mathrm{nr}} K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\partial} K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$$

$$\text{analytic } \Phi \longleftarrow\!\!\shortmid \exists\zeta \longmapsto [C^\bullet] \text{ algebraic}$$

Theorem (equivariant p-adic Artin conjecture)

Assume the equivariant Iwasawa main conjecture.

Let $\mathfrak{O}$ be a graduated $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$.

Then ζ is in the image of the map $\mathfrak{O} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})), x \mapsto [(x)]$.

The proof builds on the explicit description of the skew fields in $\mathcal{Q}(\mathcal{G})$ as well as Nichifor–Palvannan's method (2019).