An equivariant p-adic Artin conjecture

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Abstract

We formulate an equivariant version of Greenberg's p-adic Artin conjecture for smoothed equivariant p-adic Artin L-functions in the context of an arbitrary one-dimensional admissible p-adic Lie extension of a totally real number field. As opposed to existing work on this matter, we do not assume that the underlying Galois group $\mathcal{G}$ is the direct product of a finite group and a profinite group isomorphic to $\mathbb{Z}_p$. We study the conjecture by investigating the Wedderburn decomposition of the total ring of quotients of the Iwasawa algebra $\Lambda(\mathcal{G})$. From this, we deduce validity of the conjecture in several interesting cases.

Zusammenfassung

Wir formulieren eine äquivariante Fassung von Greenbergs p-adischer Artin-Vermutung für geglättete äquivariante p-adische Artinsche L-Funktionen für beliebige eindimensionale p-adische Lie-Erweiterungen eines total reellen Zahlkörpers. Im Gegensatz zu vorherigen Arbeiten wird nicht vorausgesetzt, dass die zugrunde liegende Galoisgruppe ein direktes Produkt einer endlichen Gruppe und einer proendlichen Gruppe isomorph zu $\mathbb{Z}_p$ ist. Die Vermutung wird mithilfe der Wedderburnschen Zerlegung des totalen Quotientenrings der Iwasawa-Algebra $\Lambda(\mathcal{G})$ studiert. Dies erlaubt uns die Verifizierung der Vermutung in etlichen interessanten Fällen.

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I recognise the efforts of those who contribute to freedom and accessibility of knowledge, may this be done by sharing their own work online, by answering questions on StackExchange and MathOverflow, by contributing to Wikimedia projects, or in another way. Your work is seen.

Tanādos ùtelirem.

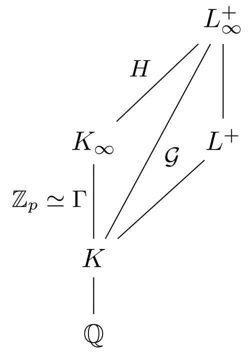
Contents

Introduction

Historical overview and recent developments

Integrality questions have accompanied the study of p-adic Artin L-functions since its inception. Indeed, in the seminal works of Pi. Cassou-Noguès [Cas79] resp. Deligne and Ribet [Rib79; DR80], it was shown that in the case of a linear totally even p-adic Artin character χ with open kernel, the associated p-adic L-function can be expressed as a power series $G_\chi(T)$ divided by an explicit polynomial.

These results of Cassou-Noguès and Deligne–Ribet were generalised to arbitrary p-adic Artin characters with open kernel by Greenberg [Gre83], employing the method of Brauer induction. This inevitably led to a weakening of the integrality property above, with $G_\chi(T)$ now being a fraction of two power series. In response to this, Greenberg posited two conjectures, namely the p-adic Artin conjecture and a strengthening of it, which assert that $G_\chi(T)$ is contained in the ring of power series over a certain ring of integers, possibly up to a constant factor. These conjectures are now theorems: the p-adic Artin conjecture follows from the main conjecture proven by Wiles [Wil90], and its strengthening was proven by Ritter and Weiss [RW04].



Let us now set the scene. Let $\mathcal{G}$ be the Galois group of an admissible one-dimensional p-adic Lie extension L^+_∞/K, where L^+/K is the maximal totally real subextension of a finite CM extension L/K such that L contains a primitive pth root of unity, and L^+_∞ is the cyclotomic $\mathbb{Z}_p$-extension of L^+. Let S be a finite set of places of K containing all places ramifying in L^+_∞/K and all infinite places. The group $\mathcal{G}$ is isomorphic to the semidirect product $H \rtimes \Gamma$ where H is a finite group and $\Gamma \simeq \mathbb{Z}_p$. The results cited above are valid for any p-adic Artin character $\chi : \mathcal{G} \to \mathbb{C}_p$ with open kernel, and the associated p-adic L-function $L_{p,S}(\chi, s)$ can be expressed as a fraction $G_{\chi,S}(T)$ of two power series divided by an explicit polynomial.

As we have seen, integrality of characterwise p-adic L-functions is settled. Strides towards integrality in an equivariant setting were made by Nichifor and Palvannan [NP19]. Their work concerns extensions cut out by mod p characters, with the assumption that $\mathcal{G}$ is a direct product of H and Γ. They considered the equivariant p-adic Artin L-function, seen as a collection of Greenberg's p-adic L-functions modified by some local Euler factors; this is an element in $\mathfrak{z}(\mathcal{Q}(\mathcal{G}))$ where $\mathfrak{z}(-)$ denotes the centre and $\mathcal{Q}(\mathcal{G}) = \mathrm{Quot}(\Lambda(\mathcal{G}))$ is the total ring of quotients of the Iwasawa algebra $\Lambda(\mathcal{G})$, which is obtained from $\Lambda(\mathcal{G})$ by inverting all central regular elements. Assuming the main conjecture, this equivariant p-adic Artin L-function is the reduced norm of an element in $K_1(\mathcal{Q}(\mathcal{G}))$. Under the assumption that the relevant Iwasawa module has a free resolution of length one, they proved that this element in $K_1(\mathcal{Q}(\mathcal{G}))$ has a representative that is a 1×1 matrix over a maximal order in $\mathcal{Q}(\mathcal{G})$. A key step in the proof is a dimension reduction statement involving Dieudonné determinants of matrices.

The assumption on $\mathcal{G}$ being a direct product makes the representation theory of $\mathcal{Q}(\mathcal{G})$ rather straightforward: indeed, in this case, the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$ is directly determined by that of $\mathbb{Q}_p[H]$. The relevance of this fact is that the reduced norm map is defined Wedderburn componentwise. Also, the maximal order in Nichifor–Palvannan's theorem is induced by the unique maximal orders in each of the skew fields in the Wedderburn decomposition. Therefore, when extending the results to $\mathcal{G}$ which are semidirect products, one first needs to describe Wedderburn components of $\mathcal{Q}(\mathcal{G})$. In the case of pro-p groups $\mathcal{G}$, this task was carried out by Lau [Lau12a]. The difference between Wedderburn decompositions of $\mathbb{Q}_p[H]$ and $\mathcal{Q}(\mathcal{G})$ can be substantial: for instance, when $\mathcal{G}$ is pro-p, all Schur indices for $\mathbb{Q}_p[H]$ are 1 due to a theorem of Schilling [CR87, (74.15)], whereas $\mathcal{Q}(\mathcal{G})$ can have nontrivial Schur indices.

The result of Nichifor and Palvannan is valid away from the trivial character. Indeed, the relevant Iwasawa module does not admit a free resolution of length one in the case of the trivial character. A possible way to remedy this is the introduction of a nonempty smoothing set T disjoint from S (satisfying a certain torsion freeness condition), and adjusting the p-adic L-function by a modified Euler factor at primes in T. In recent work of Johnston and Nickel [JN19], an (S,T)-modified Iwasawa module Y_S^T was introduced, along with a corresponding (S,T)-modified equivariant p-adic L-function and a main conjecture relating them. The module Y_S^T admits a free resolution of length one. One may formulate a main conjecture relating the class of Y_S^T in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$ to a T-smoothed version Φ_S^T of Ritter–Weiss's equivariant p-adic Artin L-function, postulating the existence of a unique element $\zeta_S^T \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\mathrm{nr}(\zeta_S^T) = \Phi_S^T$ and ζ_S^T is mapped to the negative of the class of Y_S^T under the connecting homomorphism of relative K-theory; see Conjecture 1.4.4.

The question of integrality as well as the idea behind smoothing have their origins in the global setup. In this context, the Artin conjecture asserts that for all nontrivial irreducible Artin characters χ of $\mathrm{Gal}(L/K)$, the complex Artin L-function $L_S(\chi, -)$ is holomorphic on $\mathbb{C}$. For T a finite set of places disjoint from S, one defines the (S,T)-modified Artin L-function $L_S^T(\chi, -)$; the corresponding equivariant object is the Stickelberger element $\Theta_S^T \in \mathbb{C}[\mathrm{Gal}(L/K)]$. A theorem of Siegel states that the specialisation $\Theta_S^T(0)$ is rational, that is, $\Theta_S^T(0) \in \mathbb{Q}[\mathrm{Gal}(L/K)]$. Works of Pi. Cassou-Noguès and Deligne–Ribet show that if L/K is abelian and T satisfies a torsion freeness condition as mentioned above, then this element is integral, that is, $\Theta_S^T(0) \in \mathbb{Z}[\mathrm{Gal}(L/K)]$.

New results

The main objective of this thesis is extending the results above to new cases. Analogously to Greenberg's p-adic Artin conjecture, we posit the following:

Conjecture A (equivariant p-adic Artin conjecture)**.** *Let $\Gamma_0 := \Gamma^{p^{n_0}}$ where n_0 is large enough such that Γ_0 is central in $\mathcal{G}$. Let $\mathfrak{M}$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Then the smoothed equivariant p-adic Artin L-function Φ_S^T is in the image of the composite map*

$$\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$$

where the first arrow is the natural map sending an invertible element to the class of the 1×1 matrix consisting of said element.

Under the assumption of the main conjecture, the statement can also be reformulated in terms of ζ_S^T (Conjecture 4.1.6).

As noted above, addressing this conjecture requires knowledge about the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$. Building on methods of Nickel from [Nic14], we study how indecomposable idempotents of $\mathbb{Q}_p[H]$ behave in $\mathcal{Q}(\mathcal{G})$. Before summarising our results, we introduce some notation. Let η resp. χ be irreducible characters of H resp. $\mathcal{G}$ with open kernel, and let $\mathbb{Q}_p(\eta)$ resp. $\mathbb{Q}_{p,\chi}$ be the fields obtained from $\mathbb{Q}_p$ by adjoining all values of η resp. χ on H. We say that two irreducible characters η, η' resp. χ, χ' of H resp. $\mathcal{G}$ with open kernel are equivalent, denoted $\eta \sim_{\mathbb{Q}_p} \eta'$ resp. $\chi \sim_{\mathbb{Q}_p} \chi'$, if there exists a Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)$ resp. $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p,\chi}/\mathbb{Q}_p)$ such that for all $h \in H$, $\sigma(\eta(h)) = \eta'(h)$ resp. $\sigma(\mathrm{res}^{\mathcal{G}}_H \chi(h)) = \mathrm{res}^{\mathcal{G}}_H \chi'(h)$ holds. Then the semisimple algebras $\mathbb{Q}_p[H]$ and $\mathcal{Q}(\mathcal{G})$ have Wedderburn decompositions

$$\mathbb{Q}_p[H] \simeq \prod_{\eta \in \mathrm{Irr}(H)/\sim_{\mathbb{Q}_p}} M_{n_\eta}(D_\eta) \quad \text{and} \quad \mathcal{Q}(\mathcal{G}) \simeq \prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(D_\chi)$$

where D_η resp. D_χ are skew fields with Schur index s_η resp. s_χ. Note that the skew fields D_η are well understood, owing to work of Hasse. Let χ be fixed, and let η be an irreducible constituent of $\mathrm{res}^{\mathcal{G}}_H \chi$. The extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is always a cyclic Galois p-extension. Let v_χ be the minimal exponent for which there exists a Galois automorphism $\tau \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi})$ such that for all $h \in H$, the equality $\eta(\gamma^{v_\chi} h \gamma^{-v_\chi}) = \tau(\eta(h))$ holds—in fact, there is exactly one such τ, and it is a generator of the Galois group. For the following statements, we make the additional assumption that $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified.

Lemma B. *If $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified, then $n_\chi = v_\chi n_\eta$ and $s_\chi = (\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}) s_\eta$.*

The skew fields D_χ are described as follows. Let $\mathcal{O}_{D_\eta}$ denote the unique maximal $\mathbb{Z}_p$-order in D_η. The Galois automorphism τ admits a unique extension to a $\mathbb{Q}_{p,\chi}$-automorphism of D_η of the same order. Let $\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]$ denote the ring of formal skew power series whose underlying additive group is the ring $\mathcal{O}_{D_\eta}[[X]]$ of formal power series, and multiplication is defined by $Xr = \tau(r)X + (\tau(r) - r)$ for all $r \in \mathcal{O}_{D_\eta}$.

Theorem C. *Assume that $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified. Then the skew field D_χ is isomorphic to* $\mathrm{Quot}\left(\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]\right)$.

Moreover, the preimage of the formal skew power series ring $\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]$ is the unique maximal order in D_χ. If the total ramification condition holds for all χ and η as above, then the preimage $\mathfrak{M}(\mathcal{G}) \subseteq \mathcal{Q}(\mathcal{G})$ of

$$\prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]])$$

is a maximal $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$.

The methods of Nichifor and Palvannan, which are based on Venjakob's noncommutative Weierstraß theory [Ven03], can be extended to the skew power series ring $\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]$. This leads to the following integrality theorem.

Theorem D. *Assume that for all irreducible characters χ with open kernel and all irreducible constituents $\eta \mid \mathrm{res}^{\mathcal{G}}_H \chi$, the extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified, and assume the main conjecture. Then ζ^T_S is in the image of the natural map $\mathfrak{M}(\mathcal{G}) \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))$. In particular, the equivariant p-adic Artin conjecture holds for $\mathfrak{M} = \mathfrak{M}(\mathcal{G})$.*

These theorems hold whenever indecomposable idempotents of $\mathbb{Q}_p[H]$ remain indecomposable in $\mathcal{Q}(\mathcal{G})$. This is the case when $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified, which explains the appearance of this condition in the statements above. This condition is satisfied, in particular, whenever $\mathcal{G}$ is a pro-p group (Lemma 2.4.10) or when $\mathcal{G} \simeq H \times \Gamma$ is a direct product (Lemma 2.4.11). A natural continuation of this work would be to generalise Theorem 3.3.11 to arbitrary extensions.

As we will discuss in Section 2.4.2.2, unramified pieces of the extension significantly affect decomposability of idempotents, meaning that the machinery used to derive the above results on the Wedderburn decomposition would need further refinement, an enterprise which lies beyond the scope of this thesis, and constitutes the subject of future work.

Knowledge of the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$ has another important application: it can be used to study the kernel $SK_1(\mathcal{Q}(\mathcal{G}))$ of the reduced norm map $\mathrm{nr} : K_1(\mathcal{Q}(\mathcal{G})) \to \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$. Vanishing of $SK_1(\mathcal{Q}(\mathcal{G}))$ is a special case of a conjecture of Suslin, and it is equivalent to uniqueness in the equivariant main conjecture. Reduction steps towards vanishing of $SK_1(\mathcal{Q}(\mathcal{G}))$ were proven by Ritter–Weiss [RW05] and Johnston–Nickel [JN20, §12]. In the case when $\mathcal{G}$ is a pro-p-group, Lau utilised the theory of higher local fields to study SK_1 of completed localisations [Lau12a, §3]. In Chapter 5, we extend these results to the context of Theorem 3.3.11, ultimately proving that SK_1 of equal characteristic completed localisations vanishes.

We stated the equivariant p-adic Artin conjecture for all $\Lambda(\Gamma_0)$-orders $\mathfrak{M}$ in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. The smaller $\mathfrak{M}$ is, the stronger this statement becomes, the extreme being $\mathfrak{M} = \Lambda(\mathcal{G})$. While proving this in general seems to be out of reach at the present juncture, it can be proven in the so-called p-abelian case, in which it is an easy corollary of a theorem of Johnston and Nickel [JN20].

Corollary E. *If p does not divide the order of the commutator subgroup of $\mathcal{G}$, then $\Phi_S^T \in \mathfrak{z}(\Lambda(\mathcal{G}))$. Moreover, Conjecture 4.1.3 holds with $\mathfrak{M} = \Lambda(\mathcal{G})$.*

Outline

The thesis is structured as follows.

Chapter 0 is preliminary in nature, its purpose being to recall some of the machinery used in later chapters; all results therein can be considered to be well-known (under an appropriate definition of this adjective).

In Chapter 1, we introduce smoothed equivariant p-adic Artin L-functions, following Johnston and Nickel, and formulate the corresponding version of the equivariant Iwasawa main conjecture.

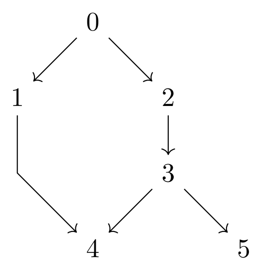
In Chapter 2, we investigate the ring structure of $\mathcal{Q}(\mathcal{G})$; this is independent of Chapter 1. We consider an indecomposable idempotent f of $\mathbb{Q}_p[H]$: then $f\mathbb{Q}_p[H]f$ is one of the skew fields D_η, and it can be endowed with a canonical Galois action. We describe the ring structure of $f\mathcal{Q}(\mathcal{G})f$ and relate this to the aforementioned Galois action. We conclude that under certain assumptions, the ring $f\mathcal{Q}(\mathcal{G})f$ is isomorphic to D_χ.

Chapter 3 continues the work of determining the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$. Under the assumptions of Chapter 2, we realise D_χ as a cyclic algebra. We also identify D_χ with the total ring of quotients of a formal skew power series ring as in Theorem 3.3.11 above.

Chapter 4 brings all of the above together. Building on the work of Nichifor and Palvannan, we prove a dimension reduction statement, which we then use to prove the integrality result Theorem 4.4.1.

Finally, in Chapter 5 we place the study of D_χ into the context of 2-dimensional higher local fields, following the approach of Lau. This allows us to make certain steps in the direction of Suslin's conjecture on the vanishing of the reduced Whitehead group $SK_1(D_\chi)$.

Dependencies between the individual chapters are visualised in the following diagram.



Chapter 0

Preliminaries

In this chapter, we fix some notation, and recall some well-known results. We claim no originality. No proofs will be given; we provide references instead.

0.1 Standing assumptions and notation

Here we collect some general notation used throughout the text. A list of specific notation can be found on page 91.

The letter p always stands for an odd prime number. The word 'ring' is short for 'not necessarily commutative ring with unity'. A domain is a ring with no zero divisors; in particular, principal ideal domains (PIDs) and unique factorisation domains (UFDs) are not necessarily commutative; we will give their respective definitions at the appropriate juncture. The term 'integral domain' is reserved for commutative domains. If $\mathscr{R}$ is a ring, $\mathrm{Quot}(\mathscr{R})$ stands for the total ring of quotients of $\mathscr{R}$, which is obtained from $\mathscr{R}$ by inverting all central regular elements; when $\mathscr{R}$ is an integral domain, this is the field of fractions, and to emphasise this, we use the notation $\mathrm{Frac}(\mathscr{R})$ instead.

We will abuse notation by writing $\oplus$ for a direct product of rings, even though this is not a coproduct in the category of rings.

For a ring $\mathscr{R}$, the ring of $n \times n$ matrices is $M_n(\mathscr{R})$. The $n \times n$ identity matrix is $\mathbf{1}_n$. This is not to be confused with the trivial character, which is denoted by $\mathbb{1}$.

The centre of a group $\mathscr{G}$ resp. a ring $\mathscr{R}$ is denoted by $\mathfrak{z}(\mathscr{G})$ resp. $\mathfrak{z}(\mathscr{R})$, and the abelianisation of a group $\mathscr{G}$ is $\mathscr{G}^{\mathrm{ab}}$. An algebraic closure of a field $\mathscr{F}$ is denoted by $\mathscr{F}^{\mathrm{c}}$. Overline means either topological closure or residue (skew) field, but never algebraic closure. If $\mathscr{S}$ is a finite set, then $\#\mathscr{S}$ denotes its cardinality.

For $n \geq 1$, μ_n stands for the group of nth roots of unity; μ_{p^∞} is the group of all p-power roots of unity. For a field $\mathscr{F}$, we write $\mu(\mathscr{F})$ for the group of all roots of unity in $\mathscr{F}$, or in other words, the torsion part of $\mathscr{F}^\times$.

In our notation, we make a clear distinction between rings of power series and completed group algebras: we use double square brackets for the former and blackboard square brackets for the latter. E.g. $\mathbb{Z}_p[[T]]$ is a ring of power series, and $\mathbb{Z}_p[\![\Gamma]\!]$ is a completed group algebra.

Script letters, like $\mathscr{G}$, (not to be confused with calligraphic ones, like $\mathcal{G}$) are used for making general statements, such as the ones above, or when citing lemmata from other works so as to avoid clash of notation.

0.2 Schur indices

A general reference on Schur indices is [CR87, §74]. We also mention the classical book [Yam74]. A good overview is provided in [Ung17].

Let $\mathcal{G}$ be a finite group. Then the group algebra $\mathbb{Q}_p[\mathcal{G}]$ is semisimple (Maschke's theorem), and its Wedderburn components correspond to irreducible characters of $\mathcal{G}$:

$$\mathbb{Q}_p[\mathcal{G}] = \bigoplus_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(\mathcal{D}_\chi) \tag{0.1}$$

Here $\mathcal{D}_\chi$ is a skew field, $\mathrm{Irr}(\mathcal{G})$ is the set of $\mathbb{Q}_p^c$-valued irreducible characters of $\mathcal{G}$, and two characers $\chi, \chi' \in \mathrm{Irr}(\mathcal{G})$ are equivalent, denoted $\chi \sim_{\mathbb{Q}_p} \chi'$, if there exists some $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p)$ such that ${}^\sigma\chi = \chi'$.

The dimension $\dim_{\mathfrak{z}(\mathcal{D}_\chi)} \mathcal{D}_\chi$ is a perfect square, and its square root is called the Schur index of the irreducible character χ resp. the (Schur) index of the skew field $\mathcal{D}_\chi$. In the literature, the term 'index' is often used when one considers skew fields in general, without any reference to a character; for us, the skew fields at hand arise from Wedderburn decompositions, and so we deem it reasonable to also refer to their indices as Schur indices.

The Schur index of an arbitrary skew field $\mathcal{D}$ is the minimal degree m such that there is a degree m extension $\mathcal{E}$ of $\mathfrak{z}(\mathcal{D})$ that is a splitting field of $\mathcal{D}$, that is, $\mathcal{E} \otimes_{\mathfrak{z}(\mathcal{D})} \mathcal{D} \xrightarrow{\sim} M_n(\mathcal{E})$ for some n. In fact, every maximal subfield of $\mathcal{D}$ is a splitting field, and its degree over $\mathfrak{z}(\mathcal{D})$ is the Schur index, see [Rei03, Theorem 7.15(i)]. Even more is true:

Proposition 0.2.1 ([Rei03, Theorem 28.5(ii)])**.** *Let $\mathcal{D}$ be a skew field, and let $\mathcal{E}/\mathfrak{z}(\mathcal{D})$ be a finite extension. Then there is an embedding of $\mathfrak{z}(\mathcal{D})$-algebras*

$$\mathcal{E} \hookrightarrow M_r(\mathcal{D})$$

for some $r \geq 1$. Let r be minimal with this property. Then $\mathcal{E}$ is a splitting field for $\mathcal{D}$ if and only if $\mathcal{E}$ is a maximal subfield inside $M_r(\mathcal{D})$ if and only if $\mathcal{E}$ is its own centraliser in $M_r(\mathcal{D})$.

Equivalently, one may define the Schur index of a character instead of the skew field associated with it. Let χ be an absolutely irreducible $\mathbb{Q}_p^c$-valued character of $\mathcal{G}$, and let

$$\psi := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p)} \sigma(\chi)$$

be the sum of all of its Galois conjugates, where $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(g) : g \in \mathcal{G})$. Then the Schur index s_χ of χ over $\mathbb{Q}_p$ is the minimal positive integer such that $s_\chi \psi$ is the character of a $\mathbb{Q}_p$-valued representation of $\mathcal{G}$. This definition readily extends to field extensions of $\mathbb{Q}_p$; however, in this work, we always consider Schur indices over $\mathbb{Q}_p$. We just remark that for a character χ as before, its Schur index is the same over $\mathbb{Q}_p$ and $\mathbb{Q}_p(\chi)$, see [CR81, Theorem 74.5(iii)].

Finally, we shall make frequent use of the following result:

Theorem 0.2.2 ([Wit52, Satz 10])**.** *If $\mathcal{G}$ is a finite group, then all Schur indices of skew fields occurring in the Wedderburn decomposition of $\mathbb{Q}_p[\mathcal{G}]$ divide $p-1$.*

0.3 Iwasawa algebras, characters and idempotents

Let $\mathcal{G}$ be a profinite group. For $\mathcal{F}/\mathbb{Q}_p$ a finite field extension with ring of integers $\mathcal{O}_\mathcal{F}$, define the Iwasawa algebra of $\mathcal{G}$ over $\mathcal{O}_\mathcal{F}$ as

$$\Lambda^{\mathcal{O}_\mathcal{F}}(\mathcal{G}) := \mathcal{O}_\mathcal{F} \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) := \mathcal{O}_\mathcal{F} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![\mathcal{G}]\!] = \mathcal{O}_\mathcal{F}[\![\mathcal{G}]\!]$$

Let $\mathcal{Q}^{\mathcal{F}}(\mathcal{G}) := \mathrm{Quot}(\mathcal{O}_{\mathcal{F}}[\![\mathcal{G}]\!])$ denote its total ring of quotients. For brevity, we will write $\mathcal{Q}(\mathcal{G}) := \mathcal{Q}^{\mathbb{Q}_p}(\mathcal{G})$ and $\mathcal{Q}^{\mathrm{c}}(\mathcal{G}) := \mathbb{Q}_p^{\mathrm{c}} \otimes_{\mathbb{Q}_p} \mathcal{Q}(\mathcal{G})$.

Let us specialise to the case when $\mathcal{G} = H \rtimes \Gamma$, where H is a finite group and $\Gamma \simeq \mathbb{Z}_p$ is isomorphic to the additive group of the p-adic integers. Then $\mathcal{G}$ is a 1-dimensional p-adic Lie group.

Let $\Gamma_0 := \Gamma^{p^{n_0}}$ where n_0 is chosen such that $\Gamma_0 \subseteq \mathcal{G}$ is central. This is the case when n_0 is large enough: indeed, since $\mathcal{G}$ is a semidirect product, failure of Γ to be central comes from the homomorphism $\varphi : \Gamma \to \mathrm{Aut}(H)$ defining conjugation by elements of Γ not being trivial. However, since H is finite, so is $\mathrm{Aut}(H)$, wherefore φ has open kernel in Γ. All such open subgroups are of the form Γ^{p^n} for some $n \geq 0$. Therefore if $\ker(\varphi) = \Gamma^{p^n}$ then $\Gamma^{p^{n_0}}$ is central in $\mathcal{G}$ whenever $n_0 \geq n$. Let $\gamma_0 := \gamma^{p^{n_0}}$.

Fix a topological generator γ of Γ. As in [JN19, §4.2] and in the proof of [RW04, Proposition 5], the Iwasawa algebra resp. its total ring of quotients admit the following decompositions:

$$\Lambda^{\mathcal{O}_{\mathcal{F}}}(\mathcal{G}) = \bigoplus_{i=0}^{p^{n_0}-1} \Lambda^{\mathcal{O}_{\mathcal{F}}}(\Gamma_0)[H]\gamma^i \tag{0.2}$$

$$\mathcal{Q}^{\mathcal{F}}(\mathcal{G}) = \bigoplus_{i=0}^{p^{n_0}-1} \mathcal{Q}^{\mathcal{F}}(\Gamma_0)[H]\gamma^i \tag{0.3}$$

Moreover, $\mathcal{Q}^{\mathcal{F}}(\mathcal{G})$ is a semisimple artinian ring, as shown in [RW04, Proposition 5(1)]. On the other hand, the Iwasawa algebra $\Lambda^{\mathcal{O}_{\mathcal{F}}}(\mathcal{G})$ is never semisimple.

As $\mathcal{Q}^{\mathcal{F}}(\mathcal{G})$ is semisimple, it can be studied further by investigating its Wedderburn decomposition. This can be done by considering characters and their associated idempotents as follows. By a complex resp. p-adic Artin character we shall mean the trace of a Galois representation of a finite dimensional vector space over $\mathbb{C}$ resp. $\mathbb{Q}_p^{\mathrm{c}}$ such that the character has open kernel. Let $\mathrm{Irr}(\mathcal{G})$ denote the set of absolutely irreducible $\mathbb{Q}_p^{\mathrm{c}}$-valued characters of $\mathcal{G}$ with open kernel, and let $\chi \in \mathrm{Irr}(\mathcal{G})$. Let $\eta \mid \mathrm{res}^{\mathcal{G}}_H \chi$ be an irreducible constituent of the restriction of χ to H. Define the fields

$$\mathcal{F}_\chi := \mathcal{F}(\chi(h) : h \in H) \tag{0.4}$$

$$\mathcal{F}(\eta) := \mathcal{F}(\eta(h) : h \in H) \tag{0.5}$$

Write $w_\chi := (\mathcal{G} : \mathrm{St}(\eta))$ where $\mathrm{St}(\eta)$ is the stabliliser of η in $\mathcal{G}$; this is a power of p since H stabilises η. The field $\mathcal{F}(\eta)$ is an extension of $\mathcal{F}_\chi$. Furthermore, the field $\mathcal{F}(\eta)$, and thus also the subfield $\mathcal{F}_\chi$, is abelian over $\mathcal{F}$, as it is contained in some cyclotomic extension.

Compliant with the notation of [Nic14, (3)], we have the following idempotents at hand:

$$e(\eta) := \frac{\eta(1)}{\#H} \sum_{h \in H} \eta(h^{-1})h \in \mathcal{F}(\eta)[H]$$

$$e_\chi := \sum_{g \in \mathcal{G}/\,\mathrm{St}(\eta)} e({}^g\eta) \in \mathcal{F}(\eta)[H]$$

$$\varepsilon(\eta) := \sum_{\sigma \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})} e({}^\sigma\eta) \in \mathcal{F}[H]$$

$$\varepsilon_\chi := \sum_{\sigma \in \mathrm{Gal}(\mathcal{F}_\chi/\mathcal{F})} e_{{}^\sigma\chi} \in \mathcal{F}[H]$$

For $g \in \mathcal{G}$ and $\sigma \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})$, the characters ${}^g\eta$ resp. ${}^\sigma\eta$ of H are given by ${}^g\eta(h) = \eta(ghg^{-1})$ resp. ${}^\sigma\eta(h) = \sigma(\eta(h))$. As in [Nic14, Lemma 1.1], there are irreducible constituents

$\eta_{(0)}, \ldots, \eta_{(\nu_\chi^{\mathcal{F}}-1)}$ of the restriction $\operatorname{res}_H^{\mathcal{G}} \chi$ such that there is a decomposition

$$\operatorname{res}_H^{\mathcal{G}} \chi = \sum_{i=0}^{\nu_\chi^{\mathcal{F}}-1} \sum_{\sigma \in \operatorname{Gal}(\mathcal{F}(\eta_{(i)})/\mathcal{F}_\chi)} {}^{\sigma}\eta_{(i)} = \sum_{g \in \mathcal{G}/\operatorname{St}(\eta)} {}^{g}\eta = \sum_{i=0}^{w_\chi-1} {}^{\gamma^i}\eta \tag{0.6}$$

This shows that irreducible constituents of $\operatorname{res}_H^{\mathcal{G}} \chi$ are all $\mathcal{G}$-conjugates of one another. Moreover, it follows that the index w_χ depends only on χ. Since H is a normal subgroup, this shows that for a given χ, the field $\mathcal{F}(\eta)$ does not depend on the choice of η. In particular, all degrees $(\mathcal{F}(\eta_{(i)}) : \mathcal{F}_\chi)$ are equal. It follows readily that the number $\nu_\chi^{\mathcal{F}}$ satisfies

$$w_\chi = \nu_\chi^{\mathcal{F}} \cdot (\mathcal{F}(\eta) : \mathcal{F}_\chi), \tag{0.7}$$

In particular, the degree $(\mathcal{F}(\eta) : \mathcal{F}_\chi)$ is a power of p because w_χ is. Note that w_χ is independent of $\mathcal{F}$. The number $\nu_\chi^{\mathcal{F}}$ depends on χ, but for a fixed χ, it is independent of the choice of η, since both w_χ and $(\mathcal{F}(\eta) : \mathcal{F}_\chi)$ are.

Two more consequences of (0.6) are the following:

$$e_\chi = \frac{\chi(1)}{\# H w_\chi} \sum_{h \in H} \chi(h^{-1}) h \in \mathcal{F}_\chi[H]$$
$$\varepsilon_\chi = \sum_{i=0}^{\nu_\chi^{\mathcal{F}}-1} \varepsilon(\eta_{(i)})$$

We return to discussing the semisimple algebra $\mathcal{Q}^{\mathcal{F}}(\mathcal{G})$. In [Nic14, Proposition 1.5], the centre is described:

$$\mathfrak{z}\left(\mathcal{Q}^{\mathcal{F}}(\mathcal{G})\right) \simeq \bigoplus_{\chi \in \operatorname{Irr}(\mathcal{G})/\sim_{\mathcal{F}}} \mathcal{Q}^{\mathcal{F}_\chi}(\Gamma'_\chi) \tag{0.8}$$

The equivalence relation $\sim_{\mathcal{F}}$ on $\operatorname{Irr}(\mathcal{G})$ is defined as follows: two characters χ, χ' are equivalent if there is a $\sigma \in \operatorname{Gal}(\mathcal{F}_\chi/\mathcal{F})$ such that ${}^{\sigma}(\operatorname{res}_H^{\mathcal{G}} \chi) = \operatorname{res}_H^{\mathcal{G}} \chi'$. Here $\Gamma'_\chi \simeq \mathbb{Z}_p$ is the profinite group generated by the element γ'_χ introduced in [Nic14, Lemma 1.2], which is a modification of the Ritter–Weiss element γ_χ from [RW04, Proposition 5(2)]. We will recall the definition of the Ritter–Weiss element in (1.3).

Furthermore, in [Nic14, Corollary 1.9], the following information is gathered about the Wedderburn decomposition:

$$\mathcal{Q}^{\mathcal{F}}(\mathcal{G}) \simeq \bigoplus_{\chi \in \operatorname{Irr}(\chi)/\sim_{\mathcal{F}}} M_{n_\chi}(D_\chi)$$

where D_χ is a skew field with centre $\mathcal{Q}^{\mathcal{F}_\chi}(\Gamma'_\chi)$, and

$$\chi(1) = n_\chi s_\chi \tag{0.9}$$

where s_χ is the Schur index of D_χ. In [Nic14, Corollary 1.13], it is shown that if η is as above, and s_η denotes its Schur index, then there are divisibilities

$$s_\chi \mid s_\eta \left(\mathcal{F}(\eta) : \mathcal{F}_\chi\right)$$
$$\frac{\eta(1)}{s_\eta} \mid n_\chi$$

In the case when $\mathcal{G}$ is a pro-p group, the Wedderburn decomposition was described by Lau in [Lau12a, Theorem 1]. The results of that article are based on Schilling's theorem (also attributed to Witt and Roquette, see the remark after [CR87, (74.15)]), which states that for a finite p-group $\mathcal{H}$, the group algebra $\mathbb{Q}_p[\mathcal{H}]$ is the direct sum of matrix algebras over fields. If $\mathcal{G}$ is pro-p, then H is a p-group, and we get that for all $\eta \in \mathrm{Irr}(H)$, the skew field D_η is a field, that is, $s_\eta = 1$. However, the skew fields D_χ may still have nontrivial Schur indices even when $\mathcal{G}$ is pro-p: an example of this phenomenon was computed by Lau, see [Lau12a, p. 1232ff].

0.4 Skew power series rings

Let $\mathcal{R}$ be a noetherian pseudocompact ring, that is, it is a complete Hausdorff noetherian topological ring with a fundamental system $(\mathcal{L}_i)_{i\in I}$ of open neighbourhoods of zero such that $\mathcal{L}_i \subseteq \mathcal{R}$ is a left ideal and each $\mathcal{R}/\mathcal{L}_i$ is a finite length $\mathcal{R}$-module. Let $\sigma \in \operatorname{End}\mathcal{R}$ be a ring endomorphism, and let $\delta : \mathcal{R} \to \mathcal{R}$ be a σ-derivation, that is, a homomorphism of additive groups such that for all $r, s \in \mathcal{R}$, $\delta(rs) = \delta(r)s + \sigma(r)\delta(s)$. Suppose that δ is σ-nilpotent. We refrain from giving the definition of σ-nilpotency in general. In the case when σ and δ commute, which is the only case we shall encounter in this work, this is equivalent to δ being topologically nilpotent, that is, for all $n \geq 1$ there is an $m \geq 1$ such that for all $k \geq m$, there is an inclusion $\delta^k(\mathcal{R}) \subseteq (\operatorname{rad}\mathcal{R})^n$, where $\operatorname{rad}\mathcal{R}$ is the Jacobson radical of $\mathcal{R}$.

For such a ring $\mathcal{R}$, endomorphism σ, and derivation δ, one can define the formal skew power series ring $\mathcal{R}[[X;\sigma,\delta]]$ as the ring with underlying additive group $\mathcal{R}[[X]]$ and multiplication defined by $Xr = \sigma(r)X + \delta(r)$ for $r \in \mathcal{R}$. Nilpotence of δ is needed to ensure that multiplication is well-defined, that is, the coefficients of any product converge in $\mathcal{R}$. We refer to [SV06, §§0–1] for details. In the sequel, we drop the adjective 'formal'.

Let $\mathcal{R}$ be a local ring with maximal ideal $\mathfrak{m} = \operatorname{rad}\mathcal{R}$ such that $\mathcal{R}$ is separated and complete with respect to the $\mathfrak{m}$-adic topology. Then skew power series over $\mathcal{R}$ possess a Weierstraß theory, as exhibited in [Ven03, §3]. For a skew power series $f(X) = \sum_{i=0}^{\infty} a_i X^i \in \mathcal{R}[[X;\sigma,\delta]]$, let

$$\operatorname{ord}^{\mathrm{red}}(f) := \inf\{i : a_i \in \mathcal{R}^\times\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

be the lowest degree term with an invertible coefficient, called the reduced order of f. Then the following hold:

Theorem 0.4.1 (Weierstraß division theorem)**.** *If $f \in \mathcal{R}[[X;\sigma,\delta]]$ has finite reduced order, then*

$$\mathcal{R}[[X;\sigma,\delta]] = \mathcal{R}[[X;\sigma,\delta]]f \oplus \bigoplus_{i=0}^{\operatorname{ord}^{\mathrm{red}} f - 1} \mathcal{R}[[X;\sigma,\delta]]X^i$$

Theorem 0.4.2 (Weierstraß preparation theorem)**.** *If $f \in \mathcal{R}[[X;\sigma,\delta]]$ has finite reduced order, then there is a unique unit $\varepsilon \in \mathcal{R}[[X;\sigma,\delta]]^\times$ and a unique distinguished skew polynomial $F \in \mathcal{R}[X;\sigma,\delta]$ such that $f = \varepsilon F$.*

Here a monic skew polynomial being distinguished means that all coefficients but the leading one are contained in the maximal ideal $\mathfrak{m}$.

0.5 Algebraic K-theory

We recall the relevant K-groups used in equivariant Iwasawa theory, mostly following the exposition in [JN14, §4.1]. Let $\mathcal{R}$ be a noetherian integral domain, $\mathcal{A}$ a finite dimensional algebra over Frac($\mathscr{R}$), and $\mathfrak{A}$ an $\mathscr{R}$-order in $\mathscr{A}$. Later we will set these to be $\mathscr{R} := \Lambda(\Gamma_0)$, $\mathscr{A} := \mathcal{Q}(\mathcal{G})$ and $\mathfrak{A} := \Lambda(\mathcal{G})$.

The groups $K_0(\mathfrak{A})$ resp. $K_0(\mathscr{A})$ are the Grothendieck groups of the abelian categories $\mathbf{P}(\mathfrak{A})$ resp. $\mathbf{P}(\mathscr{A})$ of finitely generated projective left $\mathfrak{A}$- resp. $\mathscr{A}$-modules; see [Suj13, §1] for details. We let $K_1(\mathfrak{A})$ resp. $K_1(\mathscr{A})$ denote the Whitehead groups of the rings $\mathfrak{A}$ resp. $\mathscr{A}$, as defined e.g. in [CR81, §40].

Let $\mathbf{C}^{\mathrm{b}}(\mathbf{P}(\mathfrak{A}))$ be the category of bounded complexes of finitely generated projective $\mathfrak{A}$-modules, and let $\mathbf{C}^{\mathrm{b}}_{\mathrm{tor}}(\mathbf{P}(\mathfrak{A}))$ be the full subcategory of such complexes with $\mathscr{R}$-torsion cohomology groups. We define the relative K_0-group $K_0(\mathfrak{A}, \mathscr{A})$ to be the abelian group generated by the objects of $\mathbf{C}^{\mathrm{b}}_{\mathrm{tor}}(\mathbf{P}(\mathfrak{A}))$, with the relations being $[C^\bullet] = 0$ for $C^\bullet$ acyclic, and $[C^\bullet] = [C'^\bullet] + [C''^\bullet]$ for short exact sequences $0 \to C'^\bullet \to C^\bullet \to C''^\bullet \to 0$ in $\mathbf{C}^{\mathrm{b}}_{\mathrm{tor}}(\mathbf{P}(\mathfrak{A}))$. In fact, $\mathbf{C}^{\mathrm{b}}_{\mathrm{tor}}(\mathbf{P}(\mathfrak{A}))$ is a Waldhausen category, and $K_0(\mathfrak{A}, \mathscr{A}) = K_0(\mathbf{C}^{\mathrm{b}}_{\mathrm{tor}}(\mathbf{P}(\mathfrak{A})))$; see [Wei13, II, Definition 9.1.2 and p. 185].

Remark 0.5.1. The relative K_0-group $K_0(\mathfrak{A}, \mathscr{A})$ has several equivalent definitions; the one just recalled is the one we shall use. A somewhat more explicit definition can be given by defining $K_0(\mathfrak{A}, \mathscr{A})$ as the abelian group generated by triples (P, φ, Q) where P and Q are finitely generated projective $\mathfrak{A}$-modules and $\varphi : \mathscr{A} \otimes_{\mathfrak{A}} P \xrightarrow{\sim} \mathscr{A} \otimes_{\mathfrak{A}} Q$ is an isomorphism of $\mathscr{A}$-modules. The relations are

$$(P, \varphi, Q) + (Q, \psi, R) = (P, \psi\varphi, R)$$
$$(P', \varphi', Q') + (P'', \varphi'', Q'') = (P, \varphi, Q)$$

where $P' \hookrightarrow P \twoheadrightarrow P''$ and $Q' \hookrightarrow Q \twoheadrightarrow Q''$ are exact sequences of $\mathfrak{A}$-modules, and the diagram

$$\begin{array}{ccccc} \mathscr{A} \otimes_{\mathfrak{A}} P' & \longrightarrow & \mathscr{A} \otimes_{\mathfrak{A}} P & \longrightarrow & \mathscr{A} \otimes_{\mathfrak{A}} P'' \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \mathscr{A} \otimes_{\mathfrak{A}} Q' & \longrightarrow & \mathscr{A} \otimes_{\mathfrak{A}} Q & \longrightarrow & \mathscr{A} \otimes_{\mathfrak{A}} Q'' \end{array}$$

is commutative. See [Wei13, II, Definition 2.10] for more on this. An isomorphism between this group and the one defined above is given by

$$[(P, \varphi, Q)] \mapsto \Big[\underset{-2}{P} \xrightarrow{\varphi} \underset{-1}{Q}\Big] \tag{0.10}$$

on triples (P, φ, Q) for which φ maps P to Q. Here the complex on the right is concentrated in degrees -2 and -1. For further details and other definitions, we refer the reader to [Suj13, §2]. ∘

The aforementioned K-groups are connected by the following exact sequence, termed the localisation exact sequence of K-theory:

$$K_1(\mathfrak{A}) \to K_1(\mathscr{A}) \xrightarrow{\partial} K_0(\mathfrak{A}, \mathscr{A}) \to K_0(\mathfrak{A}) \to K_0(\mathscr{A}) \tag{0.11}$$

The first resp. last arrows are induced by the inclusion $\mathfrak{A} \to \mathscr{A}$ using functoriality of Whitehead resp. Grothendieck groups. The connecting homomorphism is given as follows: if $M \in \mathrm{GL}_n(\mathscr{A})$, and $\widetilde{M} \in \mathrm{End}(\mathfrak{A})$ such that $1_{\mathscr{A}} \otimes \widetilde{M} = M$, then

$$\partial([M]) := \Big[\underset{-2}{\mathfrak{A}^n} \xrightarrow{\widetilde{M}} \underset{-1}{\mathfrak{A}^n}\Big] \tag{0.12}$$

where the complex is in degrees −2 and −1. Finally, the third arrow sends a complex $[P \xrightarrow{\varphi} Q]$ concentrated in degrees −2 and −1 to $[P] - [Q]$.

In the case $\mathcal{R} := \Lambda(\Gamma_0)$, $\mathcal{A} := \mathcal{Q}(\mathcal{G})$ and $\mathfrak{A} := \Lambda(\mathcal{G})$, the connecting homomorphism ∂ is surjective by [Wit11, Corollary 3.8]; see [JN14, §4.1] for further comments on this. Therefore the localisation sequence takes the following form:

$$K_1(\Lambda(\mathcal{G})) \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\partial} K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})) \hookrightarrow K_0(\Lambda(\mathcal{G})) \to K_0(\mathcal{Q}(\mathcal{G})) \tag{0.13}$$

Finally, we introduce some derived categorical language, which will make the formulation of some statements easier. We let $\mathbf{D}(\mathfrak{A})$ denote the derived category of $\mathfrak{A}$-modules. A complex of $\mathfrak{A}$-modules is perfect if, considered as an elemeent of $\mathbf{D}(\mathfrak{A})$, it is isomorphic to a complex in $\mathbf{C}^{\mathrm{b}}(\mathbf{P}(\mathfrak{A}))$. The full subcategory of perfect complexes with $\mathcal{R}$-torsion cohomology modules will be denoted by $\mathbf{D}^{\mathrm{perf}}_{\mathrm{tor}}(\mathfrak{A})$. These definitions give rise to a well-defined map $\mathbf{D}^{\mathrm{perf}}_{\mathrm{tor}}(\mathfrak{A}) \to K_0(\mathfrak{A}, \mathcal{A})$. In the notation $\mathbf{D}^{\mathrm{perf}}_{\mathrm{tor}}(\mathfrak{A})$, the domain $\mathcal{R}$ is suppressed; in the main body of this work, we will always have $\mathcal{R} := \Lambda(\Gamma_0)$ and $\mathfrak{A} := \Lambda(\mathcal{G})$, so this should not lead to confusion.

0.6 Maximal orders

We record some results on maximal orders; we refer to Reiner's excellent book [Rei03] for details.

In this section, let $\mathcal{R}$ be a noetherian integral domain (commutative), and let $\mathcal{K} := \operatorname{Frac} \mathcal{R}$ be its field of fractions. Let $\mathcal{A}$ be a separable $\mathcal{K}$-algebra, that is, a $\mathcal{K}$-algebra such that for any field extension $\mathcal{K}'/\mathcal{K}$, the $\mathcal{K}'$-algebra $\mathcal{A} \otimes_{\mathcal{K}} \mathcal{K}'$ is semisimple. An $\mathcal{R}$-order Δ in $\mathcal{A}$ is a subring of $\mathcal{A}$ (with the same unity element) such that Δ is a full $\mathcal{R}$-lattice in $\mathcal{A}$, that is, Δ is a finitely generated $\mathcal{R}$-submodule such that

$$\mathcal{A} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in \mathcal{K}, x_i \in \Delta, n \geq 0 \right\}$$

A maximal $\mathcal{R}$-order is an $\mathcal{R}$-order which is maximal with respect to containment.

Maximal orders always exist. An important case when they are unique is the following:

Theorem 0.6.1. *Let $\mathcal{R}$ be a complete discrete valuation ring (commutative). Let $\mathcal{D}$ be a skew field with centre containing $\mathcal{K}$ such that $\dim_{\mathcal{K}} \mathcal{D}$ is finite. Then there is a unique maximal $\mathcal{R}$-order in $\mathcal{D}$, namely the integral closure of $\mathcal{R}$ in $\mathcal{D}$.*

Theorem 0.6.1 is fairly general: it applies whenever $\mathcal{K}$ is a local field (i.e. when $\mathcal{R}$ has finite residue field) or a higher local field (see Section 0.9). In the case of local fields, the maximal order can be described even more explicitly, see Section 0.7. A note on terminology: Reiner calls all fields $\mathcal{K}$ as in Theorem 0.6.1 local fields, whereas we reserve this term for the case when $\mathcal{R}$ has finite residue field.

Maximal orders behave well with respect to taking matrix rings or direct sums:

Proposition 0.6.2 ([Rei03, Theorem 8.7])**.** *Let $\mathcal{R}$ be a noetherian integrally closed integral domain with field of fractions $\mathcal{K}$. Then if Δ is a maximal $\mathcal{R}$-order in a separable $\mathcal{K}$-algebra $\mathcal{A}$, then the matrix ring $M_n(\Delta)$ is a maximal $\mathcal{R}$-order in $M_n(\mathcal{A})$ for all $n \geq 1$.*

Proposition 0.6.3 ([Rei03, Theorem 10.5(ii)])**.** *Suppose that a separable $\mathcal{K}$-algebra $\mathcal{A}$ has decomposition $\mathcal{A} = \bigoplus_{i=1}^{n} \mathcal{A}_i$ into simple components. Then if for all $i = 1, \ldots, n$, the ring Δ_i is a maximal $\mathcal{R}$-order in $\mathcal{A}_i$, then $\bigoplus_{i=1}^{n} \Delta_i$ is a maximal $\mathcal{R}$-order in $\mathcal{A}$.*

The following two results show that the property of being a maximal order can be checked locally at height 1 primes or at the (algebraic) completion.

Proposition 0.6.4 ([Rei03, Theorem 11.4])**.** *Let $\mathcal{R}$ be a noetherian integrally closed integral domain with field of fractions $\mathcal{K}$. Let $\mathcal{A}$ be a separable $\mathcal{K}$-algebra. Then an $\mathcal{R}$-order Δ in $\mathcal{A}$ is maximal if and only if Δ is a reflexive $\mathcal{R}$-module and for each height 1 prime ideal $\mathfrak{p}$ of $\mathcal{R}$, the localisation $\Delta_{\mathfrak{p}}$ is a maximal $\mathcal{R}_{\mathfrak{p}}$-order in $\mathcal{A}$.*

Proposition 0.6.5 ([Rei03, Theorem 11.5])**.** *Let $\mathcal{R}$ be a noetherian integrally closed local integral domain with field of fractions $\mathcal{K}$. Let $\mathcal{A}$ be a separable $\mathcal{K}$-algebra. Let $\widehat{\mathcal{R}}$ be the completion of $\mathcal{R}$, let $\widehat{\mathcal{K}}$ be its field of fractions, and let $\widehat{\Delta} := \widehat{\mathcal{R}} \otimes_{\mathcal{R}} \Delta$ and $\widehat{\mathcal{A}} := \widehat{\mathcal{K}} \otimes_{\mathcal{K}} \mathcal{A}$. Then Δ is a maximal $\mathcal{R}$-order in $\mathcal{A}$ if and only if $\widehat{\Delta}$ is a maximal $\widehat{\mathcal{R}}$-order in $\widehat{\mathcal{A}}$.*

0.7 Skew fields over local fields

We begin with some generalities on cyclic algebras; see [Rei03, §30] for details. Let $\mathcal{L}/\mathcal{F}$ be a finite cyclic Galois extension of fields of degree s, with ς a generator of $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$. Let $a \in \mathcal{F}^\times$ be a unit. Then the cyclic algebra $\mathcal{A} := (\mathcal{L}/\mathcal{F}, \varsigma, a)$ is defined to be the $\mathcal{F}$-algebra

$$(\mathcal{L}/\mathcal{F}, \varsigma, a) := \bigoplus_{i=0}^{s-1} \mathcal{L}\alpha^i \tag{0.14}$$

where α is a formal nth root of a, and α obeys the multiplication rule $\alpha x \alpha^{-1} = \varsigma(x)$ for all $x \in \mathcal{L}$. Cyclic algebras need not be skew fields; a criterion for when this is the case is given by the following theorem of Wedderburn.

Theorem 0.7.1 ([Rei03, (30.7)], [Lam01, (14.9)])**.** *Let $\mathcal{L}/\mathcal{F}$ be a cyclic Galois extension with generator ς of degree s. Let $\mathcal{A} := (\mathcal{L}/\mathcal{F}, \varsigma, a)$. Then $\mathcal{A}$ is a skew field if a has order s in the norm factor group $\mathcal{F}^\times/N_{\mathcal{L}/\mathcal{F}}(\mathcal{K}^\times)$, that is, when $a, a^2, \ldots, a^{s-1} \notin N_{\mathcal{K}/\mathcal{F}}(\mathcal{K}^\times)$.*

The theory of skew fields over local fields was originally laid out by Hasse in [Has31], a presentation of which is available in [Rei03, Chapter 3]. We provide a brief review of the results, some of which are used extensively throughout this work.

Let $\mathcal{R}$ be a complete commutative discrete valuation ring, $\mathcal{K} = \mathrm{Frac}\,\mathcal{R}$ its field of fractions. Let $\mathcal{D}$ be a skew field with $\mathfrak{z}(\mathcal{D}) = \mathcal{K}$ and Schur index s. By extending the valuation from $\mathcal{K}$ to $\mathcal{D}$, one can show that $\mathcal{D}$ contains a unique maximal $\mathcal{R}$-order, which is the integral closure of $\mathcal{R}$ in $\mathcal{D}$: this is Theorem 0.6.1.

Under the additional assumption that the residue field of $\mathcal{R}$ is finite, the skew field $\mathcal{D}$ can be described explicitly: see [Rei03, §14]. In this case, $\mathcal{K}$ is a (1-dimensional) local field. Write q for the order of the residue field of $\mathcal{R}$. Let $\mathcal{W} = \mathcal{K}(\omega)$ be the cyclotomic field extension of $\mathcal{K}$ obtained by adjoining a primitive $(q^s - 1)$st root of unity ω. Then $\mathcal{W}$ is a maximal subfield of $\mathcal{D}$, called the inertia field (unique up to conjugacy in $\mathcal{D}$).

Fix a uniformiser $\pi \in \mathcal{K}$. Then there exists a uniformiser $\pi_{\mathcal{D}}$ in the maximal order such that $\pi_{\mathcal{D}}^s = \pi$, and there exists $1 \leq r \leq s$ coprime to s such that $\pi_{\mathcal{D}} \omega \pi_{\mathcal{D}}^{-1} = \omega^{q^r}$. The fraction $r/s \in \mathbb{Q}/\mathbb{Z}$ is called the Hasse invariant of $\mathcal{D}$. Let $\sigma \in \mathrm{Gal}(\mathcal{W}/\mathcal{K})$ be the automorphism defined by $\sigma(\omega) = \omega^{q^r}$; this is a power of the Frobenius and a generator of the Galois group $\mathrm{Gal}(\mathcal{W}/\mathcal{K})$. The maximal order in $\mathcal{D}$ is generated by $\pi_{\mathcal{D}}$ and ω as an $\mathcal{R}$-algebra. In other words, $\mathcal{D}$ is the cyclic algebra

$$\mathcal{D} = \big(\mathcal{K}(\omega)/\mathcal{K}, \sigma, \pi_{\mathcal{K}}\big) = \bigoplus_{i=0}^{s-1} \mathcal{K}(\omega)\pi_{\mathcal{D}}^i \tag{0.15}$$

Since $\mathscr{W}$ is a maximal subfield of $\mathscr{D}$, it is also a splitting field. A splitting isomorphism can be described explicitly as follows.

$$\begin{aligned} \mathscr{W} \otimes_{\mathscr{K}} \mathscr{D} &\xrightarrow{\sim} M_{\mathfrak{s}}(\mathscr{W}) \\ \omega \otimes 1 &\mapsto \omega \mathbf{1}_{\mathfrak{s}} \\ 1 \otimes \omega &\mapsto \operatorname{diag}\left(\omega, \sigma(\omega), \ldots, \sigma^{\mathfrak{s}-1}(\omega)\right) \\ 1 \otimes \pi_{\mathscr{D}} &\mapsto \begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ \vdots & & \ddots & \\ & & & 1 \\ \pi & & \ldots & 0 \end{pmatrix} \end{aligned} \tag{0.16}$$

Here $\omega \in \mathscr{W}$. The image of the subring $1 \otimes \mathscr{D} \subset \mathscr{W} \otimes_{\mathscr{K}} \mathscr{D}$ under this map is isomorphic to $\mathscr{D}$ (as rings).

0.8 Reduced norms and the Dieudonné determinant

We recall definitions of the reduced norm map, the Dieudonné determinant, and some of their key properties. Standard references on reduced norms include [CR81, §7D] and [Rei03, §9]. Caveat: while our definition of the reduced norm on a central simple algebra agrees with that of [Rei03, §9a], its extension to semisimple algebras differs from that in [Rei03, §9b] and [CR81, §7D]. For the Dieudonné determinant, see [Wei13, III§1] and [CR81, pp. 165–166]. A survey with a view towards Iwasawa algebras is provided in [NP19, pp. 616–617]; we shall follow the exposition there.

Let $\mathscr{K}$ be a field. Let $\mathscr{A}$ be a central simple $\mathscr{K}$-algebra. Let $\mathscr{E}/\mathscr{K}$ be a splitting field for $\mathscr{A}$, meaning that there is an isomorphism

$$\varphi : \mathscr{E} \otimes_{\mathscr{K}} \mathscr{A} \xrightarrow{\sim} M_n(\mathscr{E})$$

of $\mathscr{E}$-algebras. A splitting field always exists by [CR81, Proposition 7.25]. The reduced characteristic polynomial of a is defined to be

$$\operatorname{rcp}_{\mathscr{A}/\mathscr{K}}(a) := \operatorname{cp}(\varphi(1 \otimes a))$$

where $\operatorname{cp}(\varphi(1 \otimes a))$ is the characteristic polynomial of the matrix $\varphi(1 \otimes a)$ over the field $\mathscr{E}$. The reduced characteristic polynomial does not depend on the choice of the splitting field $\mathscr{E}$.

Recall that over a field $\mathscr{F}$, the determinant of a matrix $M \in M_n(\mathscr{F})$ is plus-minus the constant term of its characteristic polynomial, with sign depending on the degree:

$$\det M = (-1)^{\deg \operatorname{cp}(M)} \operatorname{cp}(M)(0)$$

The reduced norm, which can be seen as a noncommutative generalisation of the determinant, is defined in analogy with this description of the determinant:

$$\operatorname{nr}_{\mathscr{A}/\mathscr{K}}(a) := (-1)^{\deg \operatorname{rcp}_{\mathscr{A}/\mathscr{K}}(a)} \operatorname{rcp}_{\mathscr{A}/\mathscr{K}}(a)(0)$$

For every $n \geq 1$, the matrix ring $M_n(\mathscr{A})$ is also a central simple $\mathscr{K}$-algebra, and the above definition provides maps $\operatorname{nr}_{M_n(\mathscr{A})/\mathscr{K}}$. On invertible matrices, these take values in $\mathscr{K}^\times$, and they induce a reduced norm map on the colimit, denoted $\operatorname{nr} : K_1(\mathscr{A}) \to \mathscr{K}^\times$. The kernel of this map is denoted by $SK_1(\mathscr{A}) := \ker(\operatorname{nr})$.

The following result shows that elements in orders have integral reduced norms.

Proposition 0.8.1 ([Rei03, Theorem 10.1])**.** *Let $\mathscr{R}$ be a noetherian integrally closed integral domain, $\mathscr{K} = \operatorname{Frac}(\mathscr{R})$ its field of fractions, $\mathscr{A}$ a central simple $\mathscr{K}$-algebra, and Δ an $\mathscr{R}$-order in $\mathscr{A}$. Then the reduced characteristic polynomial of an element in Δ has coefficients in $\mathscr{R}$.*

The notion of a reduced norm extends to arbitrary semisimple $\mathcal{K}$-algebras as follows: if $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$, where each $\mathcal{A}_i$ is a central simple algebra with centre $\mathcal{K}_i$, then there is a reduced norm map $\mathrm{nr}_{\mathcal{A}_i/\mathcal{K}_i}$ in each component, and their product defines a homomorphism

$$\mathrm{nr}_{\mathcal{A}/\mathfrak{z}(\mathcal{A})} := \prod_{i=1}^{n} \mathrm{nr}_{\mathcal{A}_i/\mathcal{K}_i} : \mathcal{A} \to \mathfrak{z}(\mathcal{A}) = \prod_{i=1}^{n} \mathcal{K}_i$$

Remark 0.8.2. In the case $\mathcal{A} = \mathcal{Q}(\mathcal{G})$, a special case of a conjecture attributed to Suslin predicts that $\mathrm{nr} : K_1(\mathcal{Q}(\mathcal{G})) \to \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^{\times}$ is injective, or in other words, that $SK_1(\mathcal{Q}(\mathcal{G})) = 1$. See [Mer94, p. 250] for the conjecture on SK_1, and [RW04, Remark (E)] for why it is applicable in this case.

The conjecture was originally formulated by Suslin in [Sus06, p. 125]. This original version concerns a homomorphism φ from SK_1 of a division algebra to a certain quotient of the 4th étale cohomology group of its centre with values in $\mu_{s^2}^{\otimes 3}$ where s is the Schur index. Suslin proved that φ is an isomorphism when $s = 2$, and conjectured it to be injective for s an arbitrary prime. If the centre has cohomological dimension at most 3, the aforementioned étale cohomology group vanishes, and injectivity of φ would then force SK_1 to vanish as well.

Triviality of $SK_1(\mathcal{Q}(\mathcal{G}))$ implies uniqueness in the equivariant Iwasawa main conjecture, see Section 1.4. Progress towards vanishing of $SK_1(\mathcal{Q}(\mathcal{G}))$ has been made in [RW05, Corollary, p. 167], [Lau12a, §3], [Lau12b], and most recently in [JN20, §12]. See also Proposition 5.3.2 and Lemma 5.4.1. ◦

There is another generalisation of the usual determinant: the Dieudonné determinant. Let $\mathcal{R}$ be a semilocal ring (that is, $\mathcal{R}/\operatorname{rad}\mathcal{R}$ is semisimple artinian). Define the subgroup $W(\mathcal{R}) \leq \mathcal{R}^{\times}$ as

$$W(\mathcal{R}) := \{(1 + rs)(1 + sr)^{-1} : (1 + rs) \in \mathcal{R}^{\times}\}$$

Then the Dieudonné determinant $\det : \mathrm{GL}_n(\mathcal{R}) \to \mathcal{R}^{\times}/W(\mathcal{R})$ is the homomorphism characterised by the following two properties:

1. Elementary matrices have Dieudonné determinant 1. An elementary matrix is a matrix which differs from the identity matrix by a single off-diagonal entry.
2. If $r \in \mathcal{R}$ and $M \in GL_n(\mathcal{R})$ then $\det(\mathrm{diag}(r, 1, \ldots, 1)M) = r \det(M)$.

In particular, the Dieudonné determinant of a(n upper or lower) triangular matrix is the product of the elements in its diagonal. This makes sense since the Dieudonné determinant is only defined up to $W(\mathcal{R})$, which contains the commutator subgroup of $\mathcal{R}^{\times}$. (This is not obvious; consult either [Bre68, Theorem 3.7] or [Wei13, III, Lemma 1.3.3 and Exercise 1.1] for a proof.)

We indicate how to compute $\det A$ for $A \in M_n(\mathcal{R})$. We say that an $n \times n$ matrix is a permutation matrix if it is obtained from the identity matrix $\mathbf{1}_n$ by swapping two of its columns. This is the terminology used in [NP19]; some authors also call a product of such matrices a permutation matrix. Suppose that we can write $A = UBV$ where B is diagonal, and U and V are products of elementary, permutation and scalar matrices. In this case it is said that A admits a diagonal reduction via elementary operations. As explained above, we can compute the Dieudonné determinant of any diagonal matrix, and the same goes for elementary and scalar matrices. Permutation matrices have order at most 2, hence their Dieudonné determinants are second roots of unity.

Jacobson's elementary reduction theorem provides an important class of rings for which there is such a decomposition $A = UBV$. We say that $\mathcal{R}$ is a PID if all left ideals and all right ideals are principal, see [Jac43, Chapter 3, §1]. Then the elementary reduction theorem states the following:

Theorem 0.8.3 ([Jac43, Chapter 3, Theorem 16]). *If $\mathcal{R}$ is a PID, then every square matrix over $\mathcal{R}$ admits a diagonal reduction via elementary operations.*

In the colimit, the Dieudonné determinant naturally extends to a homomorphism $K_1(\mathcal{R}) \to \mathcal{R}^\times/W(\mathcal{R})$. Due to results of Vaserstein, there is an isomorphism $\mathcal{R}^\times/W(\mathcal{R}) \simeq (\mathcal{R}^\times)^{\mathrm{ab}}$ whenever $\mathcal{R}$ is a local ring or $2 \in \mathcal{R}^\times$, see [Wei13, III, Example 1.3.7]. Moreover, [Vas05, Theorem 2] states that in this case, the Dieudonné determinant provides an isomorphism

$$K_1(\mathcal{R}) \xrightarrow{\sim} (\mathcal{R}^\times)^{\mathrm{ab}} \tag{0.17}$$

The ring $\mathcal{Q}(\mathcal{G})$ is semisimple and therefore semilocal. The Iwasawa algebra $\Lambda(\mathcal{G})$ is finitely generated as a module over the commutative local ring $\Lambda(\Gamma_0)$, hence it is semilocal, see [CR81, Proposition 5.28(ii)]. Therefore the Dieudonné determinant is defined for both $\mathcal{Q}(\mathcal{G})$ and $\Lambda(\mathcal{G})$.

If $\mathcal{R} = \mathcal{D}$ is a skew field, then the Dieudonné determinant is a homomorphism $\det : \mathrm{GL}_n(\mathcal{D}) \to (\mathcal{D}^\times)^{\mathrm{ab}}$, and the reduced norm factors over it, see [CR81, (7.42)]

$$\begin{array}{ccc} \mathrm{GL}_n(\mathcal{D}) & \xrightarrow{\mathrm{nr}_{M_n(\mathcal{D})/\mathfrak{z}(\mathcal{D})}} & \mathfrak{z}(\mathcal{D})^\times \\ & {\scriptstyle \det} \searrow \quad \nearrow {\scriptstyle \mathrm{nr}_{\mathcal{D}/\mathfrak{z}(\mathcal{D})}} & \\ & (\mathcal{D}^\times)^{\mathrm{ab}} & \end{array} \tag{0.18}$$

We will sometimes talk about reduced norms of elements in $\Lambda(\mathcal{G})$. In this case, the map is understood to be the reduced norm map $\mathrm{nr}_{\mathcal{Q}(\mathcal{G})/\mathfrak{z}(\mathcal{Q}(\mathcal{G}))}$, as the ring $\Lambda(\mathcal{G})$ is not necessarily semisimple but it is contained in the semisimple ring $\mathcal{Q}(\mathcal{G})$. In particular, the reduced norm of such an element need not lie in $\Lambda(\mathcal{G})$, but see Corollary 4.6.5 for a special case in which this is true.

0.9 Higher local fields

We recall basics of the theory of higher local fields. A survey of this can be found in [Mor12]; for a more comprehensive overview, we refer to [FK00], where further references to the original articles are provided. Many results in [Ser80] are formulated not just for usual local fields but for any (complete) discretely valued field, and hence are valid for higher local fields as well.

The notion of a higher dimensional local field is defined recursively. A 0-dimensional local field is a finite field. For $n \geq 1$, an n-dimensional local field is a complete discretely valued field whose residue field is an $(n-1)$-dimensional local field. In particular, 1-dimensional local fields are precisely the usual local fields. The highest dimension we shall encounter in this work is 2.

As explained in [Mor12, §6], higher local fields can be constructed by successive localisation and completion. In the 2-dimensional regular case, this goes as follows. Let $\mathcal{A}$ be a 2-dimensional regular local commutative ring with maximal ideal $\mathfrak{m}$, and assume that it is essentially of finite type over $\mathbb{Z}$, that is, $\mathcal{A}$ is the localisation of a finitely generated $\mathbb{Z}$-algebra. Let $\mathfrak{p} \subset \mathcal{A}$ be a prime ideal of height 1 such that $\mathcal{A}/\mathfrak{p}$ is regular. First complete $\mathcal{A}$ in the $\mathfrak{m}$-adic topology, then localise at the prime ideal $\widehat{\mathfrak{p}}$ obtained from $\mathfrak{p}$, and finally complete with respect to $\widehat{\mathfrak{p}}$. The field of fractions of the ring $\widehat{\widehat{\mathcal{A}}_{\widehat{\mathfrak{p}}}}$ so obtained is a 2-dimensional local field.

Example 0.9.1. Let $\mathcal{A} := \mathbb{Z}[t]_{(p,t)}$ be the ring of polynomials in one variable t over $\mathbb{Z}$ localised at the ideal generated by p and t. This is a 2-dimensional regular local commutative ring with maximal ideal $\mathfrak{m} = (p,t)\mathcal{A}$. The ring $\mathbb{Z}[t]$ is a finitely generated $\mathbb{Z}$-algebra, and $\mathcal{A}$ is a localisation thereof, hence $\mathcal{A}$ is essentially of finite type over $\mathbb{Z}$. Its completion in the $\mathfrak{m}$-adic topology is $\widehat{\mathcal{A}} = \mathbb{Z}_p[[t]]$.

Let $\mathfrak{p} := t\mathbb{Z}[t]_{(p,t)}$. This is a prime ideal of height 1, and $\mathcal{A}/\mathfrak{p} \simeq \mathbb{Z}_{(p)}$, the localisation of $\mathbb{Z}$ at p. This is a regular local commutative ring (it is a DVR). Localising $\widehat{\mathcal{A}}$ at $\widehat{\mathfrak{p}}$, then completing, and taking the field of fractions, we obtain the 2-dimensional local field $\mathbb{Q}_p(\!(t)\!)$, the field of Laurent series in one variable over $\mathbb{Q}_p$. The details of the argument are laid out in [Mor12, Example 6.10(i)(a)]. ○

Besides the construction of Laurent series one encounters in the 1-dimensional case, the higher dimensional theory also involves fields of doubly infinite convergent power series, defined as follows. Let $\mathcal{F}$ be a complete discretely valued field with valuation $v_{\mathcal{F}}$. Then define

$$\mathcal{F}\{\{T\}\} := \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in \mathcal{F},\, \inf\{v_{\mathcal{F}}(a_i) : i \in \mathbb{Z}\} > -\infty,\, \lim_{i\to-\infty} a_i = 0 \right\}$$

The ring structure is defined in the obvious way, and there is a valuation $v_{\mathcal{F}\{\{T\}\}}$ on $\mathcal{F}\{\{T\}\}$ given by $v_{\mathcal{F}\{\{T\}\}}(\sum a_i T^i) := \inf\{v_{\mathcal{F}}(a_i) : i \in \mathbb{Z}\}$. The ring $\mathcal{F}\{\{T\}\}$ turns out to be a complete discretely valued field with respect to this valuation.

Example 0.9.2. As in Example 0.9.1, let $\mathcal{A} := \mathbb{Z}[t]_{(p,t)}$, but choose $\mathfrak{p} := p\mathbb{Z}[t]_{(p,t)}$ this time. This is a height 1 prime ideal, and the quotient $\mathcal{A}/\mathfrak{p} \simeq \mathbb{F}_p[t]_{(t)}$ is regular. Applying the process above, we obtain the 2-dimensional local field $\mathbb{Q}_p\{\{t\}\}$. ○

We give one last example, this one in characteristic p. As the classificiation theorem below shows, these together cover all prototypical instances of 2-dimensional higher local fields.

Example 0.9.3. Let $\mathbb{F}$ be a finite field, and let $\mathcal{A} := \mathbb{F}[t_1, t_2]_{(t_1,t_2)}$ be the localisation of the ring of polynomials in two variables t_1, t_2 at the ideal generated by t_1 and t_2. Then $\mathfrak{m} = (t_1, t_2)\mathcal{A}$ is a maximal ideal, and $\mathfrak{p} = (t_2)$ is a height 1 prime ideal. The conditions above are satisfied, and the resulting 2-dimensional local field is $\mathbb{F}(\!(t_1, t_2)\!)$. ○

Higher local fields possess a classification theorem. For 2-dimensional local fields, this takes the following form (the higher dimensional statement is completely analogous and shall not be needed).

Theorem 0.9.4. *Let $\mathcal{F}$ be a 2-dimensional local field with residue field $\overline{\mathcal{F}}$, and let $\mathbb{F}$ denote the finite field that is the residue field of the 1-dimensional local field $\overline{\mathcal{F}}$. Then one of the following is true.*

- *If $\mathcal{F}$ is of (equal) positive characteristic (and hence so are all successive residue fields), then $\mathcal{F} \simeq \mathbb{F}(\!(T_1)\!)(\!(T_2)\!)$.*
- *If $\mathcal{F}$ is of equal characteristic zero (that is, $\operatorname{char}\mathcal{F} = \operatorname{char}\overline{\mathcal{F}} = 0$), then $\mathcal{F} \simeq \overline{\mathcal{F}}(\!(T)\!)$.*
- *If $\mathcal{F}$ is of mixed characteristic (that is, $\operatorname{char}\mathcal{F} = 0$ and $\operatorname{char}\overline{\mathcal{F}} > 0$), then $\mathcal{F}$ is isomorphic to a finite extension of $\mathbb{Q}_{p,\mathbb{F}}\{\{T\}\}$ where $\mathbb{Q}_{p,\mathbb{F}}$ is the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$.*

Fields of the form $\mathbb{F}(\!(T_1)\!)(\!(T_2)\!)$, $\overline{\mathcal{F}}(\!(T)\!)$, and $\mathbb{Q}_{p,\mathbb{F}}\{\{T\}\}$ (but not their finite extensions) are referred to as standard 2-dimensional local fields.

Chapter 1

Smoothed equivariant p-adic Artin L-functions

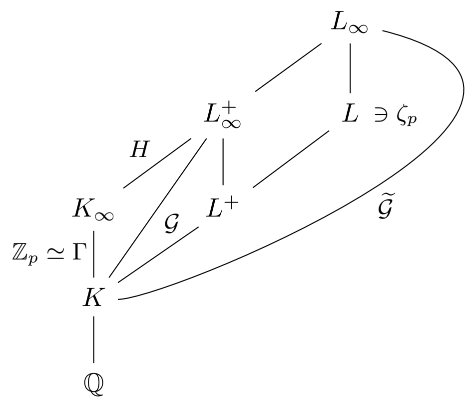
We begin this chapter by introducing the setup of our work (Section 1.1). Then we recall the definitions of characterwise resp. equivariant p-adic Artin L-functions, and introduce a smoothed version of the latter (Section 1.2). Subsequently, we recall the module Y_S^T of Johnston and Nickel (Section 1.3), and relate these analytic resp. algebraic object to each other by a smoothed equivariant main conjecture (Section 1.4).

1.1 Setup: Galois extensions and sets of places

This section details the standing assumptions that shall be kept in place for the rest of this work. In case milder assumptions also suffice, this will be noted. Some results on changing this setup can be found in Section 4.2.

As before, let p be an odd prime. Let L be a CM number field containing ζ_p, that is, L is quadratic over its maximal totally real subfield L^+. Let $K \subseteq L^+$ be a totally real number field such that L/K is a finite Galois extension. In other words, L/K is a finite Galois CM extension.

Let K_∞/K be the cyclotomic $\mathbb{Z}_p$-extension of K, and let $L_\infty = K_\infty L$ be the composite. Under Leopoldt's conjecture, the only $\mathbb{Z}_p$-extension of a totally real field is the cyclotomic one, so—at least conjecturally—focusing on the cyclotomic $\mathbb{Z}_p$-extension is not actually a restriction.



Let L_∞^+ be the maximal totally real subfield of L_∞. Define the Galois groups $\widetilde{\mathcal{G}} := \mathrm{Gal}(L_\infty/K)$, $\mathcal{G} := \mathrm{Gal}(L_\infty^+/K)$, $\Gamma := \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$, and $H := \mathrm{Gal}(L_\infty^+/K_\infty)$. Since the extension L/K is finite, H is also a finite (not necessarily abelian) group. Then $\mathcal{G} \simeq H \rtimes \Gamma$ is a semidirect product.

In the sequel, we will mostly work with the totally real extension L_∞^+/K. However, for some constructions—most notably that of Y_S^T introduced in Section 1.3—we do need p-power roots of unity, and thus L_∞. The setup described above agrees with that of §5.2 and §8 of [JN19], which is where the module Y_S^T showing up in the main conjecture (Conjecture 1.4.4 below) was first defined.

Let S be a finite set of places of K containing all places ramifying in L_∞/K. In particular, S contains all p-adic places, since these ramify in the $\mathbb{Z}_p$-extension K_∞/K, and all infinite places, since K is totally real whereas L_∞ is totally complex.

Let T be another nonempty finite set of places of K such that $S \cap T = \emptyset$. Moreover, we assume that

$$\bigcap_{w \in T(L)} \big\{x \in \mu(L) : x \equiv 1 \pmod{w}\big\} = \{1\} \tag{1.1}$$

where $T(L)$ is the set of places of L above those in T, and $\mu(L)$ denotes the group of roots of unity in L. This is a torsion freeness condition: it means that the group of S-units has no element other than 1 congruent to 1 modulo all places above T. We will refer to T as the smoothing set.

Remark 1.1.1. In other words, the sets S and T satisfy condition $\mathrm{Hyp}(S, T)$ of [JN19, §3.4]. Some special cases when (1.1) is satisfied are listed in [JN19, Remark 3.1].

Note that assumption (1.1) is indeed necessary for the definition of Y_S^T from [JN19] to be valid. Indeed, the assumption is used in the finite-level statement Proposition 7.6, which in turn is used in the infinite-level Proposition 8.2 and its Corollary 8.3, which in turn are needed for the construction of Y_S^T in Proposition 8.5. ◦

We will also use the letter T to denote a variable in power series rings, but this should not lead to confusion. The smoothing set T will usually appear as a superscript, e.g. as in Y_S^T.

1.2 p-adic Artin L-functions

In this section, we recall definitions of Artin L-functions in the complex, characterwise p-adic, and equivariant p-adic settings. We then introduce a smoothed version and study its basic properties.

The statements of this section are valid without assuming either one of the assumptions $\zeta_p \in L$, $T \neq \emptyset$ or (1.1).

1.2.1 Definitions

First, let us recall the classical definition of complex Artin L-functions for the extension L_∞^+/K; mutatis mutandis, the same definition works for L_∞/K.

Definition 1.2.1 ([Neu99, Definition VII.10.1])**.** Let χ be an irreducible complex character of $\mathcal{G}$ with open kernel (i.e. $\mathcal{G}/\ker\chi$ is finite). Define the S-truncated Artin L-function as

$$L_S(\chi, s) := \prod_{v \notin S} \det\big(1 - \varphi_{w_\infty}\mathfrak{N}(v)^{-s}\big|V_\chi\big)^{-1}$$

where

- for all v we fix a place w_∞ of L_∞^+ above it;

- V_χ is the representation affording χ, that is, V_χ is a simple $\mathbb{C}[\mathcal{G}/\ker\chi]$-module with character χ; in other words, V_χ is a $\mathbb{C}$-vector space, $\rho_\chi : \mathcal{G} \twoheadrightarrow \mathcal{G}/\ker\chi \to \operatorname{Aut} V_\chi$ an Artin representation, and $\chi = \operatorname{Tr}\rho_\chi$;
- φ_{w_∞} is the Frobenius at w_∞ (this makes sense since v is unramified).

Here $\mathfrak{N}(v)$ denotes the absolute norm of v. The definition is independent of the choices for the places w_∞: changing w_∞ leads to replacing φ_{w_∞} by a conjugate, and the determinant is conjugation invariant.

The function $L_S(\chi, s)$ is absolutely convergent for $\operatorname{Re} s > 1$, and has a meromorphic continuation to the complex plane. ◦

Let $\chi_{\mathrm{cyc}} : \operatorname{Gal}(K(\mu_{p^\infty})/K) \to \mathbb{Z}_p^\times$ be the cyclotomic character. The decomposition $\mathbb{Z}_p^\times \simeq \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ gives rise to characters

$$\begin{aligned} \omega &: \operatorname{Gal}(K(\mu_{p^\infty})/K) \to \operatorname{Gal}(K(\zeta_p)/K) \to \mu_{p-1} \\ \kappa &: \operatorname{Gal}(K(\mu_{p^\infty})/K) \to \operatorname{Gal}(K_\infty/K) \to 1 + p\mathbb{Z}_p \end{aligned}$$

Here ω is the Teichmüller character.

The construction of p-adic Artin L-functions for linear characters was given by Pi. Cassou-Noguès resp. Deligne–Ribet in [Cas79] resp. [DR80].

Theorem 1.2.2. *For each linear character $\chi : \mathcal{G} \to \mathbb{C}_p^\times$ with open kernel, there exists a unique continuous function (called the S-truncated p-adic Artin L-function associated with χ)*

$$\begin{aligned} &L_{p,S}(\chi, -) : \mathbb{Z}_p \to \mathbb{C}_p && \text{for } \chi \neq \mathbb{1} \\ &L_{p,S}(\mathbb{1}, -) : \mathbb{Z}_p - \{1\} \to \mathbb{C}_p && \end{aligned}$$

satisfying the interpolation property

$$L_{p,S}(\chi, 1-n) = L_S(\chi\omega^{-n}, 1-n) \quad \forall n \geq 1$$

Moreover, there exist power series $G_{\chi,S}(T) \in \mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]]$ such that

$$L_{p,S}(\chi, 1-s) = \frac{G_{\chi,S}(u^s - 1)}{H_\chi(u^s - 1)}$$

where γ is the fixed topological generator of Γ, and it corresponds to $1 + T$ under the (non-canonical) isomorphism $\mathcal{O}_{\mathbb{Q}_p(\chi)}[\![\Gamma]\!] \simeq \mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]]$, $u := \kappa(\gamma)$, and

$$H_\chi(T) := \begin{cases} \chi(\gamma)(1+T) - 1 & \text{if } H \subseteq \ker\chi \\ 1 & \text{otherwise} \end{cases}$$

The construction of p-adic Artin L-functions was extended to not necessarily linear Artin characters via Brauer induction by Greenberg in [Gre83]. This, however, weakens the above statement, with $G_{\chi,S}(T)$ now being a quotient of two power series. The interpolation property is also weakened from $n \geq 1$ to $n \geq 2$. Both of these weakenings are due to possibly negative coefficients in the Brauer induction theorem. In a recent preprint, Ellerbrock and Nickel verified the interpolation property at $n = 1$, see [EN22, Corollary 6.3].

Theorem 1.2.3. *Let $\chi : \mathcal{G} \to \mathbb{C}_p$ be a nontrivial character with open kernel. There exists a unique continuous function*

$$L_{p,S}(-,\chi) : \mathbb{Z}_p \to \mathbb{C}_p$$

satisfying the interpolation property

$$L_{p,S}(1-n,\chi) = L_S(1-n,\chi\omega^{-n}) \quad \forall n \geq 2$$

Moreover, there exists a fraction $G_{\chi,S}(T) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]])$ of power series such that

$$L_{p,S}(1-s,\chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)} \tag{1.2}$$

where u, T and $H_\chi(T)$ are as in Theorem 1.2.2.

Ritter and Weiss defined equivariant p-adic Artin L-functions as follows; see [RW04, §§3–4] for the original version and [JN19, §§4.4, 4.8] for the formulation used here.

Definition 1.2.4 ([RW04, Theorem 8])**.** Let $R_p(\mathcal{G})$ be the additive group generated by $\mathbb{Q}_p^c$-valued characters of $\mathcal{G}$ with open kernel. Let $\mathrm{Hom}^*(R_p(\mathcal{G}), \mathcal{Q}^c(\Gamma)^\times)$ denote the group of homomorphisms $f : R_p(\mathcal{G}) \to \mathcal{Q}^c(\Gamma)^\times$ satisfying the following transformation properties:

- for all characters ρ such that $\mathrm{res}^{\mathcal{G}}_H \rho = \mathbb{1}$ (so-called type W characters): $f(\chi \otimes \rho) = \rho^\sharp(f(\chi))$ where $\rho^\sharp \in \mathrm{Aut}(\mathcal{Q}^c(\Gamma))$ is the automorphism induced by $\rho^\sharp(\gamma) = \rho(\gamma)\gamma$ on the fixed topological generator γ of Γ;
- for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p)$: $f({}^\sigma\chi) = \sigma(f(\chi))$. ◦

Theorem 1.2.5 ([RW04, Proof of Theorem 8])**.** *There is an isomorphism*

$$\mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times \xrightarrow{\sim} \mathrm{Hom}^*\big(R_p(\mathcal{G}), \mathcal{Q}^c(\Gamma)^\times\big)$$

We give a brief description of the isomorphism in Theorem 1.2.5. Ritter and Weiss proved that there exists a unique element

$$\gamma_\chi \in \mathfrak{z}\left(\mathcal{Q}^c(\mathcal{G})e_\chi\right) \tag{1.3}$$

which acts trivially on the space V_χ and it is of the form $\gamma_\chi = g_\chi c_\chi$, where $g_\chi \in \mathcal{G}$ is mapped to $\gamma^{w_\chi} \in \Gamma$ under the projection $\mathcal{G} \twoheadrightarrow \Gamma$, and $c_\chi \in (\mathbb{Q}_p^c[H]e_\chi)^\times$; see [RW04, Proposition 5(2)]. We will refer to γ_χ as the Ritter–Weiss element. In fact, this construction also works over a sufficiently large finite extension of $\mathbb{Q}_p$ instead of $\mathbb{Q}_p^c$, see [JN19, §4.4]. Let Γ_χ denote the procyclic group generated by γ_χ. Then $\mathfrak{z}(\mathcal{Q}^c(\mathcal{G})e_\chi) \simeq \mathcal{Q}^c(\Gamma_\chi)$, which gives rise to a group homomorphism

$$j_\chi : \mathfrak{z}\left(\mathcal{Q}^c(\mathcal{G})\right) \twoheadrightarrow \mathfrak{z}(\mathcal{Q}^c(\mathcal{G})e_\chi) \simeq \mathcal{Q}^c\left(\Gamma_\chi\right) \xrightarrow{\gamma_\chi \mapsto \gamma^{w_\chi}} \mathcal{Q}^c\left(\Gamma\right)$$

Then the isomorphism of Theorem 1.2.5 is given by

$$z \mapsto (\chi \mapsto j_\chi(z)) \tag{1.4}$$

Characterwise p-adic Artin L-functions satisfy the transformation properties listed in Definition 1.2.4, see [RW04, Proposition 11]. Therefore they can be seen as elements on the right hand side of the isomorphism of Theorem 1.2.5, so the following definition makes sense:

Definition 1.2.6 ([JN19, §4.8]). Let $\mathcal{L}_{L_\infty^+/K,S} \in \mathrm{Hom}^*(R_p(\mathcal{G}), \mathcal{Q}^c(\Gamma)^\times)$ be defined by the assignment

$$\mathcal{L}_{L_\infty^+/K,S} : \chi \mapsto \frac{G_{\chi,S}(\gamma - 1)}{H_\chi(\gamma - 1)}$$

The S-truncated equivariant p-adic Artin L-function $\Phi_S(L_\infty^+/K) \in \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$ is the corresponding element under the isomorphism of Theorem 1.2.5. ◦

We are now ready to define a smoothed version of the equivariant p-adic Artin L-function, following [JN19, §8.3]. The element $\Psi_{S,T}$ there differs from our Φ_S^T by a twist by the cyclotomic character, which is in line with the fact that $\Psi_{S,T}$ corresponds to the CM extension L_∞/K whereas Φ_S^T corresponds to the totally real extension L_∞^+/K.

Definition 1.2.7. For each $v \in T$ fix a place w_∞ of L_∞^+. Define

$$\Phi_S^T(L_\infty^+/K) := \Phi_S(L_\infty^+/K) \cdot \prod_{v \in T} \mathrm{nr}\left(1 - \varphi_{w_\infty}\right)$$

Here the reduced norm map is applied to the class of the map

$$\Lambda(\mathcal{G}) \xrightarrow{1-\varphi_{w_\infty}} \Lambda(\mathcal{G})$$

in $K_1(\mathcal{Q}(\mathcal{G}))$. Note that φ_{w_∞} makes sense since T only contains unramified places. If there is no risk of confusion, we will omit the field extension from the notation and simply write Φ_S^T. ◦

It would have been possible to introduce smoothing already in the complex L-function, as is done in [MG22, (0.20) and Footnote 4]. Instead, we make the following:

Definition 1.2.8. Let $\mathcal{L}_{L_\infty^+/K,S}^T$ denote the element of $\mathrm{Hom}^*(R_p(\mathcal{G}), \mathcal{Q}^c(\Gamma)^\times)$ corresponding to Φ_S^T under the isomorphism of Theorem 1.2.5. ◦

1.2.2 Functoriality

We describe how $\mathcal{L}_{L_\infty^+/K,S}^T$ and $\Phi_S^T(L_\infty^+/K)$ transform under replacing $\mathcal{G}$ by a quotient. Such properties are available for non-smoothed equivariant p-adic Artin L-functions, see [RW04, Proposition 12]. These go back to corresponding statements for non-equivariant p-adic Artin L functions, see [Gre83], and ultimately to non-equivariant complex Artin L-functions, see [Neu99, Proposition VII§10.4(iii, iv)].

Let $L/L'/K$ be a tower of fields such that L'/K satisfies the same conditions as imposed upon L/K. Then all notions defined for L/K are also defined for L'/K. We shall distinguish the objects associated with L'/K from those defined for L/K by a prime symbol, e.g. $\mathcal{G}' := \mathrm{Gal}(L_\infty'^+/K)$. Note that $\Gamma' = \Gamma$. We choose the places relevant to the construction of smoothed p-adic L-functions such that they remain compatible with the non-primed versions, that is, we require $w_\infty \mid w'_\infty$. In particular, there is a natural map $\pi : \mathcal{G} \twoheadrightarrow \mathcal{G}'$ sending φ_{w_∞} to $\varphi_{w'_\infty}$. The natural projection π induces surjective maps $\Lambda(\mathcal{G}) \twoheadrightarrow \Lambda(\mathcal{G}')$ and $\mathcal{Q}(\mathcal{G}) \twoheadrightarrow \mathcal{Q}(\mathcal{G}')$, which we will, by abuse of notation, also denote by π. Functoriality of K-groups induces a morphism $K_1(\pi) : K_1(\mathcal{Q}(\mathcal{G})) \to K_1(\mathcal{Q}(\mathcal{G}'))$.

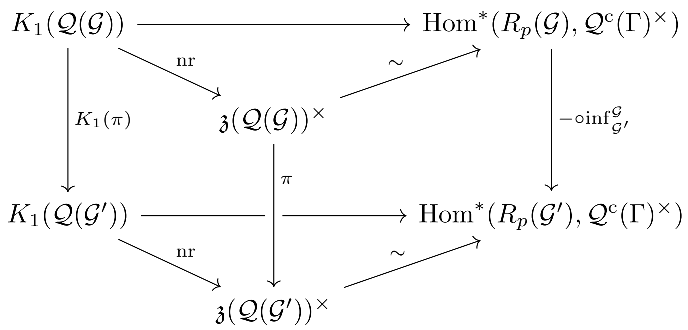
Lemma 1.2.9. *The elements $\mathcal{L}_{L_\infty^+/K,S}^T$ and $\Phi_S^T(L_\infty^+/K)$ satisfy the following transformation properties: in the setup above,*

$$\mathcal{L}_{L_\infty^+/K,S}^T \circ \inf_{\mathcal{G}'}^{\mathcal{G}} = \mathcal{L}_{L_\infty'^+/K,S}^T$$

$$\pi\left(\Phi_S^T(L_\infty^+/K)\right) = \Phi_S'^T(L_\infty'^+/K)$$

Proof. For $T = \emptyset$, this is [RW04, Proposition 12]. It remains to verify compatibility of the smoothing factors $(1 - \varphi_{w_\infty})$ on both sides.

There is a commutative diagram:



(1.5)

The square containing K_1-groups and Hom^*-groups is [RW04, Lemma 9] for deflation maps, and the two triangles are [JN19, (4.4)], originally formulated in [RW04, p. 558]. The left front square is obviously commutative.

Commutativity of the right front square can be seen from the definition of the northeast-pointing arrows: these are given by (1.4). In this proof, we write γ'_χ for the Ritter–Weiss element of the quotient extension. This notation is unfortunate, and shall not be used elsewhere: the element γ'_χ here is not the same as the element in (0.8). For commutativity of the square, one only needs to check that π maps γ_χ to γ'_χ, which follows from the uniqueness property of γ_χ, as shown in [MG22, Lemma 3.3.1(iii)].

Finally, the left vertical arrow sends $(1 - \varphi_{w_\infty})$ to $(1 - \varphi_{w'_\infty})$ because the places w_∞ and w'_∞ were chosen compatibly. The claims follow. ☐

In Section 4.2.2, these properties will be used to establish functoriality for integrality of Φ_S^T under the assumption of the main conjecture.

Remark 1.2.10. One may also consider replacing $\mathcal{G}$ by an open subgroup $\mathcal{G}'$. Then in the statement of Lemma 1.2.9, one can replace inflation by induction, as done in [RW04, Proposition 12]. See also [MG22, §3.3.2]. This, however, does not directly lead to a functoriality statement as for quotients, the crux being the definition of a map $\mathcal{Q}(\mathcal{G})^\times \to \mathcal{Q}(\mathcal{G}')^\times$. ○

1.3 The module Y_S^T

Let us now return to the setup introduced in Section 1.1.

Let $j \in \widetilde{\mathcal{G}}$ denote complex conjugation. Every $\Lambda(\widetilde{\mathcal{G}})$-module M decomposes as $M^+ \oplus M^-$, where j acts as $+1$ resp. -1 on M^+ resp. M^-. Let

$$\Lambda(\widetilde{\mathcal{G}})_- := \Lambda(\widetilde{\mathcal{G}})/(1+j)$$

The $\Lambda(\widetilde{\mathcal{G}})$-module structure on M induces $\Lambda(\mathcal{G})$- resp. $\Lambda(\widetilde{\mathcal{G}})_-$-module structures on the direct summand M^+ resp. M^-.

For a $\Lambda(\widetilde{\mathcal{G}})$-module M and $r \in \mathbb{Z}$, let $M(r)$ denote the rth Tate twist, which has the same underlying abelian group as M with group action twisted by the rth power of the cyclotomic character: $g \star m := \chi_{\mathrm{cyc}}(g)^r \cdot g * m$ where $g \in \mathcal{G}$ and $m \in M$, "$\star$" is the new action and "$*$" is the old one. We note two properties of $\Lambda(\widetilde{\mathcal{G}})$-modules which shall be used in the sequel.

Lemma 1.3.1. *Let M be a $\Lambda(\widetilde{\mathcal{G}})$-module. Then $(M(-1))^-(1) = M^+$.*

Proof. Since $\chi_{\mathrm{cyc}}(j) = -1$, we have $(M(-1))^-(1) = (M(-1)(1))^+ = M^+$. □

Lemma 1.3.2. *There is an isomorphism $\Lambda(\mathcal{G}) \simeq \Lambda(\widetilde{\mathcal{G}})_-(1)$ of $\Lambda(\widetilde{\mathcal{G}})$-modules.*

Proof. The ring $\Lambda(\widetilde{\mathcal{G}})_-$ has a natural $\Lambda(\widetilde{\mathcal{G}})$-module structure via the natural projection $\Lambda(\widetilde{\mathcal{G}}) \twoheadrightarrow \Lambda(\widetilde{\mathcal{G}})_-$. It follows that $\Lambda(\widetilde{\mathcal{G}})_-$ as a $\Lambda(\widetilde{\mathcal{G}})$-module is generated by a single element $m \in \Lambda(\widetilde{\mathcal{G}})_-$:

$$\langle m \rangle_{\Lambda(\widetilde{\mathcal{G}})} = \Lambda(\widetilde{\mathcal{G}})_-$$

Taking Tate twists, we get

$$\langle m(1) \rangle_{\Lambda(\widetilde{\mathcal{G}})} = \Lambda(\widetilde{\mathcal{G}})_-(1) \tag{1.6}$$

The element $j \in \widetilde{\mathcal{G}}$ acts as -1 on $\Lambda(\widetilde{\mathcal{G}})_- = \Lambda(\widetilde{\mathcal{G}})/(1+j)$. Since $\chi_{\mathrm{cyc}}(j) = -1$, this means that after a Tate twist, j acts trivially on $\Lambda(\widetilde{\mathcal{G}})_-(1)$. Since $\mathcal{G} = \widetilde{\mathcal{G}}/\langle j \rangle$, it follows that the action of $\Lambda(\widetilde{\mathcal{G}})$ on $\Lambda(\widetilde{\mathcal{G}})_-$ factors through $\Lambda(\mathcal{G})$.

Consequently, (1.6) can be sharpened to $\langle m(1) \rangle_{\Lambda(\mathcal{G})} = \Lambda(\widetilde{\mathcal{G}})_-(1)$, which means that there is a surjection of $\Lambda(\mathcal{G})$-modules

$$\Lambda(\mathcal{G}) \twoheadrightarrow \Lambda(\widetilde{\mathcal{G}})_-(1) \tag{1.7}$$

The group Γ_0 is a central open subgroup both in $\mathcal{G}$ and in $\widetilde{\mathcal{G}}$, and the ranks of $\Lambda(\mathcal{G})$ and $\Lambda(\widetilde{\mathcal{G}})_-(1)$ as $\Lambda(\Gamma_0)$-modules agree. Therefore (1.7) is an isomorphism. □

Let X_S denote the S-ramified Iwasawa module associated with L_∞. More explicitly, X_S is the Galois group of the maximal abelian pro-p extension of L_∞ unramified outside S over L_∞. The maximal j-invariant submodule of X_S is the S-ramified Iwasawa module X_S^+ associated with L_∞^+. This is a finitely generated torsion $\Lambda(\Gamma_0)$-module by [NSW20, Proposition 11.3.1].

Theorem 1.3.3 ([JN19, Proposition 8.5, Lemma 8.6])**.** *In the setup of Section 1.1, there exists a $\Lambda\big(\widetilde{\mathcal{G}}\big)_-$-module $Y_S^T(-1)$ sitting in an exact sequence of $\Lambda(\widetilde{\mathcal{G}})_-$-modules*

$$0 \to X_S^+(-1) \to Y_S^T(-1) \to \left(\bigoplus_{v \in T} \mathrm{ind}_{\widetilde{\mathcal{G}}_{w_\infty}}^{\widetilde{\mathcal{G}}} \mathbb{Z}_p(-1) \right)^- \to \mathbb{Z}_p(-1) \to 0 \tag{1.8}$$

Moreover, $Y_S^T(-1)$ has projective dimension ≤ 1 over $\Lambda(\widetilde{\mathcal{G}})_-$, and it is torsion over $\Lambda(\Gamma_0)$.

Since the module $Y_S^T(-1)$ has a projective resolution of length one over $\Lambda(\widetilde{\mathcal{G}})_-$, there exist a projective $\Lambda\big(\widetilde{\mathcal{G}}\big)_-$-module P and an integer $n \geq 0$ such that there is an exact sequence

$$0 \to P \to \left(\Lambda\big(\widetilde{\mathcal{G}}\big)_-\right)^n \to Y_S^T(-1) \to 0 \tag{1.9}$$

Since $Y_S^T(-1)$ is torsion over $\Lambda(\Gamma_0)$, the difference of the K_0-classes over $\Lambda(\widetilde{\mathcal{G}})_-$ of the first two terms in this short exact sequence becomes zero over the ring of quotients $\mathcal{Q}(\widetilde{\mathcal{G}})_-$:

$$\begin{aligned} K_0\left(\Lambda\big(\widetilde{\mathcal{G}}\big)_-\right) &\hookrightarrow K_0\left(\mathcal{Q}\big(\widetilde{\mathcal{G}}\big)_-\right) \\ \left[\Lambda\big(\widetilde{\mathcal{G}}\big)_-^n\right] - [P] &\mapsto 0 \end{aligned}$$

Here the map is injective by a result of Witte, see (0.13). Hence the first two terms of the projective resolution (1.9) are stably isomorphic over $\Lambda(\widetilde{\mathcal{G}})_-$. That is, there is a projective $\Lambda(\widetilde{\mathcal{G}})_-$-module Q such that $\Lambda\big(\widetilde{\mathcal{G}}\big)_-^n \oplus Q \simeq P \oplus Q$. Since projective modules are direct summands of free modules, by possibly enlarging Q, there are isomorphisms $\Lambda\big(\widetilde{\mathcal{G}}\big)_-^m \simeq \Lambda\big(\widetilde{\mathcal{G}}\big)_-^n \oplus Q \simeq P \oplus Q$ for some $m \geq 0$. Consequently, there is a projective resolution

$$0 \to \left(\Lambda(\widetilde{\mathcal{G}})_-\right)^m \to \left(\Lambda(\widetilde{\mathcal{G}})_-\right)^m \to Y_S^T(-1) \to 0$$

Twist by $(+1)$, using Lemma 1.3.2:

$$0 \to \Lambda(\mathcal{G})^m \xrightarrow{\alpha} \Lambda(\mathcal{G})^m \to Y_S^T \to 0 \tag{1.10}$$

where α is some $m \times m$ matrix. This allows us to associate with Y_S^T a class in the relative K-group $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$: considering $[Y_S^T]$ as a complex concentrated in degree 0, the resolution (1.10) shows that in the derived category $\mathbf{D}(\Lambda(\mathcal{G}))$, this is isomorphic to

$$\left[Y_S^T\right] = \Big[\underset{-1}{\Lambda(\mathcal{G})^m} \xrightarrow{\ \alpha\ } \underset{0}{\Lambda(\mathcal{G})^m}\Big]$$

where the complex on the right is concentrated in degrees -1 and 0. In particular, $[Y_S^T]$ is in $\mathbf{D}^{\mathrm{perf}}_{\mathrm{tor}}(\Lambda(\mathcal{G}))$, and thus it defines a class in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$.

1.4 Smoothed equivariant Iwasawa main conjecture

Let us first recall the equivariant Iwasawa main conjecture (EIMC) with uniqueness and without smoothing, as formulated in [JN19, Conjecture 4.3] and [JN20, Conjecture 4.7].

Definition 1.4.1. Consider the following complex:

$$C_S^\bullet(L_\infty^+/K) := \mathrm{RHom}\left(\mathrm{R}\Gamma_{\text{ét}}\left(\mathrm{Spec}\,\mathcal{O}_{L_\infty^+,S}, \underline{\mathbb{Q}_p/\mathbb{Z}_p}\right), \mathbb{Q}_p/\mathbb{Z}_p\right)$$

where $\mathcal{O}_{L_\infty^+,S}$ is the ring of S-integers of L_∞^+, consisting of elements of L_∞^+ with non-negative valuation away from S, and $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is a constant sheaf. ○

The nonzero cohomology groups of $C_S^\bullet(L_\infty^+/K)$ are

$$H^{-1}(C_S^\bullet(L_\infty^+/K)) \simeq X_S^+ \text{ and } H^0(C_S^\bullet(L_\infty^+/K)) \simeq \mathbb{Z}_p$$

Thus $C_S^\bullet(L_\infty^+/K)$ is a complex in $\mathbf{D}^{\mathrm{perf}}_{\mathrm{tor}}(\Lambda(\mathcal{G}))$. In particular, it defines a class in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$.

Conjecture 1.4.2 (non-smoothed EIMC with uniqueness)**.** *Let L/K be any finite Galois CM extension with S a finite set of places as above. There exists a unique element $\zeta_S(L_\infty^+/K) \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\mathrm{nr}\,(\zeta_S(L_\infty^+/K)) = \Phi_S(L_\infty^+/K)$. Moreover, $\partial\,(\zeta_S(L_\infty^+/K)) = -[C_S^\bullet(L_\infty^+/K)]$, where $[C_S^\bullet(L_\infty^+/K)]$ is the class of $C_S^\bullet(L_\infty^+/K)$ in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$.*

$$\begin{array}{ccccc}
 & & K_1(\mathcal{Q}(\mathcal{G})) & \xrightarrow{\ \partial\ } & K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})) \\
 & & \big\downarrow \mathrm{nr} & & \\
\exists!\,\zeta_S(L_\infty^+/K) & \longmapsto & & \longrightarrow & -[C_S^\bullet(L_\infty^+/K)] \\
\big\downarrow & & & & \\
\Phi_S(L_\infty^+/K) & & \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times & &
\end{array}$$

Remark 1.4.3. It is known that the boundary homomorphism is surjective, see (0.13). Vanishing of $SK_1(\mathcal{Q}(\mathcal{G}))$ is equivalent to the uniqueness part of the conjecture. ○

According to [JN19, Proposition 8.5], the middle two terms of (1.8), considered as 0th resp. (-1)st terms of a complex, represent $C^\bullet_S(L^+_\infty/K)(-1)$:

$$\Bigg[\underset{-1}{Y^T_S(-1)} \longrightarrow \underset{0}{\Bigg(\bigoplus_{v\in T} \operatorname{ind}^{\widetilde{\mathcal{G}}}_{\widetilde{\mathcal{G}}_{w_\infty}} \mathbb{Z}_p(-1)\Bigg)^-}\Bigg] = \left[C^\bullet_S(L^+_\infty/K)(-1)\right] \in K_0\left(\Lambda(\widetilde{\mathcal{G}})_-, \mathcal{Q}(\widetilde{\mathcal{G}})_-\right)$$

Tate twisting this by $+1$ yields the following, using Lemmata 1.3.1 and 1.3.2:

$$\Bigg[\underset{-1}{Y^T_S} \longrightarrow \underset{0}{\Bigg(\bigoplus_{v\in T} \operatorname{ind}^{\widetilde{\mathcal{G}}}_{\widetilde{\mathcal{G}}_{w_\infty}} \mathbb{Z}_p\Bigg)^+}\Bigg] = \left[C^\bullet_S(L^+_\infty/K)\right] \in K_0\left(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})\right) \tag{1.11}$$

Furthermore, [JN19, Equation (8.3)] provides the following short exact sequence of left $\Lambda(\widetilde{\mathcal{G}}_{\widetilde{w}_\infty})$-modules:

$$0 \to \Lambda\left(\widetilde{\mathcal{G}}_{\widetilde{w}_\infty}\right) \xrightarrow{1-\chi_{\mathrm{cyc}}(\varphi_{\widetilde{w}_\infty})\varphi_{\widetilde{w}_\infty}} \Lambda\left(\widetilde{\mathcal{G}}_{\widetilde{w}_\infty}\right) \to \mathbb{Z}_p(-1) \to 0 \tag{1.12}$$

where the labelled arrow is right multiplication by $(1-\chi_{\mathrm{cyc}}(\varphi_{\widetilde{w}_\infty})\varphi_{\widetilde{w}_\infty})$ and for all $v \in T$, $\widetilde{w}_\infty$ is a fixed place of L_∞ lying above w_∞. Taking $(-)_-$-parts and Tate twisting by $(+1)$ as above, we find that these factors correspond exactly to those by which we perturbed the analytic p-adic L-function in Definition 1.2.7.

In light of this, we posit the following smoothed version of Conjecture 1.4.2:

Conjecture 1.4.4 (smoothed EIMC with uniqueness)**.** *Let L/K, S and T satisfy the assumptions of Section 1.1. Then there is a unique element $\zeta^T_S(L^+_\infty/K) \in K_1(\mathcal{Q}(\mathcal{G}))$ such that* $\operatorname{nr}\left(\zeta^T_S(L^+_\infty/K)\right) = \Phi^T_S$*. Moreover,* $\partial\left(\zeta^T_S(L^+_\infty/K)\right) = [Y^T_S]$.

$$\begin{array}{ccccc}
 & & K_1(\mathcal{Q}(\mathcal{G})) & \xrightarrow{\ \partial\ } & K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})) \\
 & & \Big\downarrow \operatorname{nr} & & \\
\exists!\, \zeta^T_S & \longmapsfrom & & \longmapsto & [Y^T_S] \\
\big\downarrow & & & & \\
\Phi^T_S(L^+_\infty/K) & & \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times & &
\end{array}$$

Lemma 1.4.5. *Let the extension L/K and the sets of places S and T satisfy the assumptions of Section 1.1. Then Conjecture 1.4.2 is equivalent to Conjecture 1.4.4.*

Proof. The ingredients of the proof are all contained in the paragraphs preceding Conjecture 1.4.4; it remains to put them together.

We first deal with the existence and 'moreover' parts of the two main conjectures. Assume that Conjecture 1.4.2 holds. Then let

$$\zeta^T_S := \zeta_S \cdot \prod_{v\in T} [(1-\varphi_{w_\infty})] \in K_1(\mathcal{Q}(\mathcal{G}))$$

Its reduced norm is

$$\mathrm{nr}(\zeta_S^T) = \mathrm{nr}(\zeta_S) \cdot \prod_{v \in T} \mathrm{nr}(1 - \varphi_{w_\infty}) = \Phi_S \cdot \prod_{v \in T} \mathrm{nr}(1 - \varphi_{w_\infty}) = \Phi_S^T$$

as desired. The image under the boundary homomorphism ∂ is

$$\begin{aligned}\partial\left(\zeta_S^T\right) &= \partial\left(\zeta_S\right) + \sum_{v \in T} \partial\left(1 - \varphi_{w_\infty}\right) \\ &= -\left[C^\bullet(L_\infty^+/K)\right] + \sum_{v \in T} \partial\left(1 - \varphi_{w_\infty}\right)\end{aligned}$$

The equality (1.11) in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$ lets us rewrite the first term as the complex represented by $Y_S^T \to \left(\bigoplus_{v\in T} \mathrm{ind}_{\widetilde{\mathcal{G}}_{w_\infty}}^{\widetilde{\mathcal{G}}} \mathbb{Z}_p\right)^+$. The free resolution (1.10) allows us to replace Y_S^T by α:

$$= -\left[\underset{-2}{\Lambda(\mathcal{G})^m} \xrightarrow{\alpha} \underset{-1}{\Lambda(\mathcal{G})^m} \longrightarrow \underset{0}{\left(\bigoplus_{v\in T} \mathrm{ind}_{\widetilde{\mathcal{G}}_{w_\infty}}^{\widetilde{\mathcal{G}}} \mathbb{Z}_p\right)^+}\right] + \sum_{v \in T} \partial\left(1 - \varphi_{w_\infty}\right)$$

By the definition of ∂ and the isomorphism (0.10), the summation becomes

$$= -\left[\underset{-2}{\Lambda(\mathcal{G})^m} \xrightarrow{\alpha} \underset{-1}{\Lambda(\mathcal{G})^m} \longrightarrow \underset{0}{\left(\bigoplus_{v\in T} \mathrm{ind}_{\widetilde{\mathcal{G}}_{w_\infty}}^{\widetilde{\mathcal{G}}} \mathbb{Z}_p\right)^+}\right] + \sum_{v \in T} \left[\underset{-2}{\Lambda(\mathcal{G})} \xrightarrow{1-\varphi_{w_\infty}} \underset{-1}{\Lambda(\mathcal{G})}\right]$$

Twisting the exact sequence (1.12) by $+1$ and taking plus parts, we find that the two summations cancel each other out. Therefore $\partial\left(\zeta_S^T\right)$ is

$$= -\left[\underset{-2}{\Lambda(\mathcal{G})^m} \xrightarrow{\alpha} \underset{-1}{\Lambda(\mathcal{G})^m}\right] = \left[Y_S^T\right]$$

as desired. In conclusion, Conjecture 1.4.2 implies Conjecture 1.4.4 without uniqueness. By the same computation, Conjecture 1.4.4 implies Conjecture 1.4.4 without uniqueness by setting

$$\zeta_S := \zeta_S^T \cdot \prod_{v\in T} \left[(1 - \varphi_{w_\infty})\right]^{-1} \in K_1(\mathcal{Q}(\mathcal{G}))$$

Lastly we show that the two uniqueness statements are equivalent. Assume Conjecture 1.4.2, and let $\zeta_S^T, \zeta_S'^T \in K_1(\mathcal{Q}(\mathcal{G}))$ both satisfy Conjecture 1.4.4 without uniqueness. Then

$$\zeta_S^T \cdot \prod_{v\in T} \left[(1 - \varphi_{w_\infty})\right]^{-1} \text{ and } \zeta_S'^T \cdot \prod_{v\in T} \left[(1 - \varphi_{w_\infty})\right]^{-1}$$

both satisfy Conjecture 1.4.2, and therefore must be equal; hence so are ζ_S^T and $\zeta_S'^T$. The converse implication is proven using the same argument. □

Remark 1.4.6. An even quicker way to show that the two uniqueness statements are equivalent is to use the fact that they are both equivalent to $SK_1(\mathcal{Q}(\mathcal{G})) = 1$. ∘

Finally, we show that the 'maximal order main conjecture' (a theorem) admits a smoothed version. We first recall the statement in the non-smoothed case.

Theorem 1.4.7 ([RW04, Theorem 16, Remark (H)], [JN14, §4.5])**.** *Let $\mathfrak{M}$ be a maximal $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. If $x_S \in K_1(\mathcal{Q}(\mathcal{G}))$ is such that $\partial(x_S) = -\left[C_S^\bullet(L_\infty^+/K)\right]$, then* $\mathrm{nr}(x_S)\Phi_S^{-1} \in \mathfrak{z}(\mathfrak{M})^\times$.

Corollary 1.4.8 (smoothed maximal order main conjecture)**.** *Let $\mathfrak{M}$ be a maximal $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. If $x_S^T \in K_1(\mathcal{Q}(\mathcal{G}))$ is such that $\partial(x_S^T) = \left[Y_S^T\right]$, then* $\mathrm{nr}(x_S^T)(\Phi_S^T)^{-1} \in \mathfrak{z}(\mathfrak{M})^\times$.

Proof. The proof is by reduction to the non-smoothed version using the same techniques as for establishing equivalence of smoothed and non-smoothed main conjectures (Lemma 1.4.5). In the proof, we omit the extension from the notation $\mathcal{L}^T_{L_\infty^+/K,S}$.

There is a commutative diagram

$$\begin{array}{ccc} & K_1(\mathcal{Q}(\mathcal{G})) & \\ {}^{\mathrm{nr}}\swarrow & & \searrow^{\mathrm{Det}} \\ \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times & \xrightarrow[z\mapsto(\chi\mapsto j_\chi(z))]{\sim} & \mathrm{Hom}^*(R_p(\mathcal{G}), \mathcal{Q}^{\mathrm{c}}(\Gamma)^\times) \end{array} \tag{1.13}$$

Here the horizontal arrow is the same as in Theorem 1.2.5. We omit the definition of the homomorphism Det, and refer to [JN14, §4.3] instead. The horizontal arrow maps Φ_S to $\mathcal{L}_S$ and Φ_S^T to $\mathcal{L}_S^T$. Using the commutativity of the triangle, this yields

$$\mathcal{L}_S^T = \mathcal{L}_S \cdot \prod_v \mathrm{Det}(1-\varphi_{w_\infty})$$

Let x_S^T be as in the statement, and let

$$x_S := x_S^T \cdot \prod_v [(1-\varphi_{w_\infty})]^{-1}$$

Then $\partial(x_S) = -\left[C_S^\bullet(L_\infty^+/K)\right]$ by the computation in the proof of Lemma 1.4.5. Consequently,

$$\mathrm{Det}\left(x_S^T\right)\left(\mathcal{L}_S^T\right)^{-1} = \mathrm{Det}\left(x_S\right)\mathcal{L}_S^{-1}$$

Let $\mathbb{Z}_p^{\mathrm{c}}$ be the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p^{\mathrm{c}}$, and let $\Lambda^{\mathrm{c}}(\Gamma) := \mathbb{Z}_p^{\mathrm{c}} \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)$. Ritter and Weiss showed that $\mathrm{Det}(x_S)\mathcal{L}_S^{-1}$ is contained in $\mathrm{Hom}^*(R_p(\mathcal{G}), \Lambda^{\mathrm{c}}(\Gamma)^\times)$, see [RW04, Theorem 16]. Moreover, they showed that the horizontal arrow in (1.13) identifies $\mathfrak{z}(\mathfrak{M})^\times$ with this Hom*-group, see [RW04, Remark (H)]. The assertion now follows from commutativity of the diagram. ☐

Chapter 2

Ring structure of Wedderburn components of $\mathcal{Q}(\mathcal{G})$

In this chapter and the next one, we will investigate the Wedderburn decomposition of the semisimple ring $\mathcal{Q}(\mathcal{G})$. This will later be used to examine the reduced norm map, which is defined Wedderburn componentwise, and apply this to the study of integrality properties of Φ_S^T.

The Wedderburn decomposition of the group ring $\mathbb{Q}_p[H]$ is well understood: the skew fields occurring can be described explicitly. Therefore the same holds for $\mathcal{Q}(\Gamma_0)[H]$. A fundamental idea is that together with the decomposition (0.3), this provides a way to attack $\mathcal{Q}(\mathcal{G})$.

We first consider parts of the Wedderburn decomposition associated with an irreducible character of H (Section 2.1). We extend the Galois action from the centre to the skew fields themselves (Section 2.2), and then compare this to the action of Γ (Section 2.3). We then describe the ring structure on subrings $f_\eta^{(j)}\mathcal{Q}(\mathcal{G})f_\eta^{(j)}$ of $\mathcal{Q}(\mathcal{G})$ cut out by indecomposable idempotents $f_\eta^{(j)}$ of $\mathbb{Q}_p[H]$. Finally, we restrict our attention to the case when these idempotents remain indecomposable in $\mathcal{Q}(\mathcal{G})$ (Section 2.4). We verify this in the case when $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified. If this holds, then the aforementioned rings are in fact skew fields, and they can be identified with D_χ. This relationship will be studied in Chapter 3.

2.1 Wedderburn decompositions and their skew fields

The group ring $\mathbb{Q}_p[H]$ is semisimple by Maschke's theorem. Consider its Wedderburn decomposition

$$\mathbb{Q}_p[H] = \prod_{\eta \in \mathrm{Irr}(H)/\sim_{\mathbb{Q}_p}} M_{n_\eta}(D_\eta)$$

where the equivalence relation is as in (0.1), and each D_η is a finite dimensional skew field over its centre $\mathbb{Q}_p(\eta)$. In particular, the statements in Section 0.7 apply to D_η. Let $\mathcal{O}_{D_\eta}$ denote the unique maximal $\mathbb{Z}_p$-order in D_η.

We introduce some notation, to be used extensively in the sequel. Fix a uniformiser π_η of $\mathbb{Q}_p(\eta)$, and write π_{D_η} resp. ω_η for the elements $\pi_{\mathscr{D}}$ resp. ω of Section 0.7. Then $\pi_\eta = \pi_{D_\eta}^{s_\eta}$, and ω_η is a root of unity of order $q_\eta^{s_\eta}$, where q_η is the order of the residue field of $\mathbb{Q}_p(\eta)$, and s_η denotes the Schur index of η. Recall the well-known relationship

$$\eta(1) = s_\eta n_\eta \tag{2.1}$$

between degree, Schur index and the size of the matrix ring; see e.g. [CR87, Remark 74.10(ii)].

The Wedderburn decomposition of $\mathbb{Q}_p[H]$ induces the following decomposition of $\mathcal{Q}(\Gamma_0)[H]$:

$$\begin{aligned}\mathcal{Q}(\Gamma_0)[H] &= \mathcal{Q}(\Gamma_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[H] \\ &= \prod_{\eta\in\mathrm{Irr}(H)/\sim_{\mathbb{Q}_p}} M_{n_\eta}\big(\mathcal{Q}(\Gamma_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p(\eta)} D_\eta\big) \\ &= \prod_{\eta\in\mathrm{Irr}(H)/\sim_{\mathbb{Q}_p}} M_{n_\eta}\big(\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0) \otimes_{\mathbb{Q}_p(\eta)} D_\eta\big) \\ &= \prod_{\eta\in\mathrm{Irr}(H)/\sim_{\mathbb{Q}_p}} M_{n_\eta}\big(\widetilde{D}_\eta\big)\end{aligned}$$

where $\widetilde{D}_\eta := \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0) \otimes_{\mathbb{Q}_p(\eta)} D_\eta$.

Lemma 2.1.1. *$\widetilde{D}_\eta$ is a skew field with centre $\mathfrak{z}(\widetilde{D}_\eta) = \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)$ and Schur index s_η.*

Proof. The ring $\widetilde{D}_\eta$ is a simple algebra with centre $\mathfrak{z}(\widetilde{D}_\eta) = \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)$ by [Rei03, Theorem 7.6].

We now prove that $\widetilde{D}_\eta$ is a skew field; the argument is essentially the same as in [NP19, Lemma 2.10]. Let $\mathfrak{S}$ denote the image of the embedding

$$\Lambda^{\mathcal{O}_{\mathbb{Q}_p(\eta)}}(\Gamma_0) - \{0\} \hookrightarrow \Lambda^{\mathcal{O}_{\mathbb{Q}_p(\eta)}}(\Gamma_0) \otimes_{\mathcal{O}_{\mathbb{Q}_p(\eta)}} \mathcal{O}_{D_\eta}$$

Then $\mathfrak{S}$ is a multiplicatively closed subset, it is central in $\widetilde{D}_\eta$, and contains no zero divisors. Moreover, $\widetilde{D}_\eta$ is the localisation of $\Lambda^{\mathcal{O}_{\mathbb{Q}_p(\eta)}}(\Gamma_0) \otimes_{\mathcal{O}_{\mathbb{Q}_p(\eta)}} \mathcal{O}_{D_\eta}$ at $\mathfrak{S}$. Therefore by the previously listed properties of $\mathfrak{S}$, the localisation $\widetilde{D}_\eta$ is a skew field if and only if $\Lambda^{\mathcal{O}_{\mathbb{Q}_p(\eta)}}(\Gamma_0) \otimes_{\mathcal{O}_{\mathbb{Q}_p(\eta)}} \mathcal{O}_{D_\eta}$ is a domain. Fix an isomorphism $\Lambda(\Gamma_0) \simeq \mathbb{Z}_p[[T]]$ sending $\gamma_0 \mapsto (1+T)$. This extends to an isomorphism

$$\Lambda^{\mathcal{O}_{\mathbb{Q}_p(\eta)}}(\Gamma_0) \otimes_{\mathcal{O}_{\mathbb{Q}_p(\eta)}} \mathcal{O}_{D_\eta} \simeq \mathcal{O}_{D_\eta}[[T]]$$

The ring of formal power series over a domain is a domain. Hence $\widetilde{D}_\eta$ is a skew field.

To show the assertion on the Schur index, consider the subfield $\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0)$ of $\widetilde{D}_\eta$. Using the fact that $\mathbb{Q}_p(\eta)(\omega_\eta)$ is a splitting field for D_η, a routine computation shows that this is a splitting field for $\widetilde{D}_\eta$:

$$\begin{aligned}\widetilde{D}_\eta \otimes_{\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0) &= \Big(D_\eta \otimes_{\mathbb{Q}_p(\eta)} \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)\Big) \otimes_{\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0) \\ &= D_\eta \otimes_{\mathbb{Q}_p(\eta)} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0) \\ &= D_\eta \otimes_{\mathbb{Q}_p(\eta)} \mathbb{Q}_p(\eta)(\omega_\eta) \otimes_{\mathbb{Q}_p(\eta)} \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0) \\ &= M_{s_\eta}\left(\mathbb{Q}_p(\eta)(\omega_\eta)\right) \otimes_{\mathbb{Q}_p(\eta)} \mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0) \\ &= M_{s_\eta}\Big(\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0)\Big)\end{aligned}$$

The Schur index can be read off from the size of the matrix ring. □

The algebra $\mathcal{Q}(\mathcal{G})$ is semisimple artinian: we write

$$\mathcal{Q}(\mathcal{G}) \simeq \prod_{\chi\in\mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(D_\chi)$$

for its Wedderburn decomposition. Let s_χ denote the Schur index of D_χ: then analogously to (2.1), we have $\chi(1) = s_\chi n_\chi$, see (0.9).

Remark 2.1.2. The notions introduced above can be defined more generally over a finite extension $\mathcal{F}$ of $\mathbb{Q}_p$, with D_η resp. D_χ coming from the Wedderburn decomposition of $\mathcal{F}[H]$ resp. $\mathcal{Q}^{\mathcal{F}}(\mathcal{G})$. This is the path taken by Nickel in [Nic14], and we refer there for this treatment. We opted to work over $\mathbb{Q}_p$ instead, for the following reasons. Firstly, this case already contains the information needed to study the general context: indeed, the Wedderburn components of $\mathcal{Q}^{\mathcal{F}}(\mathcal{G})$ emerge from those of $\mathcal{Q}(\mathcal{G})$ by tensoring with $\mathcal{F}$ over $\mathbb{Q}_p$, although depending on D_χ and $\mathcal{F}$, the splitting behaviour may be nontrivial. Secondly, note that some but not all relevant invariants depend on the ground field—for instance, $s^{\mathcal{F}}$ does, but w_χ doesn't—hence working over $\mathcal{F}$ would have necessitated either constantly carrying this datum around in the notation or suppressing it, risking confusion over what depends on what. We have made one exception to this: in Section 2.3.1, we prove cyclicity of $\mathcal{F}(\eta)/\mathcal{F}_\chi$, and relate this to the invariants $s^{\mathcal{F}}$ and $v_\chi^{\mathcal{F}}$ for general $\mathcal{F}$, as this is a result that is directly relevant to those of [Nic14] but not present in that paper. ◦

Definition 2.1.3. For $1 \leq j \leq n_\eta$, let $f_\eta^{(j)}$ be the element corresponding to the $n_\eta \times n_\eta$ matrix with zeros everywhere but in the jth entry of the diagonal and 1 there, under the isomorphism

$$\varepsilon(\eta)\mathbb{Q}_p[H] \simeq M_{n_\eta}(D_\eta)$$

$$f_\eta^{(j)} \leftrightarrow \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \quad (1 \text{ in row } j, \text{ column } j) \tag{2.2}$$

We shall refer to these elements of the group ring by the name of f-idempotents; note that this is not standard terminology. ◦

These are indecomposable, not necessarily central idempotents, and the following identities hold:

$$\left(f_\eta^{(j)}\right)^2 = f_\eta^{(j)} = f_\eta^{(j)}\varepsilon(\eta), \quad \varepsilon(\eta) = \sum_{j=1}^{n_\eta} f_\eta^{(j)}$$

The last equality holds because the matrices corresponding to the $f_\eta^{(j)}$s sum up to the identity matrix in $M_{n_\eta}(D_\eta)$, which in turn corresponds to the identity element $\varepsilon(\eta)$ in $\varepsilon(\eta)\mathbb{Q}_p[H]$. Furthermore,

$$D_\eta \simeq \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} M_{n_\eta}(D_\eta) \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} \simeq f_\eta^{(j)}\mathbb{Q}_p[H]\varepsilon(\eta)f_\eta^{(j)} = f_\eta^{(j)}\mathbb{Q}_p[H]f_\eta^{(j)} \tag{2.3}$$

Indeed, multiplying an $n \times n$ matrix by the matrix corresponding to $f_\eta^{(j)}$ from both left and right just means picking the jth diagonal element.

The above constructions also work in the group ring $\mathcal{Q}(\Gamma_0)[H]$, that is, after tensoring with $\mathcal{Q}(\Gamma_0)$:

$$\widetilde{D}_\eta \simeq f_\eta^{(j)}\mathcal{Q}(\Gamma_0)[H]f_\eta^{(j)} \tag{2.4}$$

Here we tacitly identify $f_\eta^{(j)}$ with $1 \otimes f_\eta^{(j)}$, and we shall continue to do so in the sequel. These f-idempotents were used to study Wedderburn decompositions of Iwasawa algebras in [Nic14, Theorem 1.11] and in [Lau12a, Theorem 1].

Analogously, one can define $f_\chi^{(j)}$ as

$$\varepsilon_\chi \mathcal{Q}(\mathcal{G}) \simeq M_{n_\chi}(D_\chi)$$
$$f_\chi^{(j)} \leftrightarrow \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

The relationship between f-idempotents $f_\chi^{(j)}$ resp. $f_\eta^{(j')}$ associated with χ resp. η, where η is an irreducible constituent of $\operatorname{res}_H^{\mathcal{G}} \chi$, will be studied in Section 3.1.

2.2 Extending Galois action to skew fields

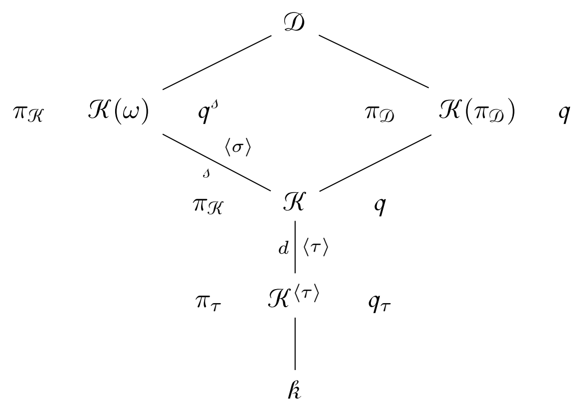
This section is concerned with the question of extending a Galois automorphism of the centre of a finite dimensional skew field over a local field to the entire skew field, under certain assumptions. Our treatment of this question is fairly general; the application to keep in mind is extending the Galois action of $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ to D_η (and subsequently to $\widetilde{D}_\eta$): this will be discussed in Section 2.3.1.

Let $\mathscr{K}$ be a local field, i.e. a finite extension of either $\mathbb{F}_p(\!(t)\!)$ or $\mathbb{Q}_p$. Fix a uniformiser $\pi_{\mathscr{K}}$ of $\mathscr{K}$, and let $\mathbb{F}_q$ be the residue field of $\mathscr{K}$, with q a power of p. Let $\mathscr{D}$ be a skew field with centre $\mathscr{K}$ and index $\mathscr{s}$. Recall the cyclic algebra description of $\mathscr{D}$ from (0.15): let ω be a primitive $(q^{\mathscr{s}}-1)$st root of unity, and let $\sigma \in \operatorname{Gal}(\mathscr{K}(\omega)/\mathscr{K})$ be the Galois automorphism defined by $\sigma(\omega) := \omega^{q^{\mathscr{r}}}$, where $\mathscr{r}/\mathscr{s}$ is the Hasse invariant of $\mathscr{D}$. Note that σ is a generator of this Galois group. Then

$$\mathscr{D} = \big(\mathscr{K}(\omega)/\mathscr{K}, \sigma, \pi_{\mathscr{K}}\big) = \bigoplus_{i=0}^{\mathscr{s}-1} \mathscr{K}(\omega)\pi_{\mathscr{D}}^i \tag{2.5}$$

where $\pi_{\mathscr{D}}^{\mathscr{s}} = \pi_{\mathscr{K}}$ and $\pi_{\mathscr{D}}\omega = \sigma(\omega)\pi_{\mathscr{D}}$.

Let $\mathscr{K}/\mathscr{k}$ be a Galois p-extension of local fields, and let $\tau \in \operatorname{Gal}(\mathscr{K}/\mathscr{k})$ be a Galois automorphism of order d (necessarily a p-power). The situation is described by the following diagram; uniformisers are on the left, residue field orders are on the right.



Furthermore, assume that the index $\mathscr{s} \mid (q_\tau - 1)$; in particular, $\mathscr{s}$ is coprime to p. Our objective is to extend τ to an automorphism of $\mathscr{D}$ with the same order d as τ: this will be accomplished in Proposition 2.2.8.

Lemma 2.2.1. *The extension $\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K}$ is a cyclic Galois extension, totally ramified of degree s.*

Proof. Indeed, $\pi_{\mathcal{D}}$ has minimal polynomial $X^{s} - \pi_{\mathcal{K}} \in \mathcal{K}[X]$, which is Eisenstein, hence the extension is totally ramified. Let ζ_s be a primitive sth root of unity: this is contained in $\mathcal{K}^{\langle\tau\rangle} \subseteq \mathcal{K}$ because $s \mid (q_\tau - 1)$. The roots of $X^{s} - \pi_{\mathcal{K}}$ are of the form $\zeta_s^i \pi_{\mathcal{D}}$, which are therefore all contained in $\mathcal{K}(\pi_{\mathcal{D}})$. Hence the extension is Galois, and a generator of the Galois group is given by $\pi_{\mathcal{K}} \mapsto \zeta_s \pi_{\mathcal{K}}$. □

The automorphism τ preserves the valuation of $\mathcal{K}$, wherefore

$$\tau(\pi_{\mathcal{K}}) = \varepsilon \cdot \pi_{\mathcal{K}} \tag{2.6}$$

for some $\varepsilon \in \mathcal{O}_{\mathcal{K}}^\times$. Under the decomposition $\mathcal{O}_{\mathcal{K}}^\times = \mu_{q-1} \times U_{\mathcal{K}}^1$, we can write $\varepsilon = \zeta \cdot u$ where $\zeta \in \mu_{q-1}$ and $u \in U_{\mathcal{K}}^1$. Let us write $N_{\langle\tau\rangle}(-)$ for the norm map from $\mathcal{K}$ to $\mathcal{K}^{\langle\tau\rangle}$.

Lemma 2.2.2. *The elements ε, ζ and u all have norm 1 in $\mathcal{K}^{\langle\tau\rangle}$:*

$$1 = N_{\langle\tau\rangle}(\varepsilon) = N_{\langle\tau\rangle}(\zeta) = N_{\langle\tau\rangle}(u)$$

Proof. Apply $N_{\langle\tau\rangle}$ to (2.6):

$$N_{\langle\tau\rangle}(\pi_{\mathcal{K}}) = N_{\langle\tau\rangle}(\varepsilon) N_{\langle\tau\rangle}(\pi_{\mathcal{K}})$$

It follows that

$$1 = N_{\langle\tau\rangle}(\varepsilon) = N_{\langle\tau\rangle}(\zeta) \cdot N_{\langle\tau\rangle}(u) \tag{2.7}$$

Since u is a 1-unit, so is its norm, that is, $N_{\langle\tau\rangle}(u) \equiv 1 \pmod{\pi_\tau}$. Then (2.7) shows that $N_{\langle\tau\rangle}(\zeta)$ must also be a 1-unit. On the other hand, ζ is a root of unity of order prime to p, and thus so is its norm. The groups of 1-units resp. roots of unity in $\mathcal{O}_{\mathcal{K}^{\langle\tau\rangle}}^\times$ have intersection the p-power roots of unity. Therefore $N_{\langle\tau\rangle}(\zeta) = 1$, which forces $N_{\langle\tau\rangle}(u) = 1$ as well. □

Lemma 2.2.3. *The element $\zeta \in \mu_{q-1}$ has order dividing $\frac{q-1}{q_\tau - 1}$, that is, $\zeta \in \mu_{(q-1)/(q_\tau-1)}$.*

Proof. By definition, the cokernel of the norm map $N_{\langle\tau\rangle}(-)$ on μ_{q-1} is the 0th Tate cohomology group:

$$\mu_{q-1} \xrightarrow{N_{\langle\tau\rangle}(-)} \mu_{q-1}^{\langle\tau\rangle} = \mu_{q_\tau - 1} \twoheadrightarrow \hat{H}^0\left(\langle\tau\rangle, \mu_{q-1}\right)$$

Since $\langle\tau\rangle$ is a p-group, and μ_{q-1} has order prime to p, this cohomology group vanishes, so $N_{\langle\tau\rangle}(-)$ is surjective on roots of unity.

It follows that the kernel of the norm map $N_{\langle\tau\rangle}(-)$ has order

$$\frac{\#\mu_{q-1}}{\#\mu_{q_\tau-1}} = \frac{q-1}{q_\tau - 1}$$

Since μ_{q-1} is a cyclic group, the kernel is the unique subgroup of this order, which is $\mu_{(q-1)/(q_\tau-1)}$. Lemma 2.2.2 finishes the proof. □

Remark 2.2.4. Lemma 2.2.3 can also be shown from Lemma 2.2.2 via an explicit computation, without using Tate cohomology. This can be done by writing $\mathcal{K}/\mathcal{K}^{\langle\tau\rangle}$ as an unramified extension $\mathcal{K}^{\mathrm{u}}/\mathcal{K}^{\langle\tau\rangle}$ followed by a totally ramified extension $\mathcal{K}/\mathcal{K}^{\mathrm{u}}$.

$$\begin{aligned} 1 &= N_{\mathcal{K}/\mathcal{K}^{\langle\tau\rangle}}(\zeta) \\ &= N_{\mathcal{K}^{\mathrm{u}}/\mathcal{K}^{\langle\tau\rangle}}\left(N_{\mathcal{K}/\mathcal{K}^{\mathrm{u}}}(\zeta)\right) \end{aligned}$$

$$
\begin{aligned}
&= N_{\mathscr{K}^{\mathrm{u}}/\mathscr{K}^{\langle\tau\rangle}}\left(\zeta^{(\mathscr{K}:\mathscr{K}^{\mathrm{u}})}\right)\\
&= \left(\prod_{i=0}^{\left(\mathscr{K}^{\mathrm{u}}:\mathscr{K}^{\langle\tau\rangle}\right)-1}\varphi^i(\zeta)\right)^{(\mathscr{K}:\mathscr{K}^{\mathrm{u}})}\\
&= \zeta^{(\mathscr{K}:\mathscr{K}^{\mathrm{u}})\left(1+q_\tau+q_\tau^2+\ldots+q_\tau^{\left(\mathscr{K}^{\mathrm{u}}:\mathscr{K}^{\langle\tau\rangle}\right)-1}\right)}\\
&= \zeta^{(\mathscr{K}:\mathscr{K}^{\mathrm{u}})\cdot\frac{q-1}{q_\tau-1}}
\end{aligned}
$$

Here φ denotes the Frobenius, which sends ζ to ζ^q. We used that the group of roots of unity of order prime to p remains the same in a totally ramified extension, and that the Galois group of an unramified extension is generated by the Frobenius. Now the degree $(\mathscr{K}:\mathscr{K}^{\mathrm{u}})$ is a p-power, but ζ has order coprime to p, and hence

$$1 = \zeta^{\frac{q-1}{q_\tau-1}}$$

as claimed. ∘

Lemma 2.2.5. *The Galois automorphism $\tau \in \mathrm{Gal}(\mathscr{K}/\mathscr{k})$ admits an extension to an element of* $\mathrm{Aut}_{\mathscr{k}}(\mathscr{K}(\pi_{\mathscr{D}}))$.

Proof. Extending τ to $\mathscr{K}(\pi_{\mathscr{D}})$ means defining $\tau(\pi_{\mathscr{D}})$. For valuation reasons, this must be of the form $\varepsilon_{\mathscr{D}}\cdot\pi_{\mathscr{D}}$ for some $\varepsilon_{\mathscr{D}} \in \mathcal{O}^\times_{\mathscr{K}(\pi_{\mathscr{D}})}$, and it has to satisfy $\tau(\pi_{\mathscr{D}}^{\mathscr{s}}) = \tau(\pi_{\mathscr{D}})^{\mathscr{s}}$. The latter is equivalent to requiring $\varepsilon_{\mathscr{D}}^{\mathscr{s}}\cdot\pi_{\mathscr{K}} = \varepsilon\cdot\pi_{\mathscr{K}}$, that is, $\varepsilon_{\mathscr{D}}^{\mathscr{s}} = \varepsilon$. So extending τ to $\mathscr{K}(\pi_{\mathscr{D}})$ means finding an $\mathscr{s}$th root of ε.

Let us once again use the decomposition $\varepsilon = \zeta\cdot u$ coming from $\mathcal{O}^\times_{\mathscr{K}} = \mu_{q-1}\times U^1_{\mathscr{K}}$. The situation is described by the following diagram.

$$
\begin{array}{ccccc}
 & & \mu_{\mathscr{s}} & & \\
 & & \downarrow & & \\
\mathcal{O}^\times_{\mathscr{K}} & \xrightarrow{\sim} & \mu_{q-1} & \times & U^1_{\mathscr{K}}\\
{\scriptstyle(-)^{\mathscr{s}}}\downarrow & & \downarrow & & \downarrow\sim\\
\left(\mathcal{O}^\times_{\mathscr{K}}\right)^{\mathscr{s}} & \xrightarrow{\sim} & \mu_{(q-1)/\mathscr{s}} & \times & U^1_{\mathscr{K}}
\end{array}
$$

Since $\mathscr{s}$ is coprime to p by assumption, the $\mathscr{s}$th-power-map is bijective on 1-units, so u has an $\mathscr{s}$th root. Moreover, we know that $\zeta \in \mu_{(q-1)/(q_\tau-1)}$ and that $\mathscr{s} \mid q_\tau - 1$, whence $\zeta \in \mu_{(q-1)/\mathscr{s}}$, which implies that ζ also has an $\mathscr{s}$th root.

In conclusion, there exists an $\varepsilon_{\mathscr{D}}$ with the desired properties; in fact, the diagram above shows that such an $\varepsilon_{\mathscr{D}}$ is contained in $\mathscr{K}$. It is easily seen that this gives rise to an $\mathscr{k}$-automorphism of $\mathscr{K}(\pi_{\mathscr{D}})$ extending τ. □

A priori, the extension of τ to $\mathscr{K}(\pi_{\mathscr{D}})$ as given in Lemma 2.2.5 is not unique: there is a choice of a root of ε involved. However, if we stipulate that the extension must have the same order d as the original τ, then the extension becomes unique.

Lemma 2.2.6. *The Galois automorphism $\tau \in \mathrm{Gal}(\mathscr{K}/\mathscr{k})$ admits a unique extension to an element $\tilde\tau \in \mathrm{Aut}_{\mathscr{k}}(\mathscr{K}(\pi_{\mathscr{D}}))$ of order d.*

Proof. As we have seen in the proof of Lemma 2.2.5, an extension of τ to $\mathcal{K}(\pi_{\mathcal{D}})$ is determined by an element $\varepsilon_{\mathcal{D}} \in \mathcal{O}_{\mathcal{K}}^{\times}$ such that

$$\varepsilon_{\mathcal{D}}^{s} = \varepsilon \tag{2.8}$$

Since $\tau^d(\pi_{\mathcal{D}}) = N_{\langle\tau\rangle}(\varepsilon_{\mathcal{D}})\pi_{\mathcal{D}}$, this extension has order d if and only if

$$N_{\langle\tau\rangle}(\varepsilon_{\mathcal{D}}) \cdot \pi_{\mathcal{D}} = \pi_{\mathcal{D}} \tag{2.9}$$

We first show uniqueness. Suppose $\varepsilon_{\mathcal{D}}, \varepsilon'_{\mathcal{D}} \in \mathcal{O}_{\mathcal{K}}^{\times}$ satisfy both (2.8) and (2.9). Comparing (2.8) for $\varepsilon_{\mathcal{D}}$ and $\varepsilon'_{\mathcal{D}}$, we find that $(\varepsilon'_{\mathcal{D}}/\varepsilon_{\mathcal{D}})^{s} = 1$. Hence $\varepsilon'_{\mathcal{D}} = \xi\varepsilon_{\mathcal{D}}$ for some $\xi \in \mu_{s}(\mathcal{K})$. From (2.9) it then follows that $N_{\langle\tau\rangle}(\xi) = 1$. Since $s \mid (q_\tau - 1)$, all sth roots of unity are already contained in $\mathcal{K}^{\langle\tau\rangle}$, that is, we have $\mu_{s}(\mathcal{K}) = \mu_{s}(\mathcal{K}^{\langle\tau\rangle})$. In particular, $\xi \in \mathcal{K}^{\langle\tau\rangle}$, and so $N_{\langle\tau\rangle}(\xi) = \xi^d$. The dth power map is bijective on $\mu_{s}(\mathcal{K})$, hence $\xi = 1$, which proves uniqueness.

Now we turn to existence. Let $\varepsilon_{\mathcal{D}} \in \mathcal{O}_{\mathcal{K}}^{\times}$ be an element satisfying (2.8); the existence of such an element is guaranteed by Lemma 2.2.5. From Lemma 2.2.2 we have that

$$1 = N_{\langle\tau\rangle}(\varepsilon) = N_{\langle\tau\rangle}(\varepsilon_{\mathcal{D}})^{s} \tag{2.10}$$

Therefore $N_{\langle\tau\rangle}(\varepsilon_{\mathcal{D}}) = \zeta_{s}^{a}$ for some a. Since d is coprime to s, the dth power map is bijective on $\mu_{s}(\mathcal{K})$, and so there exists a unique sth root of unity $\tilde{\zeta} \in \mu_{s}(\mathcal{K}^{\langle\tau\rangle})$ such that $\tilde{\zeta}^d = \zeta_{s}^{a}$. Then

$$N_{\langle\tau\rangle}\left(\tilde{\zeta}^{-1} \cdot \varepsilon_{\mathcal{D}}\right) = \tilde{\zeta}^{-d} \cdot \zeta_{s}^{a} = 1$$

Replacing the given $\varepsilon_{\mathcal{D}}$ by $\tilde{\zeta}^{-1} \cdot \varepsilon_{\mathcal{D}}$, we get the unique root of ε which makes (2.9) hold. □

Lemma 2.2.7. *$\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K}^{\langle\tau\rangle}$ is a Galois extension.*

Proof. On the one hand, the extension $\tilde{\tau}$ defined in Lemma 2.2.6 generates a subgroup

$$\langle\tilde{\tau}\rangle \subseteq \mathrm{Aut}_{\mathcal{K}^{\langle\tau\rangle}}(\mathcal{K}(\pi_{\mathcal{D}}))$$

of order d, which is a p-power. The subgroup

$$\mathrm{Gal}(\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K}) \subseteq \mathrm{Aut}_{\mathcal{K}^{\langle\tau\rangle}}(\mathcal{K}(\pi_{\mathcal{D}}))$$

has order s, which is coprime to p. Consequently, the intersection of these two subgroups is trivial. An automorphism in $\mathrm{Gal}(\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K})$ maps $\pi_{\mathcal{D}} \mapsto \zeta_{s}^{i}\pi_{\mathcal{D}}$ for some i. By the assumption $s \mid q_\tau - 1$, $\zeta_{s} \in \mathcal{K}^{\langle\tau\rangle}$, so it is fixed by τ. Moreover, as noted in the proof of Lemma 2.2.5, we also have $\varepsilon_{\mathcal{D}} \in \mathcal{K}$. As a consequence of these last two facts, it follows that $\tilde{\tau}$ commutes with all automorphisms of the extension $\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K}$. We conclude that there is an embedding

$$\langle\tilde{\tau}\rangle \times \mathrm{Gal}\left(\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K}\right) \hookrightarrow \left\{\mathcal{K}(\pi_{\mathcal{D}}) \hookrightarrow \mathbb{C}_p : \mathcal{K}^{\langle\tau\rangle}\text{-invariant homomorphisms}\right\}$$

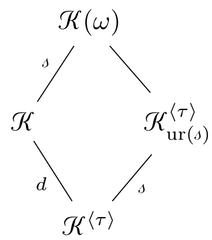
The latter set of homomorphisms has order

$$\begin{aligned}\left(\mathcal{K}(\pi_{\mathcal{D}}) : \mathcal{K}^{\langle\tau\rangle}\right) &= \left(\mathcal{K} : \mathcal{K}^{\langle\tau\rangle}\right) \cdot (\mathcal{K}(\pi_{\mathcal{D}}) : \mathcal{K}) \\ &= \#\langle\tilde{\tau}\rangle \cdot \#\,\mathrm{Gal}\left(\mathcal{K}(\pi_{\mathcal{D}})/\mathcal{K}\right)\end{aligned}$$

Therefore the embedding above is surjective, and the claim follows. □

Proposition 2.2.8. *Let $\mathcal{K}/\mathscr{k}$ be a Galois p-extension of local fields, and let $\tau \in \mathrm{Gal}(\mathcal{K}/\mathscr{k})$ have order d. Let $\mathcal{D}$ be a skew field with centre $\mathcal{K}$ and Schur index s, and assume that $s \mid (q_\tau - 1)$. Then there is a unique extension of τ to an element $\mathrm{\tau} \in \mathrm{Aut}_{\mathscr{k}}(\mathcal{D})$ of order d.*

Proof. The field extension $\mathcal{K}(\omega)/\mathcal{K}$ is unramified of degree $\mathscr{s}$. Since $\mathscr{s}$ is coprime to d, the extension arises from the unique unramified extension $\mathcal{K}^{\langle\tau\rangle}_{\mathrm{ur}(\mathscr{s})}$ of $\mathcal{K}^{\langle\tau\rangle}$ of the same degree $\mathscr{s}$ by base change.



Moreover, the extensions $\mathcal{K}(\omega)/\mathcal{K}$ and $\mathcal{K}/\mathcal{K}^{\langle\tau\rangle}$ have coprime degrees, so the two lower extensions in the diagram above must be disjoint over $\mathcal{K}^{\langle\tau\rangle}$, and there is an isomorphism

$$\mathrm{Gal}\left(\mathcal{K}(\omega)/\mathcal{K}^{\langle\tau\rangle}\right) \simeq \mathrm{Gal}\left(\mathcal{K}(\omega)/\mathcal{K}\right) \times \mathrm{Gal}\left(\mathcal{K}/\mathcal{K}^{\langle\tau\rangle}\right) \tag{2.11}$$

We let $\hat{\tau}$ be the Galois automorphism corresponding to the pair (id, τ). Note that this does *not* mean that $\hat{\tau}$ acts trivially on ω.

Define an extension $\uptau$ of τ to $\mathcal{D}$ by setting

$$\begin{aligned}\uptau(\omega) &:= \hat{\tau}(\omega)\\ \uptau(\pi_{\mathcal{D}}) &:= \tilde{\tau}(\pi_{\mathcal{D}}) = \varepsilon_{\mathcal{D}}\pi_{\mathcal{D}}\end{aligned}$$

We check that this is indeed a homomorphism, that is, it is compatible with the noncommutativity rule $\pi_{\mathcal{D}}\omega = \sigma(\omega)\pi_{\mathcal{D}}$:

$$\begin{aligned}\uptau\left(\pi_{\mathcal{D}}\omega\right) &= \varepsilon_{\mathcal{D}}\pi_{\mathcal{D}} \cdot \hat{\tau}(\omega) = \varepsilon_{\mathcal{D}} \cdot \sigma\hat{\tau}(\omega) \cdot \pi_{\mathcal{D}} \stackrel{\dagger}{=} \sigma\hat{\tau}(\omega) \cdot \varepsilon_{\mathcal{D}} \cdot \pi_{\mathcal{D}}\\ \uptau\left(\sigma(\omega)\pi_{\mathcal{D}}\right) &= \hat{\tau}\sigma(\omega) \cdot \varepsilon_{\mathcal{D}}\pi_{\mathcal{D}} \stackrel{\ddagger}{=} \sigma\hat{\tau}(\omega) \cdot \varepsilon_{\mathcal{D}}\pi_{\mathcal{D}}\end{aligned}$$

In † we used that $\varepsilon_{\mathcal{D}} \in \mathcal{K}$, which is the centre of the skew field. In ‡ we used that the group $\mathrm{Gal}(\mathcal{K}(\omega)/\mathcal{K}^{\langle\tau\rangle})$ is abelian: indeed, we have seen in (2.11) that it is the direct product of two cyclic groups.

It is clear from the definition and Lemma 2.2.6 that $\uptau$ has the prescribed order and that it is unique. □

2.3 Galois action and Γ-action

In this section, we will first use the results of Section 2.2 to endow D_η and $\widetilde{D}_\eta$ with a Galois action (Section 2.3.1). We will then explore the relation between this Galois action and the action of Γ (Section 2.3.2). After a brief interlude on how Γ acts on f-idempotents (Section 2.3.3), we will use these results to describe the ring structure of $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$ (Section 2.3.4).

2.3.1 Extending Galois action to D_η and $\widetilde{D}_\eta$

As before, let $\mathcal{F}$ be a finite extension of $\mathbb{Q}_p$. We begin this section by describing the Galois group of $\mathcal{F}(\eta)/\mathcal{F}_\chi$ in more detail: we show that it is cyclic with a certain generator which can be seen as 'canonical' with respect to the group action of Γ.

Definition 2.3.1. Let $v_\chi^{\mathcal{F}}$ be the minimal positive exponent such that $\gamma^{v_\chi^{\mathcal{F}}}$ acts as a Galois automorphism on η:

$$v_\chi^{\mathcal{F}} := \min\left\{0 < i \leq w_\chi : \exists \tau \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}), {}^{\gamma^i}\eta = {}^{\tau}\eta\right\}$$

We write $v_\chi := v_\chi^{\mathbb{Q}_p}$ when $\mathcal{F} = \mathbb{Q}_p$. ◦

Note that $v_\chi^{\mathcal{F}} \mid w_\chi$ since ${}^{\gamma^{w_\chi}}\eta = \eta$ by definition of w_χ.

Remark 2.3.2. The number v_χ was introduced in [Lau12a, p. 1223]. There is a typo in that definition: if $i = 0$ would be allowed then $\tau = \mathrm{id}$ would satisfy the conditions, and v_χ would always be zero. ◦

Proposition 2.3.3. *The Galois group* $\mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$ *is cyclic of order* $w_\chi/v_\chi^{\mathcal{F}}$*, and* $\nu_\chi^{\mathcal{F}} = v_\chi^{\mathcal{F}}$*. Any* $\tau \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})$ *such that* ${}^{\gamma^{v_\chi^{\mathcal{F}}}}\eta = {}^{\tau}\eta$ *is a generator of* $\mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$*, and in fact, there is exactly one such* τ*.*

In the sequel, the symbol τ shall be reserved to denote this unique automorphism (or an extension of it, as constructed later in this section).

Remark 2.3.4. Proposition 2.3.3 is an improvement upon [Nic14, Lemma 1.1], where it was shown that $w_\chi = \nu_\chi^{\mathcal{F}}(\mathcal{F}(\eta) : \mathcal{F}_\chi)$. Furthermore, it is a direct consequence of Proposition 2.3.3 that the fixed field L occurring in Lau's description of the centre of D_χ is in fact $\mathbb{Q}_{p,\chi}$; see Proposition 1 and Theorem 1 of [Lau12a]. ◦

Proof. Let $\tau \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})$ be as in the definition of $v_\chi^{\mathcal{F}}$. We first show that it is in fact in the subgroup $\mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$. ↯ Suppose that this is not the case, that is, τ does not preserve some element of $\mathcal{F}_\chi = \mathcal{F}(\chi(h) : h \in H)$. It follows that

$$\tau(\mathrm{res}_H^{\mathcal{G}}\,\chi) \neq \mathrm{res}_H^{\mathcal{G}}\,\chi$$

Notice that the character on the left hand side contains ${}^{\tau}\eta$ as a summand since $\eta \mid \mathrm{res}_H^{\mathcal{G}}\,\chi$, and that the right hand side contains ${}^{\gamma^{v_\chi^{\mathcal{F}}}}\eta$ as a summand since $\mathrm{res}_H^{\mathcal{G}}\,\chi = \sum_{i=0}^{w_\chi-1} {}^{\gamma^i}\eta$. We now exploit these observations:

$$\begin{aligned}
\mathrm{res}_H^{\mathcal{G}}\,\chi &= \sum_{i=0}^{w_\chi-1} {}^{\gamma^i}\eta \\
&= \sum_{i=0}^{w_\chi-1} {}^{\gamma^i}\left({}^{\gamma^{v_\chi^{\mathcal{F}}}}\eta\right) && \text{shifting } i \text{ by } v_\chi^{\mathcal{F}} \text{ modulo } w_\chi \\
&= \sum_{i=0}^{w_\chi-1} {}^{\gamma^i}\left({}^{\tau}\eta\right) && {}^{\gamma^{v_\chi^{\mathcal{F}}}}\eta = {}^{\tau}\eta \\
&= \tau\left(\sum_{i=0}^{w_\chi-1} {}^{\gamma^i}\eta\right) && {}^{\gamma^i}\left({}^{\tau}\eta\right) = {}^{\tau}\left({}^{\gamma^i}\eta\right) \\
&= \tau\left(\mathrm{res}_H^{\mathcal{G}}\,\chi\right)
\end{aligned}$$

This contradicts $\tau(\mathrm{res}_H^{\mathcal{G}}\,\chi) \neq \mathrm{res}_H^{\mathcal{G}}\,\chi$, ↯ hence $\tau \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$. In the penultimate step, the statement that the actions of τ and Γ on η commute follows directly from their definitions: indeed, for all $\gamma \in \Gamma$ and $h \in H$,

$$\left({}^{\gamma}\left({}^{\tau}\eta\right)\right)(h) = \left({}^{\tau}\eta\right)\left(\gamma h \gamma^{-1}\right) = \tau\left(\eta\left(\gamma h \gamma^{-1}\right)\right) = \tau\left({}^{\gamma}\eta(h)\right) = \left({}^{\tau}\left({}^{\gamma}\eta\right)\right)(h)$$

Consider the characters ${}^{\gamma^k}\eta$ for $0 \le k < v_\chi^{\mathcal{F}}$. These are all in separate $\mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})$-orbits: that is, for any $\psi \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})$ and $0 \le k, k' < v_\chi^{\mathcal{F}}$ distinct, we have

$$ {}^{\gamma^k}\eta \neq {}^{\psi}({}^{\gamma^{k'}}\eta) \tag{2.12} $$

↯ Indeed, suppose the contrary ${}^{\gamma^k}\eta = {}^{\psi}({}^{\gamma^{k'}}\eta)$. Without loss of generality, we may assume $k > k'$, and then ${}^{\gamma^{k-k'}}\eta = {}^{\psi}\eta$ because the γ- and Galois actions commute. But $0 < k - k' \le k < v_\chi^{\mathcal{F}}$, which contradicts minimality in the definition of $v_\chi^{\mathcal{F}}$. ↯

We will now play the two decompositions of $\mathrm{res}_H^{\mathcal{G}}\chi$ off against each other:

$$ \sum_{j=0}^{\nu_\chi^{\mathcal{F}}-1} \sum_{\psi \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)} {}^{\psi}\eta_{(j)} = \mathrm{res}_H^{\mathcal{G}}\chi = \sum_{i=0}^{w_\chi - 1} {}^{\gamma^i}\eta = \sum_{k=0}^{v_\chi^{\mathcal{F}}-1} \sum_{\ell=0}^{\frac{w_\chi}{v_\chi^{\mathcal{F}}}-1} {}^{\tau^\ell}\left({}^{\gamma^k}\eta\right) \tag{2.13} $$

where τ is still as in the definition of $v_\chi^{\mathcal{F}}$, and $\eta_{(j)}$ was defined in (0.6). Observe that the summands on the right hand side are all different, that is,

$$ {}^{\tau^\ell}({}^{\gamma^k}\eta) \neq {}^{\tau^{\ell'}}({}^{\gamma^{k'}}\eta) \text{ unless } \ell = \ell' \text{ and } k = k' $$

Indeed, for $k \neq k'$, this is (2.12). For $k = k'$, this is because ${}^{\tau^\ell}({}^{\gamma^k}\eta) = {}^{\tau^{\ell'}}({}^{\gamma^k}\eta)$ is equivalent to $\eta = {}^{\tau^{\ell'-\ell}}\eta$, which means that $\tau^{\ell'-\ell}$ fixes $\mathcal{F}(\eta)$, that is, it is the identity automorphism, and so $\ell' = \ell$. (In fact, the same argument shows that if $\psi, \psi' \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F})$ then ${}^{\psi}({}^{\gamma^k}\eta) \neq {}^{\psi'}({}^{\gamma^{k'}}\eta)$ unless $\psi = \psi'$ and $k = k'$.)

On the left hand side of (2.13), we have full Galois orbits, so this must also be true on the right hand side. It follows that $\nu_\chi^{\mathcal{F}} = v_\chi^{\mathcal{F}}$, and we may assume $\eta_{(k)} = {}^{\gamma^k}\eta$ up to renumbering. Moreover, $\mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$ is generated by τ; in particular, it is cyclic of order $w_\chi/v_\chi^{\mathcal{F}}$.

Finally, we show that there is only one $\tau \in \mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$ for which ${}^{\gamma^{v_\chi^{\mathcal{F}}}}\eta = {}^{\tau}\eta$. As we have already seen, any such element is a generator of the Galois group $\mathrm{Gal}(\mathcal{F}(\eta)/\mathcal{F}_\chi)$. The Galois group is a p-group, therefore generators are of the form τ^n with $p \nmid n$, n a positive integer. The condition in the definition of $v_\chi^{\mathcal{F}}$ reads

$$ {}^{\tau^n}\eta = {}^{\gamma^{v_\chi^{\mathcal{F}}}}\eta $$

The left hand side here is equal to ${}^{\gamma^{n v_\chi^{\mathcal{F}}}}\eta$, thus the ns satisfying this condition are precisely the ones for which $n v_\chi^{\mathcal{F}} \equiv v_\chi^{\mathcal{F}} \bmod w_\chi$, or equivalently $n \equiv 1 \bmod w_\chi/v_\chi^{\mathcal{F}}$. Since the Galois group has order $w_\chi/v_\chi^{\mathcal{F}}$, this proves uniqueness. □

Remark 2.3.5. Proposition 2.3.3 also shows that $v_\chi^{\mathcal{F}}$ does not depend on the choice of the topological generator γ: indeed, $v_\chi^{\mathcal{F}} = w_\chi/(\mathcal{F}(\eta) : \mathcal{F}_\chi)$, and neither the index $w_\chi = (\mathcal{G} : \mathrm{St}(\eta))$ nor the degree $(\mathcal{F}(\eta) : \mathcal{F}_\chi)$ depend on the choice of γ. ◦

Remark 2.3.6. We describe how τ changes upon choosing a different topological generator of Γ. Let γ' be another topological generator of Γ, and let τ' be the automorphism associated with it by Proposition 2.3.3. Then $\gamma' = \gamma^z$ for some $z \in \mathbb{Z}_p^\times$. Using the defining property of τ and τ' and the fact that $v_\chi^{\mathcal{F}}$ does not depend on the choice of a topological generator (Remark 2.3.5), we have the following sequence of equalities:

$$ {}^{\tau'}\eta = {}^{(\gamma')^{v_\chi^{\mathcal{F}}}}\eta = {}^{(\gamma^z)^{v_\chi^{\mathcal{F}}}}\eta = {}^{\left(\gamma^{v_\chi^{\mathcal{F}}}\right)^z}\eta = {}^{\tau^z}\eta $$

The uniqueness part of Proposition 2.3.3 shows that $\tau' = \tau^z$. ◦

Let $\mathscr{F} := \mathbb{Q}_p$ from now on. Write r_η/s_η for the Hasse invariant of D_η. We apply the results of Section 2.2 with $\mathscr{D} := D_\eta$ and $\mathscr{K} := \mathbb{Q}_p(\eta)$, $\pi_{\mathscr{K}} := \pi_\eta$, $q := q_\eta$, $s := s_\eta$, $r := r_\eta$, $\omega := \omega_\eta$, and $k := \mathbb{Q}_{p,\chi}$. The extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is a Galois p-extension: its degree divides the p-power w_χ, as shown in [Nic14, Lemma 1.1]; moreover, it was shown in Proposition 2.3.3 that it even turns out to be cyclic, and $\mathscr{K}^{\langle\tau\rangle} = \mathbb{Q}_{p,\chi}$. Let τ be the unique generator described there. The condition on the Schur index is satisfied by Theorem 0.2.2: we have divisibilities

$$s_\eta \mid (p-1) \mid (q_\tau - 1)$$

So the extension process is applicable.

To avoid confusion, we distinguished between various extensions of τ in Section 2.2. In light of Proposition 2.2.8, these distinctions are no longer necessary, and we shall, by abuse of notation, write τ for τ. Finally, we extend τ from D_η to $\widetilde{D}_\eta$ by letting $\tau(\gamma_0) := \gamma_0$.

Remark 2.3.7. In the case of D_η, the discussion in Section 2.2 simplifies, as the fixed field $\mathscr{K}^{\langle\tau\rangle} = \mathbb{Q}_p(\eta)^{\langle\tau\rangle}$ coincides with the base field $k = \mathbb{Q}_{p,\chi}$. Above we allowed τ to be any Galois automorphism, since the extension process still works in this more general case, and this exposition may be useful for eventual generalisations. ◦

2.3.2 δ_γ and δ_τ

Consider the abstract Wedderburn isomorphism of the central simple $\mathbb{Q}_p(\eta)$-algebras

$$\mathbb{Q}_p[H]\varepsilon(\eta) \simeq M_{n_\eta}(D_\eta) \tag{2.14}$$

Upon tensoring with $\mathcal{Q}(\Gamma_0)$, this gives rise to the following isomorphism of central simple $\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)$-algebras

$$\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta) \simeq M_{n_\eta}(\widetilde{D}_\eta) \tag{2.15}$$

On the left hand side, we have an action of conjugation by γ^{v_χ}. This is because H is a normal subgroup of $\mathcal{G}$, and $\gamma^{v_\chi} e({}^\sigma\eta)\gamma^{v_\chi} = e({}^{\tau\sigma}\eta)$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)$, so conjugation by γ^{v_χ} acts simply transitively on the summands $e({}^\sigma\eta)$ of $\varepsilon(\eta)$.

Remark 2.3.8. Note that v_χ is the minimal positive integer for which conjugation by γ^{v_χ} acts on $\mathbb{Q}_p[H]\varepsilon(\eta)$. Indeed, in the proof of Proposition 2.3.3 we have seen that for $0 \le k < v_\chi$ and $\psi, \psi' \in \mathrm{Gal}(\mathscr{F}(\eta)/\mathscr{F})$, we have ${}^\psi({}^{\gamma^k}\eta) \neq {}^{\psi'}({}^{\gamma^{k'}}\eta)$ unless $\psi = \psi'$ and $k = k'$. Hence conjugation by γ^k does not preserve $\mathbb{Q}_p[H]\varepsilon(\eta)$. ◦

On the right hand side of (2.15), there is an entry-wise action of τ. We shall now relate these two actions. Let x resp. X be elements corresponding to each other under the Wedderburn isomorphism (2.15):

$$\begin{gathered}\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta) \simeq M_{n_\eta}(\widetilde{D}_\eta)\\ x \leftrightarrow X\end{gathered}$$

We adopt the convention of denoting elements corresponding to each other by the same letter, lowercase on the left and uppercase on the right.

To emphasise where each action is coming from, we introduce the following notation. We write $\gamma^{v_\chi} x \gamma^{-v_\chi} = {}^{\gamma^{v_\chi}}x$ for the γ^{v_χ}-conjugate on the left, and $\delta_\gamma(X)$ for the corresponding element on the right. Similarly, on the right hand side we write $\tau(X)$ for the matrix obtained from X by applying τ entry-wise, and let $\delta_\tau(x)$ be the corresponding element on the left hand side. This defines automorphisms on the both sides of (2.15):

$$\delta_\gamma \in \mathrm{Aut}(M_{n_\eta}(\widetilde{D}_\eta)), \quad \delta_\tau \in \mathrm{Aut}(\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta))$$

Remark 2.3.9. Caveat: one could also let τ act coefficient-wise on the left hand side, but since all coefficients are in $\mathbb{Q}_p$, this would just be the trivial action. This is not the same as δ_τ, and we shall not use this action. ◦

Remark 2.3.10. Everything in this section really takes place over finite groups. Indeed, since χ has open kernel in $\mathcal{G}$, the action factors through a finite quotient of $\mathcal{G}$ containing H as a normal subgroup. Conjugation by γ^{v_χ} acts the same as conjugation by the image of γ^{v_χ} in this quotient group. Moreover, we could have avoided tensoring by $\mathcal{Q}(\Gamma_0)$: indeed, since Γ_0 is central, conjugation acts trivially, and τ was extended so that $\tau(\gamma_0) = \gamma_0$. We nevertheless state everything in terms of $\mathcal{Q}(\Gamma_0)$-tensored algebras, as this is the form in which results of this section will be used in the sequel. ◦

Proposition 2.3.11. *The two actions just defined agree on the respective centres. In formulæ: on the group ring side,*

$$ {}^{\gamma^{v_\chi}}(-)\big|_{\mathfrak{z}(\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta))} = \delta_\tau\big|_{\mathfrak{z}(\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta))} $$

and equivalently on the matrix ring side,

$$ \delta_\gamma\big|_{\mathfrak{z}\left(M_{n_\eta}(\widetilde{D}_\eta)\right)} = \tau\big|_{\mathfrak{z}\left(M_{n_\eta}(\widetilde{D}_\eta)\right)} $$

Proof. By definition of δ_τ and δ_γ, the two statements are equivalent, so it is enough to prove one of them: we shall prove the former. In the proof, we will work over the group ring $\mathbb{Q}_p[H]$ instead of $\mathcal{Q}(\Gamma_0)[H]$, which is permissible by Remark 2.3.10.

Consider the following commutative diagram, which we explain below.

$$ \begin{array}{ccc} \mathbb{Q}_p[H]\varepsilon(\eta) & \xrightarrow{\sim} & M_{n_\eta}(D_\eta) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Q}_p^{\mathrm{c}}[H]\varepsilon(\eta) & \xrightarrow{\sim} & \bigoplus_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} M_{\eta(1)}(\mathbb{Q}_p^{\mathrm{c}}) \\ \Big\| & & \Big\| \\ \bigoplus_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \mathbb{Q}_p^{\mathrm{c}}[H]e({}^\sigma\eta) & \xrightarrow[\bigoplus_\sigma \rho_{{}^\sigma\eta}]{\sim} & \bigoplus_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} M_{\eta(1)}(\mathbb{Q}_p^{\mathrm{c}})E({}^\sigma\eta) \end{array} $$

The first row is the Wedderburn isomorphism (2.14). The second row is obtained by extending scalars to $\mathbb{Q}_p^{\mathrm{c}}$, i.e. tensoring with $\mathbb{Q}_p^{\mathrm{c}}$ over $\mathbb{Q}_p$. It's clear what happens on the left hand side. On the right hand side, recall that $\mathbb{Q}_p^{\mathrm{c}}$ is a splitting field for D_η over its centre $\mathbb{Q}_p(\eta)$, so there is an isomorphism $\mathbb{Q}_p^{\mathrm{c}} \otimes_{\mathbb{Q}_p(\eta)} D_\eta \simeq M_{s_\eta}(\mathbb{Q}_p^{\mathrm{c}})$; also recall that $\eta(1) = n_\eta s_\eta$. Since the tensor product is taken over $\mathbb{Q}_p$ and not over $\mathbb{Q}_p(\eta)$, we get a component for each Galois automorphism in $\mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)$, or in other words, for each embedding $\mathbb{Q}_p(\eta) \hookrightarrow \mathbb{Q}_p^{\mathrm{c}}$. The vertical map on the left is inclusion. On the right, the map is induced by entry-wise application of the embedding

$$ \begin{aligned} D_\eta &\hookrightarrow D_\eta \otimes_{\mathbb{Q}_p(\eta)} \mathbb{Q}_p^{\mathrm{c}} \simeq M_{\eta(1)}(\mathbb{Q}_p^{\mathrm{c}}) \\ x &\mapsto x \otimes 1 \end{aligned} $$

in each component.

The third row of the diagram is induced from the second row by the decomposition of $\varepsilon(\eta)$ into components $e({}^\sigma\eta)$. Let $E({}^\sigma\eta)$ denote the image of $e({}^\sigma\eta)$ under the isomorphism in the second row; then $E({}^\sigma\eta)$ is the identity in the σ-component and zero elsewhere. We write

$$ \rho_{{}^\sigma\eta} : \mathbb{Q}_p^{\mathrm{c}}[H]e({}^\sigma\eta) \xrightarrow{\sim} M_{\eta(1)}(\mathbb{Q}_p^{\mathrm{c}})E({}^\sigma\eta) \xrightarrow{\sim} M_{\eta(1)}(\mathbb{Q}_p^{\mathrm{c}}) $$

for the σ-part of the map in the third row. On H, this is a representation with character ${}^{\sigma}\eta$.

Let $z \in \mathfrak{z}(\mathbb{Q}_p[H]\varepsilon(\eta))$ be a central element. Then for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)$,

$$\rho_{{}^{\sigma}\eta}(z) = \frac{1}{\eta(1)} \mathrm{Tr}\left(\rho_{{}^{\sigma}\eta}(z)\right) \cdot E\left({}^{\sigma}\eta\right) = \frac{1}{\eta(1)} {}^{\sigma}\eta(z) \cdot E\left({}^{\sigma}\eta\right)$$

where by abuse of notation, ${}^{\sigma}\eta$ denotes the $\mathbb{Q}_p^{\mathrm{c}}$-linear extension of the character ${}^{\sigma}\eta : H \to \mathbb{Q}_p^{\mathrm{c}}$ to the group ring $\mathbb{Q}_p^{\mathrm{c}}[H]$.

A central element of $M_{n_\eta}(D_\eta)$ is of the form $Z_\alpha = \alpha \mathbf{1}_{n_\eta}$ where $\alpha \in \mathfrak{z}(D_\eta) = \mathbb{Q}_p(\eta)$. Let $z_\alpha \in \mathfrak{z}(\mathbb{Q}_p[H]\varepsilon(\eta))$ be the corresponding central element in the group ring under the top horizontal map. By commutativity of the diagram, the images of Z_α and z_α in the bottom right corner coincide:

$$\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \sigma\left(Z_\alpha\right) E\left({}^{\sigma}\eta\right) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \frac{1}{\eta(1)} {}^{\sigma}\eta(z_\alpha) E\left({}^{\sigma}\eta\right) \tag{2.16}$$

We compute the image of ${}^{\gamma^{v_\chi}} z_\alpha$:

$$\begin{aligned} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \frac{1}{\eta(1)} {}^{\sigma}\eta\left({}^{\gamma^{v_\chi}} z_\alpha\right) E\left({}^{\sigma}\eta\right) &= \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \frac{1}{\eta(1)} {}^{\sigma\tau}\eta(z_\alpha) E\left({}^{\sigma}\eta\right) \\ &= \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \frac{1}{\eta(1)} {}^{\tau\sigma}\eta(z_\alpha) E\left({}^{\sigma}\eta\right) \\ &= \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \tau(\sigma(Z_\alpha)) E\left({}^{\sigma}\eta\right) \\ &= \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)} \sigma\left(Z_{\tau(\alpha)}\right) E\left({}^{\sigma}\eta\right) \end{aligned} \tag{2.17}$$

In this sequence of equations, the first step is applying the definition of τ. The second equality holds because $\mathbb{Q}_p(\eta)/\mathbb{Q}_p$ is abelian (it is contained in some cyclotomic extension). The third equality is (2.16), and the last one is by definition of Z_α and abelianity of $\mathbb{Q}_p(\eta)/\mathbb{Q}_p$.

The expression (2.17) is the image of $Z_{\tau(\alpha)}$. The corresponding group ring element is $z_{\tau(\alpha)}$, which, by definition of δ_τ, is the same as $\delta_\tau(z_\alpha)$. This concludes the proof. □

The automorphism $\delta_\tau^{-1} \circ {}^{\gamma^{v_\chi}}(-)$ of the central simple $\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0)$-algebra $\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)$ is therefore trivial on the centre. Pursuant to the Skolem–Noether theorem, see [CR81, Theorem 3.62], there is a unit $y_\eta \in \mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)^\times$ such that for all $x \in \mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)$,

$$\delta_\tau^{-1}\left({}^{\gamma^{v_\chi}} x\right) = y_\eta x y_\eta^{-1}$$

Equivalently we may write

$${}^{\gamma^{v_\chi}} x = \delta_\tau(y_\eta)\delta_\tau(x)\delta_\tau(y_\eta^{-1}) \tag{2.18}$$

The corresponding equations for matrices also hold: if $Y_\eta \in \mathrm{GL}_{n_\eta}(\widetilde{D}_\eta)$ denotes the element corresponding to y_η under (2.15), then for all $X \in M_{n_\eta}(\widetilde{D}_\eta)$,

$$\delta_\gamma(X) = \tau(Y_\eta)\tau(X)\tau(Y_\eta^{-1}) \tag{2.19}$$

As pointed out in Remark 2.3.10, everything takes place over $\mathbb{Q}_p[H]$, so in fact we may assume $y_\eta \in \mathbb{Q}_p[H]\varepsilon(\eta)^\times$ and $Y_\eta \in \mathrm{GL}_{n_\eta}(D_\eta)$. It should be noted that the Skolem–Noether theorem only determines the units y_η and Y_η up to central units in their respective ambient rings. By induction, (2.18) and (2.19) admit the following generalisations: for all $v_\chi \mid i$,

$$^{\gamma^i}x = \delta_\tau(y_\eta)\cdots\delta_\tau^{i/v_\chi}(y_\eta)\cdot\delta_\tau^{i/v_\chi}(x)\cdot\delta_\tau^{i/v_\chi}(y_\eta^{-1})\cdots\delta_\tau(y_\eta^{-1}) \tag{2.20}$$

$$\delta_\gamma^{i/v_\chi}(X) = \tau(Y_\eta)\cdots\tau^{i/v_\chi}(Y_\eta)\cdot\tau^{i/v_\chi}(X)\cdot\tau^{i/v_\chi}(Y_\eta^{-1})\cdots\tau(Y_\eta^{-1}) \tag{2.21}$$

We shall write

$$A_{i/v_\chi} := \tau(Y_\eta)\cdots\tau^{i/v_\chi}(Y_\eta) \in \mathrm{GL}_{n_\eta}(D_\eta)$$

for the conjugating element in (2.21), and

$$a_{i/v_\chi} := \delta_\tau(y_\eta)\cdots\delta_\tau^{i/v_\chi}(y_\eta) \in \mathbb{Q}_p[H]\varepsilon(\eta)^\times$$

for the corresponding element in (2.20). It follows from the definitions that if $v_\chi \mid i, j$, then

$$A_{i/v_\chi}\cdot\tau^{i/v_\chi}(A_{j/v_\chi}) = A_{(i+j)/v_\chi} \tag{2.22}$$

and similarly

$$a_{i/v_\chi}\cdot\delta_\tau^{i/v_\chi}(a_{j/v_\chi}) = a_{(i+j)/v_\chi} \tag{2.23}$$

The element $a_{p^{n_0}/v_\chi}$ is central in $\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)$. Indeed, for all $x \in \mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)$:

$$x = {}^{\gamma_0}x = {}^{\gamma^{p^{n_0}}}x = a_{p^{n_0}/v_\chi}\delta_\tau^{p^{n_0}/v_\chi}(x)a_{p^{n_0}/v_\chi}^{-1} = a_{p^{n_0}/v_\chi}xa_{p^{n_0}/v_\chi}^{-1} \tag{2.24}$$

Definition 2.3.12. Let $\gamma''_\eta := a_1^{-1}\gamma^{v_\chi} = \delta_\tau(y_\eta)\gamma^{v_\chi}$, and let Γ''_η be the procyclic group generated by γ''_η. ∘

Remark 2.3.13. The element γ''_η plays a rôle not dissimilar to the element γ_χ of Ritter–Weiss resp. γ'_χ of Nickel, see [RW04, Proposition 5] resp. [Nic14, Lemma 1.2], hence the notation. In analogy with the Ritter–Weiss construction, one may then define an element γ''_χ whose $\varepsilon(\eta)$-component is γ''_η.

Nickel showed that the centre of D_χ is isomorphic to $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)$, where Γ'_χ is the procyclic group generated by γ'_χ, see [Nic14, Proposition 1.5]. In Corollary 3.2.3, we will see that under appropriate assumptions, the centre is isomorphic to $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}((\Gamma''_\eta)^{w_\chi/v_\chi})$. This naturally raises the question of how γ'_χ is related to γ''_η. In particular, since the definition of γ''_η depends on the choice of y_η, one may ask whether this can be chosen such that $\gamma'_\chi = (\gamma''_\eta)^{w_\chi/v_\chi}$. We shall not address these questions in this work. ∘

Lemma 2.3.14. *For $j \geq 1$, we have $(\gamma''_\eta)^j = a_j^{-1}\gamma^{jv_\chi}$.*

Proof. Use induction and (2.23):

$$(\gamma''_\eta)^j = a_1^{-1}\gamma^{v_\chi}a_{j-1}^{-1}\gamma^{(j-1)v_\chi} = a_1^{-1}\delta_\tau(a_{j-1}^{-1})\gamma^{jv_\chi} = a_j^{-1}\gamma^{jv_\chi}$$ □

2.3.3 Conjugating f-idempotents

Later it will be necessary to keep track of what happens to f-idempotents under conjugation. To ease notation, we restrict our attention to delineating the behaviour of $f_\eta^{(1)}$ with respect to conjugation: the general case of $f_\eta^{(j)}$ is completely analogous.

We shall need the following observation from linear algebra. Let $\mathscr{D}$ be a skew field, and let $n \geq 1$. Let $\Pi : M_n(\mathscr{D}) \to \mathscr{D}$ be the map sending an $n \times n$ matrix $(x_{i,j})$ to its $(1,1)$-entry $x_{1,1}$. The map Π is additive and $\mathscr{D}$-linear: in other words, it is a $\mathscr{D}$-vector space homomorphism, where $\mathscr{D}$ acts on $M_n(\mathscr{D})$ by left multiplication. Note that Π fails to be multiplicative in general.

Consider the identity

$$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} x_{1,1} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

On the subspace of $n \times n$ matrices of this form, Π becomes a $\mathscr{D}$-vector space isomorphism with respect to left $\mathscr{D}$-multiplication:

$$\Pi : \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} M_n(\mathscr{D}) \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \xrightarrow{\sim} \mathscr{D} \tag{2.25}$$

On this subspace, Π is also multiplicative: indeed, the product of two such matrices is again a matrix of this shape with the $(1,1)$-entries multiplied. That is, Π is a ring isomorphism on this subspace.

Lemma 2.3.15. *Let $A, B \in \mathrm{GL}_n(\mathscr{D})$. Then there is an isomorphism of $\mathfrak{z}(\mathscr{D})$-vector spaces*

$$\begin{aligned} \Pi_{A,B} : A \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} A^{-1} M_n(\mathscr{D}) B \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} B^{-1} &\xrightarrow{\sim} \mathscr{D} \\ A \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} A^{-1} X B \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} B^{-1} &\mapsto \Pi\left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} A^{-1} X B \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \right) \end{aligned}$$

Remark 2.3.16. The source of $\Pi_{A,B}$ is a subset of $M_n(\mathscr{D})$, but it need not be a sub-$\mathscr{D}$-vector space. It is, however, a $\mathfrak{z}(\mathscr{D})$-vector space with respect to left multiplication. Different matrices X can lead to the same element on the left, but the map $\Pi_{A,B}$ is still well-defined. ◦

Proof of Lemma 2.3.15. The assertion is clear from (2.25): the map $\Pi_{A,B}$ is Π precomposed with left multiplication by A and right multiplication by B^{-1}. □

Lemma 2.3.17. *Let $A, B, C \in \mathrm{GL}_n(\mathscr{D})$. Then there is the following multiplication rule: for all $X, Y \in M_n(\mathscr{D})$,*

$$\begin{aligned} &\Pi_{A,B}\left(A \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} A^{-1} X B \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} B^{-1} \right) \cdot \Pi_{B,C}\left(B \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} B^{-1} Y C \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} C^{-1} \right) = \\ &= \Pi_{A,C}\left(A \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} A^{-1} X B \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} B^{-1} \cdot B \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} B^{-1} Y C \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} C^{-1} \right) \end{aligned}$$

Remark 2.3.18. On the right hand side, one could cancel B^{-1} and B and one of the two idempotents $\mathrm{diag}(1, 0, \ldots, 0)$ in the middle. The formula is presented in this way because this is the form in which it will be applied. ◦

Proof. The first formula in the statement is, by definition of $\Pi_{A,B}$ resp. $\Pi_{B,C}$,

$$\Pi\left(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} A^{-1} X B \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \right) \cdot \Pi\left(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} B^{-1} Y C \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \right)$$

As seen in (2.25), Π is multiplicative on these matrices, hence this is equal to the following:

$$\Pi\left(\begin{pmatrix}1&&\\&\ddots&\\&&0\end{pmatrix}A^{-1}XB\begin{pmatrix}1&&\\&\ddots&\\&&0\end{pmatrix}B^{-1}\cdot B\begin{pmatrix}1&&\\&\ddots&\\&&0\end{pmatrix}B^{-1}YC\begin{pmatrix}1&&\\&\ddots&\\&&0\end{pmatrix}\right)$$

This, in turn, is equal to the second formula by definition of $\Pi_{A,C}$, and the equality is thus established. □

Let us now specialise to the case $\mathscr{D} := \widetilde{D}_\eta$. Passing through the Wedderburn isomorphism (2.15), the ring isomorphism Π in (2.25) gives rise to a ring isomorphism

$$\pi : f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)f_\eta^{(1)} = f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]f_\eta^{(1)} \xrightarrow{\sim} \widetilde{D}_\eta \tag{2.26}$$

Let $a, b \in \mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)^\times$ be two units corresponding to $A, B \in \mathrm{GL}_n(\widetilde{D}_\eta)$. By pre- resp. postcomposing $\Pi_{A,B}$ with the Wedderburn isomorphism (2.15) resp. its inverse, we define maps $\pi_{a,b}$. These are $\mathfrak{z}(\widetilde{D}_\eta)$-vector space isomorphisms (Lemma 2.3.15) satisfying a multiplication rule analogous to Lemma 2.3.17.

Regarding the conjugation action studied in Section 2.3.2, we thus have the following:

Lemma 2.3.19. *Let $v_\chi \mid i, j$. Then there are $\mathfrak{z}(\widetilde{D}_\eta)$-vector space isomorphisms*

$$\Pi_{i,j} : \delta_\gamma^{i/v_\chi}\left(\begin{pmatrix}1&0&\\&\ddots&\\&&0\end{pmatrix}\right)\cdot M_{n_\eta}(\widetilde{D}_\eta)\cdot\delta_\gamma^{j/v_\chi}\left(\begin{pmatrix}1&0&\\&\ddots&\\&&0\end{pmatrix}\right) \xrightarrow{\sim} \widetilde{D}_\eta$$

$$\pi_{i,j} : {}^{\gamma^i}f_\eta^{(1)}\cdot\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)\cdot{}^{\gamma^j}f_\eta^{(1)} \xrightarrow{\sim} \widetilde{D}_\eta$$

Moreover, $\Pi = \Pi_{0,0}$ *and* $\pi = \pi_{0,0}$.

Proof. Let $\Pi_{i,j} := \Pi_{A_{i/v_\chi}, A_{j/v_\chi}}$ and apply Lemma 2.3.15. The isomorphism $\pi_{i,j}$ is then obtained via the Wedderburn isomorphism (2.15). □

Remark 2.3.20. The conjugate ${}^{\gamma^\ell}f_\eta^{(j)} = \gamma^\ell f_\eta^{(j)}\gamma^{-\ell}$ is also an indecomposable idempotent, but it need not be of the form $f_\eta^{(k)}$ for some k. As illustrated by the following example, this is not even the case for matrices, so there is no reason why it should hold for f-idempotents.

$$\begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}2&3\\1&2\end{pmatrix}^{-1} = \begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}2&-3\\-1&2\end{pmatrix} = \begin{pmatrix}4&-6\\2&-3\end{pmatrix}$$

○

2.3.4 Ring structure of $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$

The module $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$ is a ring: indeed, if $x, y \in \mathcal{Q}(\mathcal{G})$ then addition and multiplication rules in $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$ are as follows:

$$f_\eta^{(1)}xf_\eta^{(1)} + f_\eta^{(1)}yf_\eta^{(1)} = f_\eta^{(1)}(x+y)f_\eta^{(1)}$$
$$f_\eta^{(1)}xf_\eta^{(1)}\cdot f_\eta^{(1)}yf_\eta^{(1)} = f_\eta^{(1)}\left(xf_\eta^{(1)}y\right)f_\eta^{(1)}$$

The unity element is $f_\eta^{(1)}$. We will now describe the multiplication rule in more detail.

Under the decomposition (0.3), an element $x \in \mathcal{Q}(\mathcal{G})$ can be written as

$$x = \sum_{\ell=0}^{p^{n_0}-1} x_\ell\gamma^\ell, \quad x_\ell \in \mathcal{Q}(\Gamma_0)[H] \tag{2.27}$$

and similarly for y. When considering $f_\eta^{(1)}xf_\eta^{(1)}$, we may, without loss of generality,restrict the summation to indices ℓ divisible by v_χ. This is parallel to [Lau12a, p. 1224] and [Nic14, p. 610].

Lemma 2.3.21. *We have* $f_\eta^{(1)} x_\ell \gamma^\ell f_\eta^{(1)} = 0$ *whenever* $v_\chi \nmid \ell$*. Therefore*

$$f_\eta^{(1)} x f_\eta^{(1)} = \sum_{\ell=0}^{p^{n_0}-1} f_\eta^{(1)} x_\ell \gamma^\ell f_\eta^{(1)} = \sum_{\substack{\ell=0 \\ v_\chi \mid \ell}}^{p^{n_0}-1} f_\eta^{(1)} x_\ell \gamma^\ell f_\eta^{(1)}$$

Proof. The ε-idempotents of $\mathbb{Q}_p[H]$ are orthogonal; consequently, the definitions of $\varepsilon(\eta)$ and v_χ imply that $\varepsilon({}^{\gamma^i}\eta) \cdot \varepsilon({}^{\gamma^j}\eta) = 0$ whenever $i \not\equiv j \pmod{v_\chi}$. Assume that $v_\chi \nmid \ell$, and use this observation:

$$\begin{aligned} f_\eta^{(1)} x_\ell \gamma^\ell f_\eta^{(1)} &= f_\eta^{(1)} \varepsilon(\eta) x_\ell \gamma^\ell \varepsilon(\eta) f_\eta^{(1)} && f_\eta^{(1)} \varepsilon(\eta) = f_\eta^{(1)} = \varepsilon(\eta) f_\eta^{(1)} \\ &= f_\eta^{(1)} x_\ell \varepsilon(\eta) \gamma^\ell \varepsilon(\eta) f_\eta^{(1)} && \text{centrality of } \varepsilon(\eta) \\ &= f_\eta^{(1)} x_\ell \varepsilon(\eta) \varepsilon({}^{\gamma^\ell}\eta) \gamma^\ell f_\eta^{(1)} \\ &= 0 \end{aligned}$$

This concludes the proof. □

Lemma 2.3.21 allows us compute products in $f_\eta^{(1)} \mathcal{Q}(\mathcal{G}) f_\eta^{(1)}$: since all powers of γ occurring in the sum are divisible by v_χ, conjugation by them acts as some power of δ_τ.

$$\begin{aligned} & f_\eta^{(1)} x f_\eta^{(1)} \cdot f_\eta^{(1)} y f_\eta^{(1)} \qquad (2.28) \\ &= \left(\sum_{\substack{\ell=0 \\ v_\chi \mid \ell}}^{p^{n_0}-1} f_\eta^{(1)} x_\ell \gamma^\ell f_\eta^{(1)} \right) \left(\sum_{\substack{\ell'=0 \\ v_\chi \mid \ell'}}^{p^{n_0}-1} f_\eta^{(1)} y_{\ell'} \gamma^{\ell'} f_\eta^{(1)} \right) \\ &= \sum_{\substack{\ell,\ell'=0 \\ v_\chi \mid \ell,\ell'}}^{p^{n_0}-1} f_\eta^{(1)} \cdot x_\ell \cdot {}^{\gamma^\ell} f_\eta^{(1)} \cdot {}^{\gamma^\ell} f_\eta^{(1)} \cdot {}^{\gamma^\ell} y_{\ell'} \cdot {}^{\gamma^{\ell+\ell'}} f_\eta^{(1)} \cdot \gamma^{\ell+\ell'} \end{aligned}$$

Expand the conjugation action by using (2.20). Since τ acts trivially on $\mathrm{diag}(1, 0 \ldots, 0)$, the corresponding automorphism δ_τ also acts trivially on $f_\eta^{(1)}$ by definition.

$$\begin{aligned} &= \sum_{\substack{\ell,\ell'=0 \\ v_\chi \mid \ell,\ell'}}^{p^{n_0}-1} f_\eta^{(1)} \cdot x_\ell \cdot a_{\ell/v_\chi} f_\eta^{(1)} a_{\ell/v_\chi}^{-1} \cdot a_{\ell/v_\chi} f_\eta^{(1)} a_{\ell/v_\chi}^{-1} \cdot \qquad (2.29) \\ &\qquad \cdot a_{\ell/v_\chi} \delta_\tau^{\ell/v_\chi}(y_{\ell'}) a_{\ell/v_\chi}^{-1} \cdot a_{(\ell+\ell')/v_\chi} f_\eta^{(1)} a_{(\ell+\ell')/v_\chi}^{-1} \cdot \gamma^{\ell+\ell'} \\ &= \sum_{\substack{\ell,\ell'=0 \\ v_\chi \mid \ell,\ell'}}^{p^{n_0}-1} f_\eta^{(1)} x_\ell a_{\ell/v_\chi} f_\eta^{(1)} \cdot f_\eta^{(1)} \delta_\tau^{\ell/v_\chi}(y_{\ell'}) \delta_\tau^{\ell/v_\chi}(a_{\ell'/v_\chi}) f_\eta^{(1)} \cdot a_{(\ell+\ell')/v_\chi}^{-1} \cdot \gamma^{\ell+\ell'} \qquad (2.30) \\ &= \sum_{\substack{\ell,\ell'=0 \\ v_\chi \mid \ell,\ell'}}^{p^{n_0}-1} \pi^{-1} \left(\pi \left(f_\eta^{(1)} x_\ell a_{\ell/v_\chi} f_\eta^{(1)} \right) \cdot \pi \left(f_\eta^{(1)} \delta_\tau^{\ell/v_\chi}(y_{\ell'}) \delta_\tau^{\ell/v_\chi}(a_{\ell'/v_\chi}) f_\eta^{(1)} \right) \right) \cdot a_{(\ell+\ell')/v_\chi}^{-1} \cdot \gamma^{\ell+\ell'} \\ &\qquad (2.31) \end{aligned}$$

In the penultimate step (2.30), we used (2.23). Note that the first multiplication sign in (2.30) is multiplication in $\mathcal{Q}(\Gamma_0)[H]$, while multiplication of the πs in (2.31) is multiplication in $\widetilde{D}_\eta$, these being equivalent due to Lemma 2.3.17. Writing π^{-1} makes sense because π is a ring isomorphism, see (2.26).

Remark 2.3.22. Upon first glance, it may be unclear that the expression in (2.30) is in the ring $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$. However, since $f_\eta^{(1)}$ is an idempotent, nothing changes if we multiply (2.28) by $f_\eta^{(1)}$ on the right. Therefore we may multiply all subsequent expressions by $f_\eta^{(1)}$ on the right, thus ending up with an element of the form $f_\eta^{(1)}zf_\eta^{(1)}$. ○

Remark 2.3.23. The last step (2.31) may appear tautological: indeed, it is just an application of the ring isomorphism π and its inverse. Our reason for stating the multiplication rule in this way is the following. In Section 2.4.1, we will use the multiplication rule in our search for an inverse of a nonzero element in $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$. Writing the element sought as in (2.27), the multiplication rule will provide equations for each coefficient. We will then be able to remove the a-factor on the right, thus ending up with linear equations over the skew field $\widetilde{D}_\eta$. ○

In (2.31), we have a description of the general multiplication rule of the ring $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$. Notice that this is controlled by the automorphism δ_τ. The element γ''_η, which we have already singled out in Definition 2.3.12, plays a special rôle:

Lemma 2.3.24. *Conjugation by γ''_η acts as δ_τ on $f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]f_\eta^{(1)}$.*

Proof. Let $y_0 \in \mathcal{Q}(\Gamma_0)[H]$. Compute the action of conjugation by γ''_η on $f_\eta^{(1)}y_0f_\eta^{(1)}$ using (2.18):

$$\begin{aligned}\gamma''_\eta \cdot f_\eta^{(1)}y_0f_\eta^{(1)} \cdot (\gamma''_\eta)^{-1} &= a_1^{-1}\gamma^{v_\chi} \cdot f_\eta^{(1)}y_0f_\eta^{(1)} \cdot \gamma^{-v_\chi}a_1 \\ &= a_1^{-1} \cdot {}^{\gamma^{v_\chi}}f_\eta^{(1)} \cdot {}^{\gamma^{v_\chi}}y_0 \cdot {}^{\gamma^{v_\chi}}f_\eta^{(1)} \cdot \gamma^{v_\chi}\gamma^{-v_\chi} \cdot a_1 \\ &= a_1^{-1} \cdot a_1f_\eta^{(1)}a_1^{-1} \cdot a_1\delta_\tau(y_0)a_1^{-1} \cdot a_1f_\eta^{(1)}a_1^{-1} \cdot a_1 \\ &= f_\eta^{(1)}\delta_\tau(y_0)f_\eta^{(1)}\end{aligned}$$

□

2.4 Zero divisors in $f_\eta^{(j)}\mathcal{Q}(\mathcal{G})f_\eta^{(j)}$

The elements $f_{\eta_{(i)}}^{(j)}$ are indecomposable idempotents in the group ring $\mathbb{Q}_p[H]$, and thus

$$f_{\eta_{(i)}}^{(j)}\mathbb{Q}_p[H]f_{\eta_{(i)}}^{(j)} \simeq D_{\eta_{(i)}}$$

as witnessed by (2.3). Here the $\eta_{(i)}$ are the characters from (0.6). In this section, we study the question of whether $f_{\eta_{(i)}}^{(j)}$ remains indecomposable in $\mathcal{Q}(\mathcal{G})$. This is equivalent to the following:

Condition 2.4.1. *For all i and j, the algebra $f_{\eta_{(i)}}^{(j)}\mathcal{Q}(\mathcal{G})f_{\eta_{(i)}}^{(j)}$ is a skew field.*

Without loss of generality, we may restrict our attention to the case $j = 1$. Furthermore, we shall suppress the index i of $\eta_{(i)}$.

We will first reformulate the condition in linear algebraic terms (Section 2.4.1), which in turn will be translated to the language of cyclic algebras (Section 2.4.2). We then proceed to verify Condition 2.4.1 in the case when $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified (Section 2.4.2.1), and offer further remarks on the general case (Section 2.4.2.2), in particular showing that there are cases in which the condition does not hold.

The point of considering Condition 2.4.1 is that in this case, the skew field D_χ can be described rather explicitly: this is the objective of Chapter 3.

2.4.1 Translating the question to linear algebra

Condition 2.4.1 means that every nonzero element of $f_\eta^{(1)}\mathcal{Q}(\mathcal{G})f_\eta^{(1)}$ admits a left and right inverse. Since the two are equivalent, we shall work with left inverses, and move powers of γ to the right: this will make the formulæ arising from the multiplication rule slightly more palatable. Let $x \in \mathcal{Q}(\mathcal{G})$ be such that $f_\eta^{(1)}xf_\eta^{(1)} \neq 0$. We seek a $y \in \mathcal{Q}(\mathcal{G})$ such that $f_\eta^{(1)}yf_\eta^{(1)}$ is a left inverse of $f_\eta^{(1)}xf_\eta^{(1)}$.

In other words, we study the question whether the following equation admits a solution:

$$f_\eta^{(1)}yf_\eta^{(1)} \cdot f_\eta^{(1)}xf_\eta^{(1)} = f_\eta^{(1)}\Big(1\cdot\gamma^0 + 0\cdot\gamma^1 + \ldots + 0\cdot\gamma^{p^{n_0}-1}\Big)f_\eta^{(1)} = f_\eta^{(1)} \tag{2.32}$$

Under the decomposition (0.3), the elements x resp. y can be written as

$$x = \sum_{\ell'=0}^{p^{n_0}-1} x_{\ell'}\gamma^{\ell'} \quad \text{resp.} \quad y = \sum_{\ell=0}^{p^{n_0}-1} y_\ell\gamma^\ell, \quad x_{\ell'}, y_\ell \in \mathcal{Q}(\Gamma_0)[H] \tag{2.33}$$

In $f_\eta^{(1)}xf_\eta^{(1)}$ resp. $f_\eta^{(1)}yf_\eta^{(1)}$, these summations restrict to indices ℓ' resp. ℓ divisible by v_χ, pursuant to Lemma 2.3.21. Using the multiplication rule (2.30), the equation (2.32) can be rewritten as

$$\sum_{\substack{\ell,\ell'=0\\ v_\chi|\ell,\ell'}}^{p^{n_0}-1} \Big(f_\eta^{(1)}y_\ell a_{\ell/v_\chi}f_\eta^{(1)}\Big)\cdot\Big(f_\eta^{(1)}\delta_\tau^{\ell/v_\chi}(x_{\ell'}a_{\ell'/v_\chi})f_\eta^{(1)}\Big)\cdot a^{-1}_{(\ell+\ell')/v_\chi}\cdot\gamma^{\ell+\ell'} = f_\eta^{(1)}\gamma^0 f_\eta^{(1)} \tag{2.34}$$

Equation (2.34) is equivalent to the collection of the following equations for $0 \le k < p^{n_0}$ and $v_\chi \mid k$:

$$\sum_{\substack{\ell,\ell'=0\\ \ell+\ell'\equiv k \pmod{p^{n_0}}\\ v_\chi|\ell,\ell'}}^{p^{n_0}-1} \Big(f_\eta^{(1)}y_\ell a_{\ell/v_\chi}f_\eta^{(1)}\Big)\cdot\Big(f_\eta^{(1)}\delta_\tau^{\ell/v_\chi}(x_{\ell'}a_{\ell'/v_\chi})f_\eta^{(1)}\Big)\cdot a^{-1}_{(\ell+\ell')/v_\chi}\cdot\gamma_0^t\gamma^k = f_\eta^{(1)}\delta_{0,k}\gamma^k f_\eta^{(1)}$$

Here $\delta_{0,k}$ is the Kronecker delta, and $t = t(k,\ell)$ is defined by the equation $\ell+\ell' = k + p^{n_0}t$, that is, $t = 1$ if $k < \ell$ (or equivalently $k < \ell'$) and zero otherwise. On the level of coefficients, we have an equation for each k:

$$\sum_{\substack{\ell,\ell'=0\\ \ell+\ell'\equiv k \pmod{p^{n_0}}\\ v_\chi|\ell,\ell'}}^{p^{n_0}-1} \Big(f_\eta^{(1)}y_\ell a_{\ell/v_\chi}f_\eta^{(1)}\Big)\cdot\Big(f_\eta^{(1)}\delta_\tau^{\ell/v_\chi}(x_{\ell'}a_{\ell'/v_\chi})f_\eta^{(1)}\Big)\cdot a^{-1}_{(k+p^{n_0}t)/v_\chi}\cdot\gamma_0^t = \delta_{0,k}f_\eta^{(1)}$$

Consider the factor $a^{-1}_{(k+p^{n_0}t)/v_\chi}$: using (2.23), it can be rewritten as $\delta_\tau^{k/v_\chi}(a_{p^{n_0}t/v_\chi})a^{-1}_{k/v_\chi}$. The factor a^{-1}_{k/v_χ} depends only on k, and thus can be removed by multiplying the kth equation by a_{k/v_χ}. For $k = 0$, this is just 1, and for $k \neq 0$, the right hand side is zero: in either case, the right hand side does not change. The factor $\delta_\tau^{k/v_\chi}(a_{p^{n_0}t/v_\chi})$ is central in $\mathbb{Q}_p(\eta)[H]\varepsilon(\eta)$ by (2.24), so it may be moved inside the second factor in brackets.

In conclusion, applying the multiplication rule (2.31), we find that (2.32) is equivalent to the following system of linear equations over $\widetilde{D}_\eta$, with indeterminates $\pi(f_\eta^{(1)} y_\ell a_{\ell/v_\chi} f_\eta^{(1)})$:

$$\sum_{\substack{\ell,\ell'=0 \\ \ell+\ell'\equiv k \pmod{p^{n_0}} \\ v_\chi|\ell,\ell'}}^{p^{n_0}-1} \underbrace{\pi\left(f_\eta^{(1)} y_\ell a_{\ell/v_\chi} f_\eta^{(1)}\right)}_{\in \widetilde{D}_\eta} \cdot \underbrace{\pi\left(f_\eta^{(1)} \delta_\tau^{\ell/v_\chi}(x_{\ell'} a_{\ell'/v_\chi}) \delta_\tau^{k/v_\chi}(a_{p^{n_0}t/v_\chi})^{-1} f_\eta^{(1)}\right)}_{d_{k,\ell} \in \widetilde{D}_\eta} \gamma_0^t = \delta_{0,k} \tag{2.35}$$

Here k runs over the numbers $0 \le k < p^{n_0}$ such that $v_\chi \mid k$. The two factors in the summation are both in $\widetilde{D}_\eta$ due to Lemma 2.3.19, with the product operation being the one of $\widetilde{D}_\eta$ (see Lemma 2.3.17). Note that the second factor inside the summation, denoted by $d_{k,\ell}$, depends only on k and ℓ, since ℓ' and t are determined by these two.

Lemma 2.4.2. *Equation* (2.32) *admits a solution if and only if the system of linear equations* (2.35) *over* $\widetilde{D}_\eta$*, with* $0 \le k < p^{n_0}$ *and* $v_\chi \mid k$*, has a solution.*

Proof. It is clear from the discussion above that if $y \in \mathcal{Q}(\mathcal{G})$ is a solution to (2.32), then the equations (2.35) will be satisfied. Conversely, assume that there are $\upsilon_\ell \in \widetilde{D}_\eta$ given such that they satisfy (2.35), that is,

$$\sum_{\substack{\ell,\ell'=0 \\ \ell+\ell'\equiv k \pmod{p^{n_0}} \\ v_\chi|\ell,\ell'}}^{p^{n_0}-1} \upsilon_\ell \cdot \pi\left(f_\eta^{(1)} \delta_\tau^{\ell/v_\chi}(x_{\ell'} a_{\ell'/v_\chi}) \delta_\tau^{k/v_\chi}(a_{p^{n_0}t/v_\chi})^{-1} f_\eta^{(1)}\right) \gamma_0^t = \delta_{0,k}$$

Define y_ℓ via the Wedderburn isomorphism as follows:

$$\begin{aligned} M_{n_\eta}(\widetilde{D}_\eta) &\simeq \mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta) \\ \upsilon_\ell \cdot \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix} &\leftrightarrow y_\ell a_{\ell/v_\chi}^{-1} \end{aligned}$$

Since $\varepsilon(\eta) \in \mathbb{Q}_p[H]$, the element y_ℓ can be viewed as an element in $\mathcal{Q}(\Gamma_0)[H]$. Let

$$y := \sum_{\substack{\ell=0 \\ v_\chi|\ell}}^{p^{n_0}-1} y_\ell \gamma^\ell \in \mathcal{Q}(\mathcal{G})$$

Then by the multiplication rule (2.31), we have $\upsilon_\ell = \pi\left(f_\eta^{(1)} y_\ell a_{\ell/v_\chi} f_\eta^{(1)}\right)$, and thus the equations (2.35) are satisfied, so $f_\eta^{(1)} y f_\eta^{(1)}$ is a left inverse. □

Let $n := p^{n_0}/v_\chi$, and let M be the $n \times n$ matrix whose (i,j)th entry is

$$d_{(i-1)v_\chi,(j-1)v_\chi} \gamma_0^{t((i-1)v_\chi,(j-1)v_\chi)} = \begin{cases} d_{(i-1)v_\chi,(j-1)v_\chi}\gamma_0 & \text{if } i < j \\ d_{(i-1)v_\chi,(j-1)v_\chi} & \text{if } i \ge j \end{cases}$$

The existence of a solution to the system of linear equations (2.35) is thus equivalent to the matrix M being nonsingular.

Given a pair (ℓ, k) in the summation in (2.35), the number ℓ' is uniquely determined. Moreover, the pairs (ℓ, k) and $(\ell + v_\chi, k + v_\chi)$ yield the same ℓ', since $\ell + \ell' \equiv k \pmod{p^{n_0}}$. Therefore

$$d_{k,\ell} = \pi\left(f_\eta^{(1)} \delta_\tau^{\ell/v_\chi}(x_{\ell'} a_{\ell'/v_\chi}) \delta_\tau^{k/v_\chi}(a_{p^{n_0}t/v_\chi})^{-1} f_\eta^{(1)}\right) \tag{2.36}$$

$$d_{k+v_\chi,\ell+v_\chi} = \pi\left(f_\eta^{(1)} \delta_\tau^{(\ell+v_\chi)/v_\chi}(x_{\ell'} a_{\ell'/v_\chi}) \delta_\tau^{(k+v_\chi)/v_\chi}(a_{p^{n_0}t/v_\chi})^{-1} f_\eta^{(1)}\right) \tag{2.37}$$

Recall that δ_τ comes from the entry-wise τ-action via the Wedderburn isomorphism, and that π comes from the map Π via the Wedderburn isomorphism. Thus (2.36) and (2.37) allow us to deduce that

$$\tau(d_{k,\ell}) = d_{k+v_\chi,\ell+v_\chi} \tag{2.38}$$

where the indices are understood modulo p^{n_0}.

Let us write $m_i := d_{0,iv_\chi}$ for $0 \le i < n$. The condition $f_\eta^{(1)}xf_\eta^{(1)} \neq 0$ implies that at least one of these m_is is not zero. Indeed, at least one $f_\eta^{(1)}x_{\ell'}f_\eta^{(1)}$ is nonzero, and this is left unaltered by the application of the automorphism δ_τ or multiplication by the unit $a^{-1}_{p^{n_0}t/v_\chi}$. Then using (2.38), we find that $M \in M_n(\widetilde{D}_\eta)$ is of the following shape; recall that τ acts trivially on γ_0:

$$M = \begin{pmatrix} m_0 & m_1\gamma_0 & m_2\gamma_0 & \cdots & m_{n-2}\gamma_0 & m_{n-1}\gamma_0 \\ \tau(m_{n-1}) & \tau(m_0) & \tau(m_1)\gamma_0 & \ddots & \tau(m_{n-3})\gamma_0 & \tau(m_{n-2})\gamma_0 \\ \tau^2(m_{n-2}) & \tau^2(m_{n-1}) & \tau^2(m_0) & \ddots & \tau^2(m_{n-4})\gamma_0 & \tau^2(m_{n-3})\gamma_0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \tau^{n-2}(m_2) & \tau^{n-2}(m_3) & \tau^{n-2}(m_4) & \cdots & \tau^{n-2}(m_0) & \tau^{n-2}(m_1)\gamma_0 \\ \tau^{n-1}(m_1) & \tau^{n-1}(m_2) & \tau^{n-1}(m_3) & \cdots & \tau^{n-1}(m_{n-1}) & \tau^{n-1}(m_0) \end{pmatrix} \tag{2.39}$$

We conclude that Condition 2.4.1 is the special case of the following:

Condition 2.4.3. *Let $M \in M_n(\widetilde{D}_\eta)$ be as in (2.39), with at least one of $m_0, \dots, m_{n-1}$ not zero. Then M is nonsingular.*

In fact, since the as are units and π from (2.26) is an isomorphism, any such matrix M arises from some x, and thus Condition 2.4.3 is equivalent to Condition 2.4.1.

2.4.2 Reduction to Wedderburn's theorem

In this section, we reformulate Condition 2.4.3 further. Let us write $m'_i := \tau^i(m_{n-i})$ where $0 \le i < n$ and $n-i$ is understood modulo n (that is, to be 0 when $i = 0$). Then the transpose of M is

$$M^\top = \begin{pmatrix} m'_0 & m'_1 & m'_2 & \cdots & m'_{n-2} & m'_{n-1} \\ \tau(m'_{n-1})\gamma_0 & \tau(m'_0) & \tau(m'_1) & \ddots & \tau(m'_{n-3}) & \tau(m'_{n-2}) \\ \tau^2(m'_{n-2})\gamma_0 & \tau^2(m'_{n-1})\gamma_0 & \tau^2(m'_0) & \ddots & \tau^2(m'_{n-4}) & \tau^2(m'_{n-3}) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \tau^{n-2}(m'_2)\gamma_0 & \tau^{n-2}(m'_3)\gamma_0 & \tau^{n-2}(m'_4)\gamma_0 & \cdots & \tau^{n-2}(m'_0) & \tau^{n-2}(m'_1) \\ \tau^{n-1}(m'_1)\gamma_0 & \tau^{n-1}(m'_2)\gamma_0 & \tau^{n-1}(m'_3)\gamma_0 & \cdots & \tau^{n-1}(m'_{n-1})\gamma_0 & \tau^{n-1}(m'_0) \end{pmatrix} \tag{2.40}$$

Of course, M is nonsingular if and only if $M^\top$ is, and the condition in Condition 2.4.3 translates to at least one m'_i not being zero.

Consider the following algebra: let

$$A := \bigoplus_{i=0}^{n-1} \widetilde{D}_\eta \left(\gamma_0^{1/n}\right)^i$$

as a left $\widetilde{D}_\eta$-module. Make A an algebra by mandating the multiplication rule $\gamma_0^{1/n}d = \tau(d)\gamma_0^{1/n}$ for all $d \in \widetilde{D}_\eta$. The algebra A has a basis over $\widetilde{D}_\eta$ given by powers of $\gamma_0^{1/n}$. In this basis, right multiplication by the nonzero element $\sum_{i=0}^{n-1} m_i'(\gamma_0^{1/n})^i$ is given by the matrix $M^\top$ in (2.40). It follows that Condition 2.4.3 is equivalent to A being a skew field.

Remark 2.4.4. In the definition of A, we consider $\gamma_0^{1/n}$ as a formal nth root of γ_0. By fiat, this has the same conjugation action as γ^{v_χ} does on η. However, since $n = p^{n_0}/v_\chi$, the identity $\gamma_0^{1/n} = \gamma^{v_\chi}$ holds in Γ. Therefore it behooves us to identify $\gamma_0^{1/n}$ with γ^{v_χ} for the rest of this section, with the tacit understanding that this is purely formal, and that things do not take place in Γ or $\mathcal{G}$. ◦

Notice that A looks like a cyclic algebra, except for the fact that this terminology is reserved for the case when $\widetilde{D}_\eta$ is a field. We will show that A is in fact a cyclic algebra over an appropriate field; this will allow us to utilise the well-developed theory of cyclic algebras in this case.

To this end, let

$$\widehat{D}_\eta := D_\eta \otimes_{\mathbb{Q}_p(\eta)} \mathcal{Q}^{\mathbb{Q}_p(\eta)}\left(\Gamma_0^{\frac{w_\chi}{v_\chi n}}\right) = D_\eta \otimes_{\mathbb{Q}_p(\eta)} \mathcal{Q}^{\mathbb{Q}_p(\eta)}\left(\Gamma^{w_\chi}\right)$$

Recall the cyclic algebra description of D_η:

$$D_\eta = \bigoplus_{\ell=0}^{s_\eta-1} \mathbb{Q}_p(\eta)(\omega_\eta)\pi_{D_\eta}^\ell$$

with multiplication rule $\pi_{D_\eta}\omega_\eta = \sigma(\omega_\eta)\pi_{D_\eta}$ where σ is some power of the Frobenius (it acts as $\sigma(\omega_\eta) = \omega_\eta^{q_\eta^{r_\eta}}$ where r_η/s_η is the Hasse invariant of D_η). This provides a cyclic algebra description for $\widehat{D}_\eta$:

$$\widehat{D}_\eta = \bigoplus_{\ell=0}^{s_\eta-1} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}\left(\Gamma^{w_\chi}\right)\pi_{D_\eta}^\ell$$

Therefore we may write A as

$$A = \bigoplus_{i=0}^{n-1} \widetilde{D}_\eta(\gamma_0^{1/n})^i = \bigoplus_{j=0}^{\frac{w_\chi}{v_\chi}-1} \widehat{D}_\eta(\gamma_0^{1/n})^j = \bigoplus_{j=0}^{\frac{w_\chi}{v_\chi}-1}\bigoplus_{\ell=0}^{s_\eta-1} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}\left(\Gamma^{w_\chi}\right)\pi_{D_\eta}^\ell\left(\gamma^{v_\chi}\right)^j \tag{2.41}$$

Conjugation by π_{D_η} acts as σ whereas conjugation by γ^{v_χ} acts as τ.

Lemma 2.4.5. *Combining the two direct sums into one, we get the following cyclic algebra description:*

$$A = \bigoplus_{k=0}^{\frac{w_\chi}{v_\chi}s_\eta-1} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}\left(\Gamma^{w_\chi}\right)\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^k \tag{2.42}$$

Conjugation by $\pi_{D_\eta}\gamma^{v_\chi}$ acts as $\sigma\tau$, and

$$\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^{\frac{w_\chi}{v_\chi}s_\eta} = N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\pi_\eta)\gamma^{w_\chi s_\eta}$$

Remark 2.4.6. In the conjugation action, it matters not whether we write $\sigma\tau$ or $\tau\sigma$, as the two automorphisms commute. Indeed, both are defined by extending Galois automorphisms of the abelian extension $\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_{p,\chi}$ to act trivially on γ^{w_χ}. ◦

Proof. Consider the algebra

$$\sum_{k=0}^{\frac{w_\chi}{v_\chi}s_\eta-1} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}\left(\Gamma^{w_\chi}\right)\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^k \tag{2.43}$$

It is clear that this is contained in the double sum in (2.41), since expanding $(\pi_{D_\eta}\gamma^{v_\chi})^k$ gives an element of $\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma^{w_\chi})$ times $\pi_{D_\eta}^k\gamma^{v_\chi k}$. We will show that the converse containment also holds.

According to Lemma 2.2.5, τ acts on π_{D_η} as multiplication by a unit in $\mathcal{O}_{\mathbb{Q}_p(\eta)}^\times$. Therefore if we expand $(\pi_{D_\eta}\gamma^{v_\chi})^{s_\eta}$ by moving all γ^{v_χ}s to the right, we get

$$\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^{s_\eta} = u\pi_{D_\eta}^{s_\eta}\left(\gamma^{v_\chi}\right)^{s_\eta}$$

for some $u \in \mathcal{O}_{\mathbb{Q}_p(\eta)}^\times$. Then $u\pi_{D_\eta}^{s_\eta} \in \mathbb{Q}_p(\eta)^\times$, and so $(\gamma^{v_\chi})^{s_\eta}$ is contained in (2.43). Since w_χ is a p-power whereas s_η is coprime to p, there exists an integer t_η such that $s_\eta t_\eta \equiv 1 \bmod w_\chi/v_\chi$. Hence

$$\left(\gamma^{v_\chi}\right)^{s_\eta t_\eta} = \gamma^{v_\chi}\cdot\left(\gamma^{w_\chi}\right)^e$$

for some $e \geq 0$. We conclude that γ^{v_χ} is contained in (2.43).

Similarly, there is a unit $u' \in \mathcal{O}_{\mathbb{Q}_p(\eta)}^\times$ such that

$$\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^{w_\chi/v_\chi} = u'\pi_{D_\eta}^{w_\chi/v_\chi}\gamma^{w_\chi}$$

Again using coprimality of s_η to p, an argument similar to the previous one shows that π_{D_η} is contained in (2.43).

Therefore the double direct sum in (2.41) is equal to (2.43). Both algebras have dimension $s_\eta w_\chi/v_\chi$ over $\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma^{w_\chi})$. Hence the sum in (2.43) is actually direct.

Finally, let us expand the product $\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^{\frac{w_\chi}{v_\chi}s_\eta}$ by gathering the γ^{v_χ} terms to the right. We get

$$\begin{aligned}
\left(\pi_{D_\eta}\gamma^{v_\chi}\right)^{\frac{w_\chi}{v_\chi}s_\eta} &= \left(\pi_{D_\eta}\cdot\tau(\pi_{D_\eta})\cdots\tau^{\frac{w_\chi}{v_\chi}-1}(\pi_{D_\eta})\right)^{s_\eta}\gamma^{w_\chi s_\eta} && \gamma^{v_\chi}\text{ acts via }\tau\\
&= \left(\pi_{D_\eta}^{s_\eta}\cdot\tau\left(\pi_{D_\eta}^{s_\eta}\right)\cdots\tau^{\frac{w_\chi}{v_\chi}-1}\left(\pi_{D_\eta}^{s_\eta}\right)\right)\gamma^{w_\chi s_\eta}\\
&= \left(\pi_\eta\cdot\tau\left(\pi_\eta\right)\cdots\tau^{\frac{w_\chi}{v_\chi}-1}\left(\pi_\eta\right)\right)\gamma^{w_\chi s_\eta} && \pi_{D_\eta}^{s_\eta}=\pi_\eta\\
&= N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\pi_\eta)\gamma^{w_\chi s_\eta} && \langle\tau|_{\mathbb{Q}_p(\eta)}\rangle = \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi})
\end{aligned}$$

The proof is concluded. ☐

The fixed field of the automorphism $\sigma\tau$ of $\mathbb{Q}_p(\eta)(\omega_\eta)$ is $\mathbb{Q}_{p,\chi}$. Hence the centre of A is $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{w_\chi})$. Therefore in the usual notation of cyclic algebras, Lemma 2.4.5 means that A is of the form

$$A = \left(\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}\left(\Gamma^{w_\chi}\right)\Big/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left(\Gamma^{w_\chi}\right), \sigma\tau, N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\pi_\eta)\cdot\gamma^{w_\chi s_\eta}\right)$$

Fix an isomorphism $\mathbb{Z}_p[\![\Gamma^{w_\chi}]\!] \simeq \mathbb{Z}_p[[T]]$ with γ^{w_χ} corresponding to $(1+T)$. Under this isomorphism, we have

$$\begin{aligned}
\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma^{w_\chi}) &\simeq \mathrm{Frac}(\mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[T]]) =: \mathfrak{K}\\
\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{w_\chi}) &\simeq \mathrm{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]) =: \mathfrak{F}
\end{aligned}\tag{2.44}$$

The element $N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\pi_\eta)\cdot\gamma^{w_\chi s_\eta} \in \mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{w_\chi})$ is then identified with

$$a := N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\pi_\eta)(1+T)^{s_\eta} \in \mathfrak{F}$$

Wedderburn's theorem (Theorem 0.7.1) gives a sufficient (but not necessary) condition as to whether a cyclic algebra is a skew field: in this case, it states that A is a skew field whenever

a has order $\frac{w_\chi}{v_\chi}s_\eta$ in the norm factor group $\mathfrak{F}^\times/N_{\mathfrak{K}/\mathfrak{F}}(\mathfrak{K}^\times)$. Since s_η and w_χ/v_χ are coprime, in order to be able to apply Wedderburn's theorem, it suffices to show that the order of a is divisible by both of them.

Lemma 2.4.7. *The order of a in the norm factor group is divisible by s_η.*

Proof. Recall the augmentation exact sequence:

$$0 \to T\mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[T]] \to \mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[T]] \xrightarrow{\text{aug}} \mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)} \to 0$$
$$T \longmapsto 0$$

Localising at the kernel, which is a prime ideal, allows us to extend the augmentation map to

$$\text{aug} : \mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[T]]_{(T)} \to \mathbb{Q}_p(\eta)(\omega_\eta)$$

As the Galois action is trivial on T, the augmentation map is compatible with the norm maps, that is, for all $x \in \mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[T]]_{(T)}$,

$$\text{aug}\left(N_{\mathfrak{K}/\mathfrak{F}}(x)\right) = N_{\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_{p,\chi}}(\text{aug}(x))$$

Suppose that there is an $\alpha \in \mathfrak{K}$ such that $N_{\mathfrak{K}/\mathfrak{F}}(\alpha) = a^i$. Since $T \nmid a^i$, such an element α is in fact contained in $\mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[T]]_{(T)}$. Thus the augmentation map is defined on α, and

$$N_{\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_{p,\chi}}(\text{aug}(\alpha)) = \text{aug}(a^i) \tag{2.45}$$

Using the definition of a as well as transitivity of norms in a tower of extensions, we get

$$N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}\left(N_{\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_p(\eta)}(\text{aug}(\alpha))\right) = N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\pi_\eta^i) \tag{2.46}$$

In particular, the two sides of (2.46) have the same valuation in $\mathbb{Q}_{p,\chi}$. Recall the fact that if $\mathcal{K}/\mathbb{Q}_p$ is a finite field extension and $\mathcal{L}/\mathcal{K}$ is a finite Galois extension, and $x, y \in \mathcal{L}$ such that the valuations $|N_{\mathcal{L}/\mathcal{K}}(x)|_{\mathcal{K}} = |N_{\mathcal{L}/\mathcal{K}}(y)|_{\mathcal{K}}$ agree, then $|x|_{\mathcal{L}} = |y|_{\mathcal{L}}$: indeed, this is a direct consequence of the fact that units have absolute value 1. Applying this fact to (2.46), we find that

$$\text{ord}_{\pi_\eta} N_{\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_p(\eta)}(\text{aug}(\alpha)) = i$$

The extension $\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_p(\eta)$ is unramified of degree s_η, therefore $\text{ord}_{\pi_\eta} N_{\mathbb{Q}_p(\eta)(\omega_\eta)/\mathbb{Q}_p(\eta)}(-)$ has image in $s_\eta\mathbb{Z}$. In particular, $s_\eta \mid i$. ☐

Therefore Condition 2.4.3 holds if (but not necessarily only if) w_χ/v_χ also divides the order of a. The extensions $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ and $\mathbb{Q}_{p,\chi}(\omega)/\mathbb{Q}_{p,\chi}$ have coprime degrees w_χ/v_χ resp. s_η, and their compositum is $\mathbb{Q}_p(\eta)(\omega_\eta)$. Write

$$\mathfrak{L} := \text{Frac}(\mathcal{O}_{\mathbb{Q}_p(\eta)}[[T]]) \quad \text{and} \quad \mathfrak{L}' := \text{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}(\omega_\eta)}[[T]]) \tag{2.47}$$

Then $\mathfrak{L}/\mathfrak{F}$ and $\mathfrak{L}'/\mathfrak{F}$ also have the same coprime degrees, and their compositum is $\mathfrak{K}$. Therefore the norm factor group of $\mathfrak{K}/\mathfrak{F}$ decomposes as follows, see [Mot15, Theorem 1(ii)]:

$$\mathfrak{F}^\times/N_{\mathfrak{K}/\mathfrak{F}}(\mathfrak{K}^\times) \simeq \mathfrak{F}^\times/N_{\mathfrak{L}/\mathfrak{F}}(\mathfrak{L}^\times) \times \mathfrak{F}^\times/N_{\mathfrak{L}'/\mathfrak{F}}(\mathfrak{L}'^\times)$$

Therefore a has order divisible by w_χ/v_χ in $\mathfrak{F}^\times/N_{\mathfrak{K}/\mathfrak{F}}(\mathfrak{K}^\times)$ if its image has order divisible by w_χ/v_χ in $\mathfrak{F}^\times/N_{\mathfrak{L}/\mathfrak{F}}(\mathfrak{L}^\times)$.

2.4.2.1 Proof in the totally ramified case

We prove that Condition 2.4.3 holds in the case when $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified. Remarks on the general case will be made in Section 2.4.2.2.

Lemma 2.4.8. *Suppose that $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified. Then the order of a in the norm factor group is divisible by w_χ/v_χ.*

Proof. In the norm factor group $\mathfrak{F}^\times/N_{\mathfrak{L}/\mathfrak{F}}(\mathfrak{L}^\times)$, the class of a is agrees with that of $(1+T)^{s_\eta}$. So suppose that $(1+T)^{s_\eta i}$ is the norm of some $\alpha(T)\in\mathfrak{L}$.

We first show that without loss of generality, we may assume $\alpha(T)\in\mathcal{O}_{\mathbb{Q}_p(\eta)}[[T]]$. Indeed, the Weierstraß preparation theorem allows us to write

$$\alpha(T)=\pi_\eta^\ell\cdot\frac{F(T)}{G(T)}$$

where $\ell\in\mathbb{Z}$ and $F(T)\in\mathcal{O}_{\mathbb{Q}_p(\eta)}[[T]]$ and

$$G(T)=\prod_{j=0}^{k}P_j(T)\in\mathcal{O}_{\mathbb{Q}_p(\eta)}[T]$$

is a product of distinguished irreducible polynomials $P_j(T)$. The norm of $\alpha(T)$ is the product of its Galois conjugates; it follows that $\ell=0$. Moreover, since $(1+T)^{s_\eta i}$ has no denominator and since each $P_j(T)$ is irreducible, for each j there exists a Galois conjugate $\tilde{P}_j(T)$ of $P_j(T)$ such that $\tilde{P}_j(T)\mid F(T)$. Then

$$\alpha(T)\cdot\prod_{j=1}^{k}\frac{P_j(T)}{\tilde{P}_j(T)}\in\mathcal{O}_{\mathbb{Q}_p(\eta)}[[T]]$$

has the same norm as $\alpha(T)$. From now on, we assume that $\alpha(T)\in\mathcal{O}_{\mathbb{Q}_p(\eta)}[[T]]$.

Since α has integral coefficients, it is convergent at every element x of the maximal ideal $\mathfrak{m}_\chi$ of $\mathbb{Q}_{p,\chi}$. Moreover, since the Galois action is trivial on T as well as on $\mathbb{Q}_{p,\chi}$, for all $x\in\mathfrak{m}_\chi$ we have

$$N_{\mathfrak{L}/\mathfrak{F}}(\alpha(T))\big|_{T=x}=N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}(\alpha(x))=(1+x)^{s_\eta i}\tag{2.48}$$

For totally ramified cyclic extensions of local fields, all Tate cohomology groups of the unit group are cyclic of order equal to the degree, see [EN16, Corollary 2.11]. In particular,

$$\hat{H}^0\left(\mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}),\mathcal{O}_{\mathbb{Q}_p(\eta)}^\times\right)\simeq\mathbb{Z}/\tfrac{w_\chi}{v_\chi}\mathbb{Z}\tag{2.49}$$

The left hand side is the unit norm factor group $\mathcal{O}_{\mathbb{Q}_{p,\chi}}^\times\big/N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}\mathcal{O}_{\mathbb{Q}_p(\eta)}^\times$. If $\#\overline{\mathbb{Q}_{p,\chi}}$ denotes the order of the residue field of $\mathbb{Q}_{p,\chi}$, then there is a decomposition

$$\mathcal{O}_{\mathbb{Q}_{p,\chi}}^\times\simeq\mu_{\#\overline{\mathbb{Q}_{p\chi}}-1}\times U_{\mathbb{Q}_{p,\chi}}^1$$

The group of roots of unity here has order coprime to p, while w_χ/v_χ is a p-power. Therefore the isomorphism (2.49) on the unit norm factor group descends to principal units:

$$U_{\mathbb{Q}_{p,\chi}}^1\Big/N_{\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}}\left(U_{\mathbb{Q}_p(\eta)}^1\right)\simeq\mathbb{Z}/\tfrac{w_\chi}{v_\chi}\mathbb{Z}$$

Let $u\in U_{\mathbb{Q}_{p,\chi}}^1$ be a principal unit whose image in this factor group is a generator. Then u^{s_η} is also a generator, since s_η is coprime to p. Evaluating (2.48) at $T:=u-1$, we get that $u^{s_\eta i}$ is a norm. Therefore $w_\chi/v_\chi\mid i$, as was to be shown. ☐

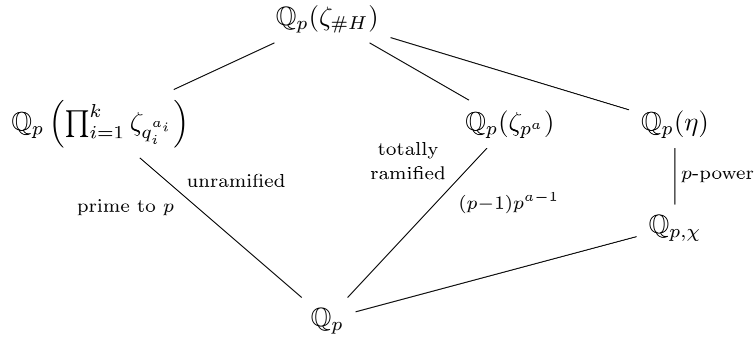
The work done in this section is summarised in the following result.

Theorem 2.4.9. *If $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified, then Condition 2.4.1 holds, that is, $f_\eta\mathcal{Q}(\mathcal{G})f_\eta$ is a skew field.* ☐

The proof of Condition 2.4.1 given above works whenever $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified. This is the case, for instance, when $\mathcal{G}$ is a pro-p-group: indeed, in this case, $\mathbb{Q}_p(\eta)$ is contained in some cyclotomic field $\mathbb{Q}_p(\zeta_{p^m})$, which is totally ramified over $\mathbb{Q}_p$. In particular, the results above constitute a generalisation of those in [Lau12a, §2]. In fact, even more is true:

Lemma 2.4.10. *Suppose H is such that $p \nmid q-1$ holds for every prime factor $q \mid \#H$. Then for every $\chi \in \operatorname{Irr}(\mathcal{G})$ and every irreducible constituent $\eta \mid \operatorname{res}^{\mathcal{G}}_H$, the extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified.*

Proof. For any given χ and η, the values of η are contained in $\mathbb{Q}_p(\zeta_{\#H})$. Let us write $\#H = p^a\prod_{i=1}^k q_i^{a_i}$ where the q_is are pairwise distinct prime numbers distinct from p. The extension $\mathbb{Q}_p(\prod_{i=1}^k \zeta_{q_i}^{a_i})/\mathbb{Q}_p$ is unramified of degree dividing $\prod_{i=1}^k(q_i-1)q_i^{a_i-1}$, which is prime to p by assumption. On the other hand, the extension $\mathbb{Q}_p(\zeta_{p^a})/\mathbb{Q}_p$ is totally ramified of degree $(p-1)p^{a-1}$. These extensions are disjoint over $\mathbb{Q}_p$, and their compositum is the field $\mathbb{Q}_p(\zeta_{\#H})$. Consequently, the inertia group $I(\mathbb{Q}_p(\zeta_{\#H})/\mathbb{Q}_p)$ of $\mathbb{Q}_p(\zeta_{\#H})/\mathbb{Q}_p$ maps onto that of $\mathbb{Q}_p(\zeta_{p^a})/\mathbb{Q}_p$, and for degree reasons, must in fact have order $(p-1)p^{a-1}$.



The extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ in question is a subquotient of $\mathbb{Q}_p(\zeta_{\#H})/\mathbb{Q}_p$, and therefore its inertia group is a homomorphic image of a subgroup of $I(\mathbb{Q}_p(\zeta_{\#H})/\mathbb{Q}_p)$. Moreover, the extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ has p-power degree. The assertion follows. ☐

Finally, we note that Theorem 2.4.9 encompasses the direct product case:

Lemma 2.4.11. *If $\mathcal{G} \simeq H \times \Gamma$ is a direct product, then for all $\chi \in \operatorname{Irr}(\mathcal{G})$ and all irreducible constituents $\eta \mid \operatorname{res}^{\mathcal{G}}_H \chi$, the fields $\mathbb{Q}_p(\eta)$ and $\mathbb{Q}_{p,\chi}$ coincide. In particular, Condition 2.4.1 holds.*

Proof. Irreducible characters of $\mathcal{G}$ with open kernel are of the form $\chi = \eta \times \chi'$ where η resp. χ' are irreducible characters of H resp. Γ with open kernel: this is [Isa76, Theorem 4.21]. Therefore $\operatorname{res}^{\mathcal{G}}_H \chi = \eta$, and $\mathbb{Q}_p(\eta) = \mathbb{Q}_{p,\chi}$ follows. In particular, the extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified, and thus Theorem 2.4.9 is applicable. ☐

2.4.2.2 Remarks on the general case

The argument in the proof of Lemma 2.4.8 does not generalise to not necessarily totally ramified extensions, because (2.49) may fail. The following counterexample shows two things. One, that the extension is not always totally ramified, so the above discussion does not cover all cases. And two, that this is not a mere technical difficulty: there are extensions for which Condition 2.4.1 fails.

Example 2.4.12. Let $p = 3$, $H \simeq C_7$ and $\mathcal{G} \simeq C_7 \rtimes \mathbb{Z}_3$. The action of $\mathbb{Z}_3$ on C_7 is the action that factors through $\mathbb{Z}_3 \twoheadrightarrow \mathbb{Z}_3/3\mathbb{Z}_3 \simeq C_3$ such that conjugation by the generator $(1+3\mathbb{Z}_3)$ acts as the 4th power map on C_7. Consider a 3-dimensional irreducible character χ of $\mathcal{G}$ factoring through $C_7 \rtimes C_3$; see [Gn, C7:C3] for the character table. There are two such characters, both of them with image $\{3, \frac{1}{2}(-1 \pm \sqrt{-7})\}$ on H. Therefore $\mathbb{Q}_{3,\chi} = \mathbb{Q}_3(\sqrt{-7})$ is a quadratic extension of $\mathbb{Q}_3$.

Looking at the character table of the cyclic group H, it is easily verified that the restriction of each of these two χs to H can be expressed as the sum of three nontrivial characters of C_7. Indeed, if ζ_7 denotes a primitive 7th root of unity, then $\zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{1}{2}(-1 + \sqrt{-7})$ and $\zeta_7^3 + \zeta_7^5 + \zeta_7^6 = \frac{1}{2}(-1 - \sqrt{-7})$. Each of these characters η have $\mathbb{Q}_3(\eta) = \mathbb{Q}_3(\zeta_7)$, the unique unramified sextic extension of $\mathbb{Q}_3$.

Therefore the extension $\mathbb{Q}_3(\eta)/\mathbb{Q}_{3,\chi}$ is unramified of degree 3: in particular, it is not totally ramified.

The commutator subgroup $\mathcal{G}' \simeq C_7$ has order coprime to p. A result of Johnston and Nickel stated in Theorem 4.6.3 then shows that all skew fields in the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$ are fields, that is, $s_\chi = 1$. However, the upcoming Corollary 3.1.3 states that Condition 2.4.1 implies $s_\chi = (\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi})s_\eta$. Therefore Condition 2.4.1 cannot hold in this case. ◦

In light of Example 2.4.12, one may consider generalisations of Condition 2.4.1 which also account for the unramified part of $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$.

Remark 2.4.13. It should be noted that the condition in Wedderburn's theorem (Theorem 0.7.1) is sufficient but not necessary. One may ask whether a more direct proof of Condition 2.4.3—at least in the totally ramified case—is possible, exploiting the special shape of the matrix M.

In the realm of linear algebra over fields, if a matrix can be written as a polynomial of another matrix (with coefficients in the field), then the eigenvalues of the first matrix can be obtained from the eigenvalues of the second matrix by applying the polynomial to these.

The matrix M can indeed be written as a polynomial of a certain 'nice enough' matrix, but two obstacles arise: first, everything happens over skew fields now, and second, the coefficients of the polynomial are not scalars but diagonal matrices of the form $\mathrm{diag}(m, \tau(m), \dots, \tau^{n-1}(m))$. The first obstacle can be done away with by using splitting maps, but the second one does not seem to have a solution unless $\tau = 1$. ◦

Chapter 3

Description of the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$

In this chapter, we study the Wedderburn components $M_{n_\chi}(D_\chi)$ of $\mathcal{Q}(\mathcal{G})$, assuming Condition 2.4.1. We provide formulæ for the dimensions n_χ as well as for the Schur indices s_χ (Section 3.1), and describe D_χ explicitly as a cyclic algebra (Section 3.2). Furthermore, we study formal skew power series rings over $\mathcal{O}_{D_\eta}$, and realise D_χ as the total ring of quotients of such a ring (Section 3.3). Both of these descriptions of D_χ are in terms of the skew fields D_η, which in turn were described in Section 0.7.

Our results generalise those of Lau in [Lau12a, §2], which focused on the case when $\mathcal{G}$ is a pro-p group. Progress towards the case of general $\mathcal{G}$ was made by Nickel in [Nic14, §1].

3.1 Schur indices and dimensions of $M_{n_\eta}(D_\eta)$ and $M_{n_\chi}(D_\chi)$

In [Nic14, Corollary 1.13], Nickel proved the following divisibilities:

$$s_\chi \mid s_\eta(\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}) \tag{3.1}$$

$$n_\eta \mid n_\chi \tag{3.2}$$

In this section, we determine the missing factor in the second inequality under the assumption of Condition 2.4.1, and deduce that the first divisibility is sharp in this case.

Our first statement is a mere reformulation of a divisibility in Nickel's proof of (3.2), the only new ingredient being the fact that $(\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}) = w_\chi/v_\chi$ from Proposition 2.3.3, which is not present in [Nic14].

Lemma 3.1.1. *There is a divisibility $n_\eta v_\chi \mid n_\chi$ (without the assumption of Condition 2.4.1).*

Proof. Nickel showed that

$$w_\chi n_\eta s_\eta = w_\chi \eta(1) = \chi(1) = n_\chi s_\chi \,\Big|\, n_\chi s_\eta(\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}) = n_\chi s_\eta \frac{w_\chi}{v_\chi}$$

The assertion follows. □

In particular, we have an inequality $v_\chi n_\eta \leq n_\chi$. This is sharp whenever Condition 2.4.1 holds:

Lemma 3.1.2. *Assume that Condition 2.4.1 holds. Then $v_\chi n_\eta = n_\chi$.*

Proof. On the one hand, we may express the primitive central idempotent ε_χ of $\mathcal{Q}(\mathcal{G})$ as

$$\varepsilon_\chi = \sum_{i=0}^{v_\chi - 1} \varepsilon(\eta_{(i)}) = \sum_{i=0}^{v_\chi - 1} \sum_{j=1}^{n_{\eta_{(i)}}} f_{\eta_{(i)}}^{(j)} \tag{3.3}$$

where the $\eta_{(i)}$s are as in (0.6), and we used that $\nu_\chi^{\mathbb{Q}_p} = v_\chi$, see Proposition 2.3.3. The skew field $D_{\eta_{(i)}}$ has centre $\mathbb{Q}_p(\eta_{(i)}) = \mathbb{Q}_p(\eta)$, which is the same for every i (see Section 0.3). The $\eta_{(i)}$s are all $\mathcal{G}$-conjugates of one another, see (0.6), so in particular, the dimensions $\eta_{(i)}(1)$ all agree. Moreover, it is clear from the character-based description of the Schur index in Section 0.2 that $\mathcal{G}$-conjugate characters have the same Schur index, so $s_{\eta_{(i)}} = s_\eta$. Therefore

$$\dim_{\mathbb{Q}_p(\eta)} M_{n_{\eta_{(i)}}}(D_{\eta_{(i)}}) = s_{\eta_{(i)}}^2 n_{\eta_{(i)}}^2 = \eta_{(i)}(1)^2 = \eta(1)^2 = s_\eta^2 n_\eta^2$$

It follows that $n_{\eta_{(i)}} = n_\eta$ is the same for all i. So (3.3) is an expression of ε_χ as a sum of $v_\chi n_\eta$ idempotents. These are indecomposable by Condition 2.4.1. Equivalently, all right ideals $f_\eta^{(j)}\mathcal{Q}(\mathcal{G})$ are simple right modules by [CR81, (3.18.iii)]. This gives rise to a strictly descending chain of submodules of $\mathcal{Q}(\mathcal{G})\varepsilon_\chi$ of length $v_\chi n_\eta$, with the factor modules being simple: in other words, this is a composition series for $\mathcal{Q}(\mathcal{G})\varepsilon_\chi$.

On the other hand,

$$\varepsilon_\chi = \sum_{i=1}^{n_\chi} f_\chi^{(j)} \tag{3.4}$$

is another decomposition of ε_χ. The idempotents $f_\chi^{(j)}$ are indecomposable in $\mathcal{Q}(\mathcal{G})$, so there is a composition series of length n_χ. Since any two composition series have the same length by [CR81, (3.9)], the assertion follows. ☐

As a consequence of Lemma 3.1.2, we obtain that the divisibility (3.1) is sharp:

Corollary 3.1.3. *Assume that Condition 2.4.1 holds. Then* $s_\chi = (\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi})s_\eta$.

Proof. One just needs to combine the information on the objects at hand.

$$\begin{aligned} s_\chi &= \frac{w_\chi s_\eta n_\eta}{n_\chi} && \text{shown in the proof of [Nic14, Corollary 1.13]} \\ &= \frac{w_\chi s_\eta}{v_\chi} && \text{Lemma 3.1.2} \\ &= (\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi})s_\eta && \text{by (0.7)} \end{aligned}$$

The assertion follows. ☐

Using these results, we can identify the skew field $f_\eta^{(j)}\mathcal{Q}(\mathcal{G})f_\eta^{(j)}$ studied in Section 2.4 with the skew field D_χ in the χ-component of the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$. More practical descriptions of D_χ will be given in Sections 3.2 and 3.3.

Corollary 3.1.4. *Assume that Condition 2.4.1 holds. Then for every irreducible constituent* $\eta \mid \mathrm{res}_H^{\mathcal{G}} \chi$ *and* $1 \le j \le n_\eta$*, there is an isomorphism of rings*

$$D_\chi \simeq f_\eta^{(j)}\mathcal{Q}(\mathcal{G})f_\eta^{(j)}$$

In particular, the right hand side is independent of the choice of η *and* j.

Proof. The χ-part of the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$ is

$$\varepsilon_\chi \mathcal{Q}(\mathcal{G}) \varepsilon_\chi \simeq M_{n_\chi}(D_\chi)$$

and since ε_χ is an idempotent, we can also write

$$\varepsilon_\chi \mathcal{Q}(\mathcal{G}) \varepsilon_\chi = \varepsilon_\chi^2 \mathcal{Q}(\mathcal{G}) \varepsilon_\chi^2 \simeq \varepsilon_\chi M_{n_\chi}(D_\chi) \varepsilon_\chi \tag{3.5}$$

As noted in the proof of Lemma 3.1.2, we have the following decomposition:

$$\varepsilon_\chi = \sum_{i=1}^{v_\chi} \sum_{j=1}^{n_\eta} f_{\eta_{(i)}}^{(j)}$$

where the $\eta_{(i)}$s are as in (0.6). Use this decomposition to expand the ε_χs on the right hand side of (3.5). Since the idempotents $f_{\eta_{(i)}}^{(j)}$ are indecomposable, the submodules

$$f_{\eta_{(i)}}^{(j)} \mathcal{Q}(\mathcal{G}) f_{\eta_{(i)}}^{(j)}$$

are simple, and $\mathcal{Q}(\mathcal{G})\varepsilon_\chi$ is their direct sum because the idempotents $f_{\eta_{(i)}}^{(j)}$ are orthogonal. Hence these are the composition factors of $\mathcal{Q}(\mathcal{G})\varepsilon_\chi$ in the first composition series in the proof of Lemma 3.1.2. On the other hand, the second composition series in that proof, coming from the idempotents $f_\chi^{(j)}$, has composition factors isomorphic D_χ. The Jordan–Hölder theorem, see [CR81, Theorem 3.11], states that any two composition series have the same composition factors up to reordering, and so $f_\eta^{(j)} \mathcal{Q}(\mathcal{G}) f_\eta^{(j)} \simeq D_\chi$ follows. □

3.2 Description of D_χ as a cyclic algebra

This section provides a description of D_χ which formally looks like a cyclic algebra as in (0.14), but with $\mathscr{L}$ not necessarily a field, and ς not necessarily a generator of the automorphism group.

Theorem 3.2.1. *Assume that Condition 2.4.1 holds. Then D_χ admits the following description: there is an isomorphism of rings*

$$D_\chi \simeq \bigoplus_{\substack{i=0 \\ v_\chi | i}}^{p^{n_0}-1} \widetilde{D}_\eta \cdot (\gamma''_\eta)^{i/v_\chi} = \bigoplus_{i=0}^{\frac{p^{n_0}}{v_\chi}-1} \widetilde{D}_\eta \cdot (\gamma''_\eta)^i$$

with conjugation by γ''_η acting as τ on $\widetilde{D}_\eta$.

Proof. The second sum is simply a rewriting of the first one. We now establish the first isomorphism. The proof uses the ring isomorphism $D_\chi \simeq f_\eta^{(1)} \mathcal{Q}(\mathcal{G}) f_\eta^{(1)}$ from Corollary 3.1.4 as well as our discussion of the multiplication rule in Section 2.3.2.

$$\begin{aligned} D_\chi &\simeq f_\eta^{(1)} \mathcal{Q}(\mathcal{G}) f_\eta^{(1)} && \text{Corollary 3.1.4} \\ &\simeq f_\eta^{(1)} \left(\bigoplus_{i=0}^{p^{n_0}-1} \mathcal{Q}(\Gamma_0)[H]\gamma^i \right) f_\eta^{(1)} && \text{by (0.3)} \\ &\simeq \bigoplus_{\substack{i=0 \\ v_\chi | i}}^{p^{n_0}-1} f_\eta^{(1)} \mathcal{Q}(\Gamma_0)[H]\gamma^i f_\eta^{(1)} && \text{Lemma 2.3.21} \end{aligned}$$

$$\simeq \bigoplus_{\substack{i=0\\ v_\chi | i}}^{p^{n_0}-1} f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]\cdot {}^{\gamma^i} f_\eta^{(1)}\cdot\gamma^i$$

$$\simeq \bigoplus_{\substack{i=0\\ v_\chi | i}}^{p^{n_0}-1} f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]\cdot a_{i/v_\chi} f_\eta^{(1)} a_{i/v_\chi}^{-1}\cdot\gamma^i \qquad \text{by (2.20)}$$

$$\simeq \bigoplus_{\substack{i=0\\ v_\chi | i}}^{p^{n_0}-1} f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H] f_\eta^{(1)}\cdot a_{i/v_\chi}^{-1}\gamma^i \qquad a_{i/v_\chi} \text{ is a unit}$$

$$\simeq \bigoplus_{\substack{i=0\\ v_\chi | i}}^{p^{n_0}-1} f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H] f_\eta^{(1)}\cdot (\gamma_\eta'')^{i/v_\chi} \qquad \text{Lemma 2.3.14}$$

$$\simeq \bigoplus_{\substack{i=0\\ v_\chi | i}}^{p^{n_0}-1} \widetilde{D}_\eta\cdot (\gamma_\eta'')^{i/v_\chi} \qquad \text{by (2.4)}$$

As we have seen in Lemma 2.3.24, conjugation by γ_η'' acts as δ_τ on $f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]f_\eta^{(1)}$, which, by definition of δ_τ, becomes the action of τ on $\widetilde{D}_\eta$. □

The description of D_χ in Theorem 3.2.1 resembles that of a cyclic algebra. Just as in Section 2.4.2, we can turn this into an isomorphism with a true cyclic algebra. For this, recall that $\widetilde{D}_\eta$ is cyclic:

$$\widetilde{D}_\eta = \bigoplus_{\ell=0}^{s_\eta-1} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0)\pi_{D_\eta}^\ell$$

with multiplication rule $\pi_{D_\eta}\omega_\eta = \sigma(\omega_\eta)\pi_{D_\eta}$. Substituting this into the right hand side of the isomorphism in Theorem 3.2.1 above, and turning the two direct summations into one, we get the following.

Corollary 3.2.2. *Assume that Condition 2.4.1 holds. Then there is a ring isomorphism from D_χ to the following cyclic algebra:*

$$D_\chi \simeq \bigoplus_{k=0}^{\frac{p^{n_0}}{v_\chi}s_\eta-1} \mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma_0)\cdot\left(\pi_{D_\eta}\gamma_\eta''\right)^k$$

with conjugation by $\pi_{D_\eta}\gamma_\eta''$ acting as $\sigma\tau$ on $\widetilde{D}_\eta$.

Proof. The proof is identical to that of Lemma 2.4.5, with Γ^{w_χ}, γ^{v_χ}, and w_χ/v_χ replaced by Γ_0, γ_η'', and p^{n_0}/v_χ, respectively. □

We describe the centre in terms of γ_η''.

Corollary 3.2.3. *Assume that Condition 2.4.1 holds. Then the centre of D_χ is ring isomorphic to*

$$\mathfrak{z}(D_\chi) \simeq \mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma_\eta'')^{w_\chi/v_\chi}\right)$$

Proof. Consider the algebra

$$A := \bigoplus_{\substack{i=0 \\ v_\chi | i}}^{p^{n_0}-1} \widetilde{D}_\eta \cdot (\gamma''_\eta)^{i/v_\chi}$$

as in Theorem 3.2.1. The element $(\gamma''_\eta)^{w_\chi/v_\chi}$ is central in A because τ has order w_χ/v_χ. The subfield $\mathbb{Q}_{p,\chi}$ is central because it is the fixed field of τ. It follows that $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right)$ is a central subfield in A, that is,

$$\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right) \subseteq \mathfrak{z}(A) \tag{3.6}$$

The dimension of A as a $\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right)$-vector space is as follows:

$$\begin{aligned}
\dim_{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right)}(A) &= \frac{p^{n_0}}{v_\chi} \cdot \dim_{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right)}\left(\widetilde{D}_\eta\right) && \text{definition of } A \\
&= \frac{p^{n_0}}{v_\chi} \cdot \frac{\dim_{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma_0)}\left(\widetilde{D}_\eta\right)}{\dim_{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma_0)}\left(\mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right)\right)} \\
&= \frac{p^{n_0}}{v_\chi} \cdot \frac{\dim_{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma_0)}\left(\widetilde{D}_\eta\right)}{\frac{p^{n_0}}{v_\chi \cdot \frac{w_\chi}{v_\chi}}} && (\dagger) \\
&= \frac{w_\chi}{v_\chi} \cdot \dim_{\mathbb{Q}_{p,\chi}}(D_\eta) && \text{definition of } \widetilde{D}_\eta \\
&= \left(\frac{w_\chi}{v_\chi} s_\eta\right)^2 = s_\chi^2 && \text{Corollary 3.1.3}
\end{aligned}$$

The step marked (†) is due to the fact that $(\gamma''_\eta)^{p^{n_0}/v_\chi}$ differs from γ_0 by a central unit $a_{p^{n_0}/v_\chi}$, as can be seen from applying Lemma 2.3.14 with $j := p^{n_0}/v_\chi$ and (2.24).

Since $A \simeq D_\chi$ by Theorem 3.2.1, the Schur index of A is s_χ, meaning that $\dim_{\mathfrak{z}(A)}(A) = s_\chi^2$. This together with (3.6) shows that the containment (3.6) is in fact an equality. The proof finishes by applying the ring isomorphism of Theorem 3.2.1. □

In [Nic14, Proposition 1.5], Nickel gave an unconditional description of the centre of D_χ:

$$\mathfrak{z}(D_\chi) \simeq \mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi) \tag{3.7}$$

Here Γ'_χ is a procyclic group generated by an element γ'_χ. The element γ'_χ was defined in [Nic14, Lemma 1.2]: it is a Galois equivariant modification of the Ritter–Weiss element γ_χ from (1.3) by a 1-unit.

Nickel also showed that if E is a finite field extension of $\mathbb{Q}_p$ such that one (and hence every) irreducible constituent η of $\operatorname{res}^{\mathcal{G}}_H \chi$ is afforded by an E-representation, then $\mathcal{Q}^E(\Gamma'_\chi)$ is a splitting field for D_χ, see [Nic14, Theorem 1.11]. The minimal fields over which η is afforded are precisely the maximal subfields of D_η: this is shown in the proof of [Isa76, Theorem 10.17]. In particular, η is afforded by a $\mathbb{Q}_p(\eta)(\omega_\eta)$-representation. The extension $\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma'_\chi)/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)$ is cyclic, and we conclude that D_χ is a cyclic algebra even without the assumption of Condition 2.4.1. For this, recall that if $\mathcal{E}/\mathcal{F}$ is a field extension, then the relative Brauer group $\operatorname{Br}(\mathcal{E}/\mathcal{F})$ can be identified with the group $H^2(\operatorname{Gal}(\mathcal{E}/\mathcal{F}), \mathcal{E}^\times)$ of equivalence classes of crossed product algebras, see [Rei03, Theorem 29.12]; and if $\mathcal{E}/\mathcal{F}$ is cyclic, then every crossed product algebra is cyclic, see [Rei03, Theorem 30.3]. We have just proved the following:

Proposition 3.2.4. *The skew field D_χ has Brauer class in* $\operatorname{Br}\left(\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma'_\chi)/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)\right)$. □

Note that while $\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma'_\chi)$ is a splitting field, it need not be a (maximal) subfield of D_χ. Indeed, if it were, then its degree over the centre would equal s_χ, but we have seen a counterexample of this in Example 2.4.12.

3.3 Description of D_χ via skew power series

In this section, we first study the skew power series ring $\mathcal{O}_{D_\eta}[[X;\tau,\delta]]$ on its own, without assuming Condition 2.4.1 (Section 3.3.1). Then we will use the description of D_χ in Theorem 3.2.1, assuming Condition 2.4.1, to identify D_χ with the total ring of quotients of this skew power series ring (Section 3.3.2). We finish the section by discussing further properties of the skew power series ring (Section 3.3.3).

3.3.1 Properties of the skew power series ring

First we will check that the relevant skew power series ring is well defined (see Section 0.4), then proceed by describing its centre, which will be used in establishing the aforementioned identification in Theorem 3.3.11.

Proposition 3.3.1. *Let $\mathcal{O}_{D_\eta}$ denote the unique maximal $\mathcal{O}_{\mathbb{Q}_p(\eta)}$-order in D_η. This is a noetherian pseudocompact ring, an τ is a topological automorphism of it.*

Let $\delta := \tau - \mathrm{id}$. Then δ is a continuous left τ-derivation, it commutes with τ, and it is τ-nilpotent. Thus the skew power series ring $\mathcal{O}_{D_\eta}[[X;\tau,\delta]]$ is well-defined.

For the proof of Proposition 3.3.1, we will need the following lemma.

Lemma 3.3.2. *For $n \geq 0$ and $d \in \mathcal{O}_{D_\eta}$:*

$$\delta^n(d) = \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} \tau^\ell(d)$$

Proof. The proof is by induction. The assertion is true for $n = 0$ and $n = 1$. Assume it has been proven for n. Then

$$\begin{aligned}
\delta^{n+1}(d) = \delta\left(\delta^n(d)\right) &= \tau\left(\sum_{\ell=0}^{n} (-1)^{n-\ell}\binom{n}{\ell}\tau^\ell(d)\right) - \sum_{\ell=0}^{n}(-1)^{n-\ell}\binom{n}{\ell}\tau^\ell(d) \\
&= \sum_{\ell=0}^{n+1}(-1)^{(n+1)-\ell}\left(\binom{n}{\ell} + \binom{n}{\ell-1}\right)\tau^\ell(d) \\
&= \sum_{\ell=0}^{n+1}(-1)^{(n+1)-\ell}\binom{n+1}{\ell}\tau^\ell(d)
\end{aligned}$$

□

Proof of Proposition 3.3.1. The ring $\mathcal{O}_{D_\eta}$ is a noetherian ring with unity. It is pseudocompact: the two-sided ideals $L_i := \mathcal{O}_{D_\eta}\pi^i_{D_\eta}$ for $i \geq 0$ form a fundamental system of open neighbourhoods of zero, and for any $i \geq 0$, $\mathcal{O}_{D_\eta}/L_i$ is of finite length over $\mathcal{O}_{D_\eta}$. Indeed,

$$\mathcal{O}_{D_\eta}/L_i \supset \mathcal{O}_{D_\eta}\pi_{D_\eta}/L_i \supset \mathcal{O}_{D_\eta}\pi^2_{D_\eta}/L_i \supset \ldots \supset \mathcal{O}_{D_\eta}\pi^{i-1}_{D_\eta}/L_i \supset \mathcal{O}_{D_\eta}\pi^i_{D_\eta}/L_i = 0$$

is a finite composition series. The ring $\mathcal{O}_{D_\eta}$ is complete with respect to the π_{D_η}-adic topology, and the ideal $\pi_{D_\eta}\mathcal{O}_{D_\eta}$ is invariant under τ by construction (cf. Lemma 2.2.5), hence τ is a topological automorphism.

The endomorphism $(\tau - \mathrm{id})$ is indeed a τ-derivation:

$$\begin{aligned}\forall x, y \in \mathcal{O}_{D_\eta} : (\tau - \mathrm{id})(xy) &= \tau(xy) - xy \\ &= \tau(x)y - xy + \tau(x)\tau(y) - \tau(x)y \\ &= (\tau - \mathrm{id})(x) \cdot y + \tau(x) \cdot (\tau - \mathrm{id})(y)\end{aligned}$$

It is continuous because τ is.

Write e for the ramification index of $\mathbb{Q}_p(\eta)(\pi_{D_\eta})/\mathbb{Q}_p$, that is, $\pi_{D_\eta}^e$ is p times a unit in $\mathbb{Q}_p(\eta)(\pi_{D_\eta})$. Let $d \in \mathcal{O}_{D_\eta}$, and apply Lemma 3.3.2 with $n := w_\chi/v_\chi$. Since p is odd, and w_χ/v_χ is a power of p, and τ has order w_χ/v_χ, the first and last terms of the sum cancel out, and all remaining terms contain $\binom{w_\chi/v_\chi}{\ell}$ for $0 < \ell < w_\chi/v_\chi$. Since w_χ/v_χ is a power of p, these coefficients are all divisible by p, and therefore $\pi_{D_\eta}^e \mid \delta^{w_\chi/v_\chi}(d)$. Using induction and the fact that both τ and the identity preserve π_{D_η}-adic valuations, we obtain that

$$\pi_{D_\eta}^{ei} \mid \delta^{iw_\chi/v_\chi}(d) \qquad \forall i \geq 1 \tag{3.8}$$

Moreover, we have that

$$\pi_{D_\eta}^{ei} \mid \delta^k(d) \qquad \forall k \geq \frac{iw_\chi}{v_\chi} \tag{3.9}$$

since both τ and the identity preserve π_{D_η}-adic valuations.

We use the observations of the previous paragraph to show τ-nilpotence of δ. The ring $\mathcal{O}_{D_\eta}$ has Jacobson radical $\mathcal{O}_{D_\eta}\pi_{D_\eta}$. Since τ and δ commute, τ-nilpotence is equivalent to topological nilpotence, that is, we need to show that for all $n \geq 1$ there is an $m \geq 1$ such that for all $k \geq m$: $\delta^k(\mathcal{O}_{D_\eta}) \subseteq \mathcal{O}_{D_\eta}\pi_{D_\eta}^n$. Let n be given. If m is chosen to be large enough, or more precisely, chosen such that $m \geq iw_\chi/v_\chi$ where i is such that $ei \geq n$, then the divisbilities (3.8) and (3.9) above show the claim. Thus τ-nilpotence is established.

As explained in Section 0.4, the skew power series ring $\mathcal{O}_{D_\eta}[[X;\tau,\delta]]$ is then well-defined. □

Remark 3.3.3. The statements above form a special case of [SV06, Lemma 1.6], where the proof is given more generally in the case when τ and δ commute. ○

The multiplication rule $Xd = \tau(d)X + \delta(d)$ admits the following generalisation to higher powers of X.

Lemma 3.3.4. *For $n \geq 0$ and $d \in \mathcal{O}_{D_\eta}$:*

$$X^n d = \sum_{i=0}^{n} \binom{n}{i} \tau^i \delta^{n-i}(d) X^i$$

Proof. The proof is by induction. For $n = 0$, the claim is vacuous, and for $n = 1$, it holds by definition. Note that τ and δ commute. Now assume that the claim has been proven for n. Then

$$\begin{aligned}X^{n+1}d = X \cdot X^n d &= X \sum_{i=0}^{n} \binom{n}{i} \tau^i \delta^{n-i}(d) X^i \\ &= \sum_{i=0}^{n} \left(\tau\left(\binom{n}{i} \tau^i \delta^{n-i}(d) \right) X^{i+1} + \delta\left(\binom{n}{i} \tau^i \delta^{n-i}(d) \right) X^i \right) \\ &= \sum_{i=0}^{n} \left(\binom{n}{i} \tau^{i+1} \delta^{n-i}(d) X^{i+1} + \binom{n}{i} \tau^i \delta^{n-i+1}(d) X^i \right)\end{aligned}$$

$$= \sum_{i=0}^{n+1} \left(\binom{n}{i-1} + \binom{n}{i} \right) \tau^i \delta^{(n+1)-i}(d) X^i$$
$$= \sum_{i=0}^{n+1} \binom{n+1}{i} \tau^i \delta^{(n+1)-i}(d) X^i \qquad \square$$

Proposition 3.3.5. *The centre of* $\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]$ *contains* $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi} - 1]]$.

Remark 3.3.6. We will see in Corollary 3.3.8 that this is in fact the full centre. ∘

To tackle the sums of products of binomial coefficients appearing in the proof of Proposition 3.3.5, we shall need the following lemma.

Lemma 3.3.7 ([Knu97, §I.2.6, (23)])**.** *Let* $r, n \in \mathbb{Z}$, $r \geq 0$, $s \in \mathbb{R}$. *Then*

$$\sum_{k \in \mathbb{Z}} (-1)^{r-k} \binom{r}{k} \binom{k+s}{n} = \binom{s}{n-r}$$

Proof of Proposition 3.3.5. The automorphism τ fixes $\mathbb{Q}_{p,\chi}$, therefore every element of $\mathcal{O}_{\mathbb{Q}_{p,\chi}}$ is central. It remains to show that for all $d \in \mathcal{O}_{D_\eta}$,

$$(1+X)^{w_\chi/v_\chi} d = d(1+X)^{w_\chi/v_\chi}$$

We expand both sides using the binomial theorem, and show that the respective coefficients agree. The right hand side is

$$d(1+X)^{w_\chi/v_\chi} = \sum_{i=0}^{w_\chi/v_\chi} d \binom{w_\chi/v_\chi}{i} X^i \tag{3.10}$$

The left hand side is

$$\begin{aligned}
(1+X)^{w_\chi/v_\chi} d &= \sum_{j=0}^{w_\chi/v_\chi} \binom{w_\chi/v_\chi}{j} X^j d \\
&= \sum_{j=0}^{w_\chi/v_\chi} \binom{w_\chi/v_\chi}{j} \sum_{i=0}^{j} \binom{j}{i} \tau^i \delta^{j-i}(d) X^i \qquad \text{Lemma 3.3.4} \\
&= \sum_{i=0}^{w_\chi/v_\chi} \sum_{j=i}^{w_\chi/v_\chi} \binom{w_\chi/v_\chi}{j} \binom{j}{i} \tau^i \delta^{j-i}(d) X^i \qquad (3.11)
\end{aligned}$$

The coefficient of X^i in (3.11) is

$$\begin{aligned}
\sum_{j=i}^{w_\chi/v_\chi} \binom{w_\chi/v_\chi}{j} \binom{j}{i} \tau^i \delta^{j-i}(d) &= \sum_{j=i}^{w_\chi/v_\chi} \sum_{\ell=0}^{j-i} (-1)^{j-i-\ell} \binom{w_\chi/v_\chi}{j} \binom{j}{i} \binom{j-i}{\ell} \tau^{i+\ell}(d) \\
&= \sum_{\ell=0}^{w_\chi/v_\chi - i} \left(\sum_{j=i+\ell}^{w_\chi/v_\chi} (-1)^{j-i-\ell} \binom{w_\chi/v_\chi}{j} \binom{j}{i} \binom{j-i}{\ell} \right) \tau^{i+\ell}(d)
\end{aligned}$$

where we used Lemma 3.3.2 and then exchanged the order of summations. We claim that the coefficient in the brackets is

$$\sum_{j=i+\ell}^{w_\chi/v_\chi} (-1)^{j-i-\ell} \binom{w_\chi/v_\chi}{j} \binom{j}{i} \binom{j-i}{\ell} = \begin{cases} \binom{w_\chi/v_\chi}{i} & \text{if } \ell = w_\chi/v_\chi - i \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

For this, recall the following well-known property of binomial coefficients, see e.g. [Knu97, §I.2.6, (20)]:

$$\binom{w_\chi/v_\chi}{j} \binom{j}{i} = \binom{w_\chi/v_\chi}{i} \binom{w_\chi/v_\chi - i}{j-i}$$

Using this, and then applying Lemma 3.3.7 with $\jmath := 0$, we find that the left hand side of (3.12) is

$$\begin{aligned} \sum_{j=i+\ell}^{w_\chi/v_\chi} (-1)^{j-i-\ell} \binom{w_\chi/v_\chi}{j} \binom{j}{i} \binom{j-i}{\ell} &= \binom{w_\chi/v_\chi}{i} \sum_{j=i+\ell}^{w_\chi/v_\chi} (-1)^{j-i-\ell} \binom{w_\chi/v_\chi - i}{j-i} \binom{j-i}{\ell} \\ &= \binom{w_\chi/v_\chi}{i} \binom{0}{\ell - (w_\chi/v_\chi - i)} \end{aligned}$$

The second binomial coefficient here is 1 if $\ell = w_\chi/v_\chi - i$ and zero otherwise, which proves (3.12).

Therefore the coefficient of X^i in (3.11) is

$$\binom{w_\chi/v_\chi}{i} \tau^{w_\chi/v_\chi}(d) = \binom{w_\chi/v_\chi}{i} d$$

where we used that τ has order w_χ/v_χ. This agrees with the coefficient of X^i in (3.10) for all i. The proof is concluded. □

Corollary 3.3.8. *The centre of* $\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]$ *is* $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]$. *Moreover, the skew field* $\mathrm{Quot}(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]])$ *has Schur index* $\frac{w_\chi}{v_\chi} s_\eta$.

For the proof, we introduce the following notation.

$$\begin{aligned} \mathfrak{O} &:= \mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]] \\ \mathfrak{D} &:= \mathrm{Quot}(\mathfrak{O}) \\ \mathcal{O}_{\mathfrak{E}} &:= \mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[(1+X)^{w_\chi/v_\chi}-1]] \\ \mathfrak{E} &:= \mathrm{Frac}\,(\mathcal{O}_{\mathfrak{E}}) \\ \mathfrak{L} &:= \mathrm{Frac}\left(\mathcal{O}_{\mathbb{Q}_p(\eta)}[[T]]\right) \\ \mathcal{O}_{\mathfrak{F}} &:= \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]] \\ \mathfrak{F} &:= \mathrm{Frac}\left(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]\right) \end{aligned}$$

For brevity, we will write $T := (1+X)^{w_\chi/v_\chi}-1$. Then $\mathfrak{L}$ resp. $\mathfrak{F}$ are isomorphic to the fields in (2.47) resp. (2.44).

Proof. First, observe that $\mathfrak{O}$ is a (noncommutative) domain, because $\mathcal{O}_{D_\eta}$ is a domain and τ is an automorphism: this is [Ven03, Corollary 2.10(i)]. Therefore the total ring of quotients $\mathfrak{D}$ is a skew field. Proposition 3.3.5 then shows that the square of the Schur index of this skew field divides the dimension over $\mathfrak{F}$. This dimension is

$$\dim_{\mathfrak{F}} \mathfrak{D} = \left(\frac{w_\chi}{v_\chi}\right)(D_\eta : \mathbb{Q}_{p,\chi}) = \left(\frac{w_\chi}{v_\chi}\right)^2 s_\eta^2 \tag{3.13}$$

On the other hand, $\mathfrak{D}$ is a left $\mathfrak{E}$-vector space of dimension

$$\dim_{\mathfrak{E}} \mathfrak{D} = \left(\frac{w_\chi}{v_\chi}\right)(D_\eta : \mathbb{Q}_p(\eta)(\omega_\eta)) = \left(\frac{w_\chi}{v_\chi}\right) s_\eta \tag{3.14}$$

Note that $\mathfrak{E}$ is a subfield of $\mathfrak{D}$. We will show that it is in fact a maximal (self-centralising) subfield. Assuming this for a moment, we deduce the statement of the Corollary. The dimension $\dim_{\mathfrak{E}} \mathfrak{D}$ equals the Schur index of $\mathfrak{D}$. This in turn equals $\dim_{\mathfrak{z}(\mathfrak{D})} \mathfrak{E}$. Since $\mathfrak{z}(\mathfrak{D}) \supseteq \mathfrak{F}$ by Proposition 3.3.5, the equations (3.13) and (3.14) above show that this containment is in fact an equality, as desired. The assertion about the Schur index follows.

Consider the composition

$$\begin{aligned} \varphi : D_\eta &\to \mathbb{Q}_p(\eta)(\omega_\eta) \otimes_{\mathbb{Q}_p(\eta)} D_\eta \xrightarrow{\sim} M_{s_\eta}(\mathbb{Q}_p(\eta)(\omega_\eta)) \\ d &\mapsto 1 \otimes d \end{aligned}$$

where the second map is the splitting isomorphism of (0.16). As showed in the proof of [Rei03, Theorem 14.6], this map is an isomorphism onto its image. Moreover, D_η is a vector space over $\mathbb{Q}_p(\eta)(\omega_\eta)$ with basis $\pi_{D_\eta}^i$ for $0 \le i < s_\eta$, and the image $\varphi(D_\eta)$ has $\mathbb{Q}_p(\eta)(\omega_\eta)$-basis $\varphi(\pi_{D_\eta}^i)$ for $0 \le i < s_\eta$.

Extending scalars, the map φ gives rise to a homomorphism

$$\hat{\varphi} : \mathrm{Quot}\left(\mathcal{O}_{D_\eta}[[T]]\right) \to \mathfrak{E} \otimes_{\mathfrak{L}} \mathrm{Quot}\left(\mathcal{O}_{D_\eta}[[T]]\right) \xrightarrow{\sim} M_{s_\eta}(\mathfrak{E})$$

by letting $\hat{\varphi}$ be the identity on T.

We define an $\mathfrak{L}$-vector space homomorphism

$$\Phi : \mathfrak{D} \to \mathfrak{E} \otimes_{\mathfrak{L}} \mathfrak{D} \to M_{\frac{w_\chi}{v_\chi} s_\eta}(\mathfrak{E}) \tag{3.15}$$

The first map sends an element $d \in \mathfrak{D}$ to $1 \otimes d$. The second map is the unique $\mathfrak{E}$-linear map determined by the following two properties: for $g \in \mathrm{Quot}(\mathcal{O}_{D_\eta}[[T]])$, it maps $1 \otimes g$ to the block diagonal matrix

$$\Phi(g) := \mathrm{diag}\left(\hat{\varphi}(g), \tau(\hat{\varphi}(g)), \tau^2(\hat{\varphi}(g)), \ldots, \tau^{\frac{w_\chi}{v_\chi}-1}(\hat{\varphi}(g))\right)$$

and it maps $1 \otimes X$ to

$$\Phi(X) := \begin{pmatrix} -\mathbf{1}_{s_\eta} & \mathbf{1}_{s_\eta} & & & \\ & -\mathbf{1}_{s_\eta} & \mathbf{1}_{s_\eta} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \mathbf{1}_{s_\eta} \\ & & & -\mathbf{1}_{s_\eta} & \mathbf{1}_{s_\eta} \\ (1+X)^{w_\chi/v_\chi}\mathbf{1}_{s_\eta} & & & & -\mathbf{1}_{s_\eta} \end{pmatrix} = \begin{pmatrix} & \mathbf{1}_{s_\eta} & & & \\ & & \mathbf{1}_{s_\eta} & & \\ & & & \ddots & \\ & & & & \mathbf{1}_{s_\eta} \\ & & & & \mathbf{1}_{s_\eta} \\ (1+X)^{w_\chi/v_\chi}\mathbf{1}_{s_\eta} & & & & \end{pmatrix} - \mathbf{1}_{\frac{w_\chi}{v_\chi} s_\eta}$$

where $\mathbf{1}_{s_\eta}$ is the $s_\eta \times s_\eta$ identity matrix. Since every element $G(X) \in \mathfrak{D}$ can be written as

$$G(X) = \sum_{i=0}^{w_\chi/v_\chi - 1} g_i(T)(1+X)^i$$

where $g_i(T) \in \mathrm{Quot}(\mathcal{O}_{D_\eta}[[T]])$, extending these assignments as

$$\Phi(G(X)) := \sum_{i=0}^{w_\chi/v_\chi - 1} \Phi(g_i(T))\Phi(1+X)^i \tag{3.16}$$

indeed defines a homomorphism as in (3.15). Moreover, for all $d \in D_\eta$, the multiplication rule

$$\Phi(1+X)\Phi(d) = \tau(\Phi(d))\Phi(1+X)$$

of skew power series rings is satisfied: indeed,

$$\begin{aligned}
&\Phi(1+X)\Phi(d) \\
&= \begin{pmatrix} & \mathbf{1}_{s_\eta} & & \\ & & \mathbf{1}_{s_\eta} & \\ & & & \ddots \\ & & & & \mathbf{1}_{s_\eta} \\ (1+X)^{w_\chi/v_\chi}\mathbf{1}_{s_\eta} & & & & \end{pmatrix} \begin{pmatrix} \varphi(d) & & & & \\ & \tau(\varphi(d)) & & & \\ & & \tau^2(\varphi(d)) & & \\ & & & \ddots & \\ & & & & \tau^{w_\chi/v_\chi - 1}(\varphi(d)) \end{pmatrix} \\
&= \begin{pmatrix} & \tau(\varphi(d)) & & \\ & & \tau^2(\varphi(d)) & \\ & & & \ddots \\ & & & & \tau^{w_\chi/v_\chi - 1}(\varphi(d)) \\ (1+X)^{w_\chi/v_\chi}\varphi(d) & & & & \end{pmatrix} \\
&= \begin{pmatrix} \tau(\varphi(d)) & & & & \\ & \tau^2(\varphi(d)) & & & \\ & & \tau^3(\varphi(d)) & & \\ & & & \ddots & \\ & & & & \varphi(d) \end{pmatrix} \begin{pmatrix} & \mathbf{1}_{s_\eta} & & \\ & & \mathbf{1}_{s_\eta} & \\ & & & \ddots \\ & & & & \mathbf{1}_{s_\eta} \\ (1+X)^{w_\chi/v_\chi}\mathbf{1}_{s_\eta} & & & & \end{pmatrix} \\
&= \tau(\Phi(d))\Phi(1+X)
\end{aligned}$$

So Φ is an $\mathfrak{L}$-algebra homomorphism. Since $\mathfrak{D}$ is a skew field, the kernel of Φ is either zero or $\mathfrak{D}$, and it is clear from the definition of Φ that it is not the zero map. Hence Φ is injective.

At last, we turn to showing that $\mathfrak{E}$ is a maximal subfield of $\mathfrak{D}$. Since Φ is an isomorphism onto its image, this is equivalent to showing that $\Phi(\mathfrak{E})$ is a maximal subfield in $\Phi(\mathfrak{D})$.

Let α be a primitive element of the field extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$. Then the Galois conjugates $\tau^j(\alpha)$ are pairwise distinct for $0 \le j < w_\chi/v_\chi$. Let σ be as in (0.16). Then the elements $\sigma^i(\omega_\eta)$ are also pairwise distinct for $0 \le i < s_\eta$. Since τ and σ have coprime orders, we conclude that the diagonal matrix $\Phi(\alpha\omega_\eta)$ has pairwise distinct entries in its diagonal.

Suppose $A \in \Phi(\mathfrak{D})$ centralises $\Phi(\mathfrak{E})$. Then in particular, it commutes with $\Phi(\alpha\omega_\eta)$. By the previous observation, this forces A to be diagonal. But it is clear from the definition of Φ that diagonal matrices have preimage in $\mathfrak{E}$, hence $A \in \Phi(\mathfrak{E})$, proving that $\mathfrak{E}$ is a maximal (self-centralising) subfield of $\mathfrak{D}$. □

Remark 3.3.9. The above realisation of $\mathfrak{D}$ through matrices is based on ideas of Hasse, which he used in his work on skew fields over local fields. See [Has31, Satz 40] for the original proof, or [Rei03, Theorem 14.6] for a more modern reference. ○

As the centre of $\mathfrak{D}$ is now determined by Corollary 3.3.8, we shall use the notation

$$\begin{aligned}
\mathfrak{z}(\mathfrak{D}) &= \mathrm{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi} - 1]]) \\
\mathcal{O}_{\mathfrak{z}(\mathfrak{D})} &:= \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi} - 1]]
\end{aligned}$$

instead of $\mathfrak{F}$ and $\mathcal{O}_{\mathfrak{F}}$ from now on.

Remark 3.3.10. Up to this point, all results is this section are about skew power series rings, and they are independent of Condition 2.4.1. Note that they could have been stated in a slightly more general setting: one can replace D_η by any finite dimensional skew field over a local field, and τ by an extension of a Galois automorphism as in Section 2.2. This justifies suppression of χ and η from the notations $\mathfrak{D}$, $\mathfrak{O}$ et cetera. ∘

3.3.2 Identification with D_χ

The main result of Section 3.3 is the following:

Theorem 3.3.11. *Assume that Condition 2.4.1 holds. Then the algebra from Theorem 3.2.1 is isomorphic (as a ring) to the total ring of quotients of a skew power series ring, and the centre is isomorphic to the field of fractions of a power series ring, as described by the following commutative diagram:*

$$\begin{array}{ccccc}
D_\chi & \simeq & \displaystyle\bigoplus_{\substack{i=0 \\ v_\chi \mid i}}^{p^{n_0}-1} \widetilde{D}_\eta \cdot (\gamma''_\eta)^{i/v_\chi} & \xrightarrow{\ \sim\ } & \operatorname{Quot}\left(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]\right) = \mathfrak{D} \\
\big\uparrow & & \big\uparrow & & \big\uparrow \\
\mathfrak{z}(D_\chi) & \simeq & \mathcal{Q}^{\mathbb{Q}_{p,\chi}}\left((\Gamma''_\eta)^{w_\chi/v_\chi}\right) & \xrightarrow{\ \sim\ } & \operatorname{Frac}\left(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]\right) = \mathfrak{z}(\mathfrak{D})
\end{array}$$

The top horizontal map is the identity on $\mathcal{O}_{D_\eta}$ and sends $\gamma''_\eta \mapsto 1+X$. The bottom horizontal map is the identity on $\mathcal{O}_{\mathbb{Q}_{p,\chi}}$ and sends $(\gamma''_\eta)^{w_\chi/v_\chi} \mapsto (1+X)^{w_\chi/v_\chi}$.

Proof. Since γ''_η is sent to $(1+X)$, the multiplication rule $\gamma''_\eta d = \tau(d)\gamma''_\eta$ for $d \in D_\eta$ becomes $(1+X)d = \tau(d)(1+X)$, which is equivalent to $Xd = \tau(d)X + (\tau - \mathrm{id})(d)$: this is indeed the multiplication rule in $\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]$. As we have seen in Corollary 3.2.3, the centre is generated by $(\gamma''_\eta)^{w_\chi/v_\chi}$; since γ_0 is central, the image of γ_0 is therefore determined by the image of γ''_η. Ergo the top horizontal map is well-defined and a ring homomorphism. The lower horizontal map is a well-defined isomorphism induced by the classical isomorphism between the Iwasawa algebra over $\mathbb{Z}_p$ and the ring of formal power series over $\mathbb{Z}_p$. Commutativity of the diagram follows directly from the definition of the arrows within.

It remains to show that the top horizontal map is an isomorphism. It is clearly injective.

Due to Theorem 3.2.1, the crossed product algebra on the left is a left vector space over its centre, of dimension s_χ^2. On the right hand side, the ring $\operatorname{Quot}(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]])$ is a skew field with centre $\operatorname{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]])$ (Corollary 3.3.8). Since the dimensions agree, it follows that the top horizontal map must be an isomorphism. □

Remark 3.3.12. This argument provides another proof of Corollary 3.3.8. Indeed, from Proposition 3.3.5 we get an inequality for the dimensions, and injectivity of the map shows that this must be an equality, and therefore the dimension of the skew field $\operatorname{Quot}(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]])$ over the field $\operatorname{Frac}\left(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]\right)$ equals the square of its Schur index. ∘

We define Σ_χ to be the preimage of $\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]$ under the isomorphism in Theorem 3.3.11. Its centre is

$$\mathfrak{z}(\Sigma_\chi) \simeq \mathfrak{z}\left(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]\right) = \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]$$

In Lemma 3.3.13 we will see that Σ_χ is a maximal order in D_χ.

3.3.3 Further properties of the skew power series ring

We finish this section by a collection of results about the skew power series ring $\mathcal{O}_{D_\eta}[[X;\tau,\delta]]$. As these only concern the skew power series ring, there is no need to assume Condition 2.4.1. If one assumes Condition 2.4.1, then these results carry over to (appropriate subrings of) D_χ under the isomorphism of Theorem 3.3.11.

Lemma 3.3.13. *$\mathfrak{O}$ is a maximal $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$-order in $\mathfrak{D}$.*

Proof. The statement follows from [Ven03, Corollary 2.10(iii)], which states that if $\mathcal{R}$ is a complete local ring with maximal ideal $\mathfrak{m}$, $\mathcal{A} = \mathcal{R}[[T;\sigma,\delta]]$ a ring of skew power series over $\mathcal{R}$, $\operatorname{gr}_{\mathfrak{m}} \mathcal{R}$ a noetherian maximal order in $\operatorname{Quot}(\operatorname{gr}_{\mathfrak{m}} \mathcal{R})$, and the reduction $\overline{\sigma}$ of σ to $\mathcal{R}/\mathfrak{m}$ is an automorphism, then $\mathcal{A}$ is a noetherian maximal order in $\operatorname{Quot}(\mathcal{A})$.

We check that the conditions of this Corollary are satisfied. In Venjakob's notation, $\mathcal{R} := \mathcal{O}_{D_\eta}$, $\mathfrak{m} := \pi_{D_\eta}\mathcal{O}_{D_\eta}$, $\mathcal{A} := \mathfrak{O}$. The associated graded ring $\operatorname{gr}_{\pi_{D_\eta}\mathcal{O}_{D_\eta}} \mathcal{O}_{D_\eta}$ is isomorphic to the polynomial ring $\overline{\mathcal{O}_{D_\eta}}[t]$ where $\overline{\mathcal{O}_{D_\eta}} = \mathcal{O}_{D_\eta}/\pi_{D_\eta}\mathcal{O}_{D_\eta}$ is the residue field; a priori, this should be a residue *skew* field, but seeing that it is the same as $\overline{\mathbb{Q}_p(\eta)}(\omega_\eta)$ shows that it is actually a field. Just as in the commutative case, the isomorphism is given by

$$\operatorname{gr}_{\pi_{D_\eta}\mathcal{O}_{D_\eta}} \mathcal{O}_{D_\eta} = \bigoplus_{i=0}^{\infty} \mathcal{O}_{D_\eta}\pi_{D_\eta}^i/\mathcal{O}_{D_\eta}\pi_{D_\eta}^{i+1} \xrightarrow{\sim} \overline{\mathbb{Q}_p(\eta)}(\omega_\eta)[t]$$
$$\left[\pi_{D_\eta}^i \bmod \pi_{D_\eta}^{i+1}\right] \mapsto t^i$$

The polynomial ring $\overline{\mathbb{Q}_p(\eta)}(\omega_\eta)[t]$ is noetherian, and a PID, hence a normal ring (see [Sta23, Lemma 00GZ]), and thus a maximal order in its field of fractions $\overline{\mathbb{Q}_p(\eta)}(\omega_\eta)(t)$.

Moreover, $\mathcal{O}_{D_\eta}$ is indeed complete in the $\pi_{D_\eta}\mathcal{O}_{D_\eta}$-adic topology. Finally, the map τ is an automorphism, and therefore so is its reduction to the residue field $\overline{\mathbb{Q}_p(\eta)}(\omega_\eta)[t]$. Thus the cited Corollary is indeed applicable. □

Lemma 3.3.14. *Let $\mathfrak{p} \subset \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ be a prime ideal of height 1. Then there is a unique maximal $\widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}$-order in $\widehat{\mathfrak{D}}_{\mathfrak{p}}$, where $\widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}$ is the ring obtained by first localising $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ at $\mathfrak{p}$ and then taking completion in the $\mathfrak{p}$-adic topology, and $\widehat{\mathfrak{D}}_{\mathfrak{p}} := \operatorname{Frac}(\widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}) \otimes_{\mathfrak{z}(\mathfrak{D})} \mathfrak{D}$.*

Proof. The localisation $\mathcal{O}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}$ of $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ at $\mathfrak{p}$ is a discrete valuation ring, see [Rei03, Theorem 4.25(i)]. Therefore its completion $\widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}$ is a complete Hausdorff discrete valuation ring. Then the assertion is a special case of Theorem 0.6.1. □

Remark 3.3.15. One can also give a direct proof using Weierstraß theory. Moreover, later in Lemma 5.1.1 we will see that the field of fractions of $\widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}$ is a 2-dimensional higher local field. ○

Proposition 3.3.16. *$\mathfrak{O}$ is the unique maximal $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$-order in $\mathfrak{D}$.*

Proof. Let Δ be a maximal $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$-order in $\mathfrak{D}$. Then by Propositions 0.6.4 and 0.6.5, the completed localisation $\widehat{\Delta}_{\mathfrak{p}} = \widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}} \Delta$ is a maximal $\widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}}$-order in $\widehat{\mathfrak{D}}_{\mathfrak{p}}$ for all height 1 prime ideals $\mathfrak{p} \subset \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$.

According to Lemma 3.3.13, $\mathfrak{O}$ is a maximal $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$-order in $\mathfrak{D}$. Lemma 3.3.14 shows that for all height 1 prime ideals $\mathfrak{p} \subset \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$,

$$\widehat{\Delta}_{\mathfrak{p}} = \widehat{\mathfrak{O}}_{\mathfrak{p}}$$

where $\widehat{\mathfrak{O}}_\mathfrak{p} = \widehat{\mathcal{O}}_{\mathfrak{z}(\mathfrak{D}),\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}} \mathfrak{D}$ is the completed localisation of $\mathfrak{D}$ at $\mathfrak{p}$. We conclude that for all height 1 prime ideals $\mathfrak{p} \subset \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$,

$$\Delta_\mathfrak{p} = \mathfrak{D} \cap \widehat{\Delta}_\mathfrak{p} = \mathfrak{D} \cap \widehat{\mathfrak{D}}_\mathfrak{p} = \mathfrak{D}_\mathfrak{p}$$

Consequently, we have the following chain of equalities:

$$\Delta = \bigcap_\mathfrak{p} \Delta_\mathfrak{p} = \bigcap_\mathfrak{p} \mathfrak{D}_\mathfrak{p} = \mathfrak{D}$$

The first and the last equalities are due to [Rei03, Theorem 4.25(iv)] and the fact that Δ and $\mathfrak{D}$ are reflexive by Proposition 0.6.4. □

Lemma 3.3.17. *Let $E \subseteq D_\eta$ be a maximal subfield. Then* $\mathrm{Frac}\left(\mathcal{O}_E[[(1+X)^{w_\chi/v_\chi} - 1]]\right)$ *is a maximal subfield of $\mathfrak{D}$. In particular,* $\mathrm{Frac}\left(\mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[(1+X)^{w_\chi/v_\chi} - 1]]\right)$ *resp. its image are maximal subfields.*

Remark 3.3.18. For $E = \mathbb{Q}_p(\eta)(\omega_\eta)$, this was already shown in the proof of Corollary 3.3.8. ∘

Proof of Lemma 3.3.17. The ring in question is clearly a subfield. Its dimension over the centre is $(E : \mathbb{Q}_{p,\chi}) = (E : \mathbb{Q}_p(\eta))(\mathbb{Q}_p(\eta) : \mathbb{Q}_{p,\chi}) = s_\eta \cdot \frac{w_\chi}{v_\chi} = s_\chi$. The assertion follows from [Rei03, Corollary 28.10]. □

As an aside, we state a result regarding $\mathfrak{D}$ resp. Σ_χ which shall not be used later. For this, we need to recall some notions about noncommutative rings. For us, a UFD is a noetherian domain $\mathcal{A}$ such that all height 1 primes $\mathfrak{p}$ of $\mathcal{A}$ are principal, and the quotient ring $\mathcal{A}/\mathfrak{p}$ is a domain. In the context of the following proof only, a PID will be understood to be a ring whose left and right ideals are all principal. This is in line with the terminology of [Ven03]; this definition of a PID is more relaxed than the one used in the rest of the thesis, e.g. in Theorem 0.8.3.

Lemma 3.3.19. $\mathfrak{D} = \mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]$ *is a UFD.*

Proof. The assertion follows directly from [Ven03, Corollary 7.4]. This states that if $\mathcal{R}$ is a local PID with maximal ideal generated by $\pi \in \mathcal{R}$ such that $\mathcal{R}$ is complete with respect to the π-adic topology, and if σ is an endomorphism of $\mathcal{R}$ such that its reduction to $\mathcal{R}/\pi\mathcal{R}$ is an automorphism, then $\mathcal{R}[[Y; \sigma, \delta]]$ is a UFD for all σ-derivations δ such that the multiplication law is well-defined.

Indeed, this result is applicable in our setting: the base ring $\mathcal{O}_{D_\eta}$ is a local principal ideal domain with π_{D_η} generating the maximal ideal, with respect to which $\mathcal{O}_{D_\eta}$ is complete; and τ is an automorphism of $\mathcal{O}_{D_\eta}$, thus it also induces an automorphism on $\mathcal{O}_{D_\eta}/\pi_{D_\eta}\mathcal{O}_{D_\eta}$. □

Chapter 4

The equivariant p-adic Artin conjecture

Firstly, we state equivariant versions of Greenberg's p-adic Artin conjecture (Section 4.1), which we will also refer to as an integrality conjecture. Then in Section 4.2, we exhibit some general properties of integrality. Using the description of D_χ in Chapter 3, we will then study integrality with respect to the maximal order Σ_χ: in Section 4.3 we prove a technical 'dimension reduction' result, which will then be used in Section 4.4 to establish integrality of $\zeta_S^T(L_\infty^+/K)$ for extensions for which Condition 2.4.1 holds. Sections 4.5 and 4.6 are devoted to the study of special classes of extensions.

4.1 Conjectures

In this section, we work in the setup of Section 1.2, except for Conjecture 4.1.6 and Lemma 4.1.7, which refer to ζ_S^T and thus require the more restrictive assumptions of Section 1.1.

As we recalled in Section 1.2, p-adic Artin L-functions attached to an arbitrary $\mathbb{Q}_p^c$-valued Artin character were introduced by Greenberg in [Gre83]. In that paper, Greenberg proposed the following p-adic version of the Artin conjecture for $S = S_p$, an integrality-type statement for p-adic Artin L-functions:

Theorem 4.1.1 (p-adic Artin conjecture)**.** *For a character $\chi : \mathcal{G} \to \mathbb{Q}_p^c$ with open kernel,*

$$G_{\chi,S}(T) \in \mathbb{Z}_p[[T]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^c$$

where $G_{\chi,S} \in \mathrm{Quot}(\mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]])$ is the object appearing in the numerator of the expression for the p-adic Artin L-function in (1.2)*.*

In [Gre83, Proposition 5], Greenberg showed that the appropriate characterwise main conjecture implies the p-adic Artin conjecture. Since the main conjecture is known due to the work of Wiles, so is the p-adic Artin conjecture, see [Wil90, Theorems 1.1 and 1.2].

Greenberg also considered the following stronger version, and proved that it follows from the main conjecture under the additional assumption of the vanishing of Iwasawa's μ-invariant, see [Gre83, p. 87]. The statement was proven unconditionally by Ritter and Weiss in [RW04, Remark (G)].

Theorem 4.1.2. *For a character $\chi : \mathcal{G} \to \mathbb{Q}_p^{\mathrm{c}}$ with open kernel,*

$$G_{\chi,S}(T) \in \mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]]$$

In analogy with these conjectures, we posit the following equivariant version of the p-adic Artin conjecture in the setup of Section 1.2.

Conjecture 4.1.3 (equivariant p-adic Artin conjecture)**.** *Let $\mathfrak{M}$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Then the smoothed equivariant p-adic Artin L-function Φ_S^T is in the image of the composite map*

$$\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times \tag{4.1}$$

where the first arrow is the natural map sending an invertible element to the class of the 1×1 matrix consisting of said element.

Remark 4.1.4. The smaller $\mathfrak{M}$ is, the stronger the statement of Conjecture 4.1.3 becomes. In particular, it is at its strongest resp. weakest when $\mathfrak{M}$ is $\Lambda(\mathcal{G})$ resp. a maximal order. ∘

Remark 4.1.5. Let $\mathfrak{M}$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Assuming Conjecture 4.1.3 for this $\mathfrak{M}$, we obtain that $\Phi_S^T \in \mathfrak{z}(\mathfrak{M})$ because of Proposition 0.8.1. In the case when $\mathfrak{M}$ is a maximal order, this conclusion is made plausible by the following unconditional argument.

Assume that $\mathfrak{M}$ is a maximal order. Let $\alpha \in M_m(\Lambda(\mathcal{G}))$ be as in (1.10). On the one hand, since $M_m(\Lambda(\mathcal{G})) \subseteq M_m(\mathfrak{M})$, and the latter is a maximal $\Lambda(\mathcal{G})$-order in $M_m(\mathcal{Q}(\mathcal{G}))$ by Proposition 0.6.2, the reduced norm $\mathrm{nr}(\alpha)$ is contained in $\mathfrak{M}$ by Proposition 0.8.1. On the other hand, $\partial([(\alpha)]) = [Y_S^T]$, and so Corollary 1.4.8 is applicable: $\mathrm{nr}(\alpha) \cdot (\Phi_S^T)^{-1} \in \mathfrak{z}(\mathfrak{M})^\times$. It follows that $\Phi_S^T \in \mathfrak{M}$ holds unconditionally. ∘

We also formulate an integrality conjecture regarding the element ζ_S^T, in the setup of Section 1.1.

Conjecture 4.1.6. *Assume Conjecture 1.4.4 without assuming uniqueness; let $\zeta_S^T \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\mathrm{nr}\, \zeta_S^T = \Phi_S^T$ and $\partial(\zeta_S^T) = -[Y_S^T]$. Let $\mathfrak{M}$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Then ζ_S^T is in the image of the natural map*

$$\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))$$

Lemma 4.1.7. *Assume Conjecture 1.4.4 including uniqueness of ζ_S^T. Then Conjecture 4.1.3 is equivalent to Conjecture 4.1.6 for the same order $\mathfrak{M}$.*

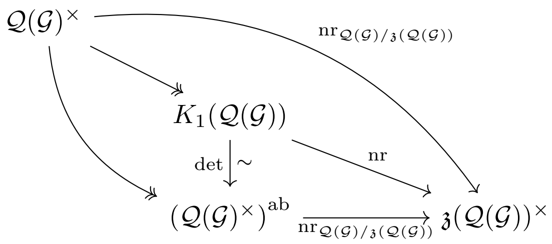
Proof. Let ζ_S^T be a preimage of Φ_S^T. If this is in the image of the natural map, then $\mathrm{nr}(\zeta_S^T) = \Phi_S^T$ is in the image of the composite map (4.1), so Conjecture 4.1.6 implies Conjecture 4.1.3.

For the converse, let ζ_S^T be the unique preimage of Φ_S^T under the reduced norm map, and let $x \in \mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times$ be a preimage of Φ_S^T under the map (4.1). Then the class $[(x)] \in K_1(\mathcal{Q}(\mathcal{G}))$ is also a preimage of Φ_S^T under the reduced norm, therefore $[(x)] = \zeta_S^T$ by uniqueness of ζ_S^T. □

Remark 4.1.8. One can also consider the weakening of the above conjectures without involving an order $\mathfrak{M}$. That is, one may ask whether Φ_S^T comes from a 1×1 matrix over $\mathcal{Q}(\mathcal{G})$, or more explicitly, whether it is in the image of the map

$$\mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times \tag{4.2}$$

This is not known in general. However, it is true that Φ_S^T is in the image of (4.2) if and only if it is in the image of $\mathrm{nr} : K_1(\mathcal{Q}(\mathcal{G})) \to \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$. Indeed, by (0.18) there is a commutative diagram



The vertical arrow is an isomorphism by (0.17). Since the map $\mathcal{Q}(\mathcal{G})^\times \twoheadrightarrow (\mathcal{Q}(\mathcal{G})^\times)^{\mathrm{ab}}$ is surjective, so is the natural map $\mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))$ by commutativity of the left triangle. The claimed equivalence follows. In particular, Φ_S^T is in the image of (4.2) if Conjecture 1.4.4 holds. ◦

4.2 Generalities

We note some generalities on the behaviour of Conjecture 4.1.3. Using the same methods as in the proof of Lemma 4.1.7, these can be reformulated for ζ_S^T under the assumption of Conjecture 1.4.4 (without uniqueness).

4.2.1 Expanding S and T

In this section, we describe how integrality behaves under adding more places to either S or T. We will work with a fixed extension L_∞^+/K, which we suppress from the notation for the sake of brevity, and simply write $\Phi_S^T := \Phi_S^T(L_\infty^+/K)$.

Lemma 4.2.1. *Let $\mathfrak{M} \subset \mathcal{Q}(\mathcal{G})$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Let S and T satisfy the hypotheses above, and let v^* be a place of K not in $S \cup T$. If Φ_S^T is in the image of the map* (4.1) *then so is $\Phi_S^{T\cup\{v^*\}}$.*

Proof. By Definition 1.2.7, the relationship between the two smoothed equivariant p-adic L-functions is $\Phi_S^{T\cup\{v^*\}} = \Phi_S^T \cdot \mathrm{nr}\left(1 - \varphi_{w_\infty^*}\right)$ where w_∞^* is the (previously) fixed place of L_∞^+ above v^*. The first term is in the image of the map by assumption, while the second one is the reduced norm of the class of the map given by left multiplication by the element $\left(1 - \varphi_{w_\infty^*}\right) \in \Lambda(\mathcal{G})$. This element is invertible in $\mathcal{Q}(\mathcal{G})$, and $\Lambda(\mathcal{G}) \subseteq \mathfrak{M}$, hence the claim. ☐

Lemma 4.2.2. *Retain the assumptions of Lemma 4.2.1, and assume Conjecture 1.4.4 without uniqueness. Then if Φ_S^T is in the image of the map* (4.1) *then so is $\Phi_{S\cup\{v^*\}}^T$.*

Proof. Let $L_{\infty,S}^+(p)^{\mathrm{ab}}$ be the maximal S-ramified abelian p-extension of L_∞^+ and analogously $L_{\infty,S\cup\{v^*\}}^+(p)^{\mathrm{ab}}$ the maximal $S \cup \{v^*\}$-ramified abelian p-extension of L_∞^+. Then

$$
\begin{aligned}
X_S^+ &= \mathrm{Gal}\left(L_{\infty,S}^+(p)^{\mathrm{ab}}/L_\infty^+\right) \\
X_{S\cup\{v^*\}}^+ &= \mathrm{Gal}\left(L_{\infty,S\cup\{v^*\}}^+(p)^{\mathrm{ab}}/L_\infty^+\right) \\
X_S^{+,v^*} &:= \mathrm{Gal}\left(L_{\infty,S\cup\{v^*\}}^+(p)^{\mathrm{ab}}/L_{\infty,S}^+(p)^{\mathrm{ab}}\right) \simeq \begin{cases} 0 & \text{if } q^* \not\equiv 1 \pmod p \\ \mathrm{ind}_{\mathcal{G}_{w_\infty}}^{\mathcal{G}} \mathbb{Z}_p(1) & \text{if } q^* \equiv 1 \pmod p \end{cases}
\end{aligned}
$$

Here q^* denotes the residue field order of v^*. The last isomorphism is [RW02, (4.∗)]. Alternatively, this isomorphism can also be deduced from the first exact sequence in [NSW20, Theorem 11.3.5] and the snake lemma; the theorem is applicable because the weak Leopoldt conjecture is known to be true for cyclotomic $\mathbb{Z}_p$-extensions, and the kernel in the theorem is what is needed for the desired isomorphism due to the last part of the first remark on [NSW20, p. 648].

In the case $q^* \not\equiv 1 \pmod p$, we thus have $X_S^+ = X_{S\cup\{v^*\}}^+$, and so (1.8) shows that $Y_S^T = Y_{S\cup\{v^*\}}^T$. Using the localisation exact sequence (0.11) and Conjecture 1.4.4, we find that ζ_S^T and $\zeta_{S\cup\{v^*\}}^T$ differ by an element in $K_1(\Lambda(\mathcal{G}))$. As in (0.17), there is an isomorphism $K_1(\Lambda(\mathcal{G})) \simeq (\Lambda(\mathcal{G})^\times)^{\mathrm{ab}}$ given by the Dieudonné determinant. In particular, there is a representative for the aforementioned element of $K_1(\Lambda(\mathcal{G}))$ in $\Lambda(\mathcal{G})^\times$ and thus in $\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times$.

It remains to treat the case $q^* \equiv 1 \pmod p$. Using (1.8), we find that the following holds in the relative K_0-group $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$:

$$\begin{aligned}
[Y_{S\cup\{v^*\}}^T] - [Y_S^T] &= \left[\mathrm{ind}_{\mathcal{G}_{w_\infty^*}}^{\mathcal{G}} \mathbb{Z}_p(1)\right] \\
&= \Big[\underset{-2}{\Lambda(\mathcal{G})} \xrightarrow{1-\varphi_{w_\infty^*}} \underset{-1}{\Lambda(\mathcal{G})}\Big] \\
&= \partial\left(1 - \varphi_{w_\infty^*}\right)
\end{aligned}$$

where the second step is (1.12) and the last one is (0.12). Therefore by the 'moreover' part of Section 1.4, we deduce that ζ_S^T and $\zeta_{S\cup\{v^*\}}^T$ differ by the product of an element in $K_1(\Lambda(\mathcal{G}))$ and the class $\left[1 - \varphi_{w_\infty^*}\right]$. The former can be dealt with as in the $q^* \not\equiv 1 \pmod p$ case. As for the latter, we have seen in the proof of Lemma 4.2.1 that it is in the image of the natural map.

The proof is finished by taking reduced norms. □

4.2.2 Functoriality: changing the top field

Consider the setup for changing the top field in Section 1.2.2. Recall that this means that $L/L'/K$ is a tower of fields such that L'/K satisfies the same conditions as L/K, so there is a natural projection $\pi : \mathcal{G} \twoheadrightarrow \mathcal{G}'$ which induces $\mathcal{Q}(\mathcal{G}) \twoheadrightarrow \mathcal{Q}(\mathcal{G}')$ and $K_1(\pi) : K_1(\mathcal{Q}(\mathcal{G})) \to K_1(\mathcal{Q}(\mathcal{G}'))$. Let $\mathfrak{M}$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$, and let $\mathfrak{M}' := \pi(\mathfrak{M})$. Then by surjectivity of the natural maps, $\mathfrak{M}'$ is a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G}')$ containing $\Lambda(\mathcal{G}')$. Write $\Phi_S^T := \Phi_S^T(L_\infty^+/K)$ and $\Phi_S'^T := \Phi_S^T(L_\infty'^+/K)$ for brevity.

Proposition 4.2.3. *Assume Conjecture 1.4.4. If Φ_S^T is in the image of the composite map*

$$\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$$

then $\Phi_S'^T$ is in the image of the composite map

$$\mathfrak{M}' \cap \mathcal{Q}(\mathcal{G}')^\times \to K_1(\mathcal{Q}(\mathcal{G}')) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}'))^\times$$

Proof. The maps in the statement fit into a commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times & \longrightarrow & K_1(\mathcal{Q}(\mathcal{G})) & \xrightarrow{\mathrm{nr}} & \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle K_1(\pi)} & & \downarrow{\scriptstyle \pi} \\
\mathfrak{M}' \cap \mathcal{Q}(\mathcal{G}')^\times & \longrightarrow & K_1(\mathcal{Q}(\mathcal{G}')) & \xrightarrow{\mathrm{nr}} & \mathfrak{z}(\mathcal{Q}(\mathcal{G}'))^\times
\end{array}$$

By Lemma 1.2.9, we have $\pi(\Phi_S^T) = \Phi_S'^T$. Hence if Φ_S^T is the image of some $x \in \mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times$ under the top row, then $\Phi_S'^T$ is the image of $\pi(x)$ under the bottom row. □

4.3 Dimension reduction for integral matrices over skew power series rings

In this section, we study matrices over the skew field $\mathfrak{D} = \mathrm{Quot}(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]])$. The main result is Proposition 4.3.1, which allows one to do dimension reduction: by this we mean that an invertible matrix with entries in a maximal order can be replaced by a smaller matrix with the same Dieudonné determinant. This is a generalisation of a result of Nichifor and Palvannan, see [NP19, Proposition 2.13].

In that paper, the skew fields come from the Wedderburn decomposition of an Iwasawa algebra over a direct product $\mathcal{G} = H \times \Gamma$, whereas we allow $\mathcal{G}$ to be a semidirect product. The difference between these two setups is substantial. In the direct product case, one has $\Lambda(\mathcal{G}) = \Lambda(\Gamma)[H]$, whence the Wedderburn components D_χ of $\mathcal{Q}(\mathcal{G})$ can be obtained from the Wedderburn components D_η of $\mathbb{Q}_p[H]$ by tensoring with $\mathcal{Q}(\Gamma)$. In this case, the skew field D_χ has a maximal order isomorphic to the power series ring $\mathcal{O}_{D_\eta}[[X]]$. In the semidirect product case, the description of the Wedderburn components becomes a more laborious endeavour: indeed, this is the entire goal of our Chapters 2 and 3. Assuming the validity of Condition 2.4.1, D_χ admits a maximal order isomorphic to the skew power series ring $\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]$ (see Lemma 3.3.13).

The contents of this section, are purely about skew power series rings. We will use the results of Sections 3.3.1 and 3.3.3; in particular, we do not assume Condition 2.4.1. The outline is the same as in [NP19, §2.2].

Recall the following notation from Section 3.3:

$$\begin{aligned} \mathfrak{O} &= \mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]] \\ \mathfrak{D} &= \mathrm{Quot}(\mathfrak{O}) \\ \mathcal{O}_{\mathfrak{E}} &= \mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[(1+X)^{w_\chi/v_\chi}-1]] \\ \mathfrak{E} &= \mathrm{Frac}\,\mathcal{O}_{\mathfrak{E}} \end{aligned}$$

Recall that $\mathfrak{D}$ is a skew field, see the proof of Corollary 3.3.8. Under the assumption of Condition 2.4.1, $\mathfrak{D}$ is isomorphic to D_χ. According to Corollary 3.3.8, its centre is

$$\mathfrak{z}(\mathfrak{D}) = \mathrm{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]])$$

Finally, recall the notation

$$\mathcal{O}_{\mathfrak{z}(\mathfrak{D})} = \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]$$

The field $\mathfrak{E}$ is a maximal (self-centralising) subfield in $\mathfrak{D}$, see Lemma 3.3.17. Recall that $\mathfrak{O}$ is a maximal order, see Lemma 3.3.13.

Proposition 4.3.1. *Let $m \geq 1$, $n \geq 1$, and*

$$A \in M_m(M_n(\mathfrak{O})) \cap \mathrm{GL}_m(M_n(\mathfrak{D}))$$

Then there exists some $C \in M_n(\mathfrak{O}) \cap \mathrm{GL}_n(\mathfrak{D})$ such that $\det A = \det C$ where $\det$ is the Dieudonné determinant.

Remark 4.3.2. An equivalent formulation of Proposition 4.3.1 would be to state it for $n = 1$ instead of for all $n \geq 1$. In fact, the proof goes by showing it for $n = 1$, and then creating an $n \times n$ matrix by taking this as the $(1,1)$-entry, filling up the rest of the diagonal with ones, and setting all off-diagonal entries to be zero. We stated the Proposition this way because this is the form in which it will be needed in the proof of Theorem 4.4.1. ◦

Proof. We begin with the following observation:

Claim 4.3.3. *The reduced norm of A is integral:*

$$\mathrm{nr}_{M_{mn}(\mathfrak{D})/\mathfrak{z}(\mathfrak{D})} A \in \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$$

Proof. Actually, even more is true: the reduced characteristic polynomial of A has coefficients in $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$. This follows directly by applying Proposition 0.8.1 with $\mathscr{R} := \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$, $\mathscr{K} = \mathfrak{z}(\mathfrak{D})$, $\mathscr{A} := M_{mn}(\mathfrak{D})$, and $\Delta := M_{mn}(\mathfrak{O})$. These satisfy the conditions of Proposition 0.8.1: indeed, Lemma 3.3.13 states that $\mathfrak{O}$ is a maximal $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$-order in $\mathfrak{D}$, and consequently $M_{mn}(\mathfrak{O})$ is a maximal $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$-order in $M_{mn}(\mathfrak{D})$ by Proposition 0.6.2. □

Recall that we call a ring a PID if all left ideals and all right ideals are principal.

Claim 4.3.4. $\mathfrak{O}\left[\pi_{D_\eta}^{-1}\right]$ *is a (noncommutative) PID.*

Proof of Claim 4.3.4. We will show that all left ideals of $\mathfrak{O}[\pi_{D_\eta}^{-1}]$ are principal; the right version can be shown in the same way, as every statement we will use has a right counterpart. The proof uses Weierstraß theory of skew power series rings, for which we refer back to Section 0.4.

Let $I \subseteq \mathfrak{O}[\pi_{D_\eta}^{-1}]$ be a nontrivial left ideal. Then $I \cap \mathfrak{O} \neq (0)$: indeed, if $0 \neq x \in I$, then there is an $a \gg 0$ such that $\pi_{D_\eta}^a x \in I \cap \mathfrak{O}$. Therefore not all reduced orders are ∞; let $f \in I \cap \mathfrak{O}$ be an element of minimal reduced order. Choosing f so that it is in $\mathfrak{O}$ is necessary in order to be able to apply the Weierstraß preparation theorem (Theorem 0.4.2). This allows us to write f uniquely as

$$f = \varepsilon F$$

where $\varepsilon \in \mathfrak{O}^\times$ is a unit and $F \in \mathfrak{O}$ is a distinguished skew polynomial. It follows that $F \in I$, and $\mathrm{ord}^{\mathrm{red}}(F) = \mathrm{ord}^{\mathrm{red}}(f)$. No such theorem is available in $\mathfrak{O}[\pi_{D_\eta}^{-1}]$.

Clearly $\mathfrak{O}[\pi_{D_\eta}^{-1}]F \subseteq I$. We will show that the converse inclusion also holds, whence I is the principal left ideal generated by F. For this, let $g \in I$ be arbitrary. Weierstraß division (Theorem 0.4.1) carries over from $\mathfrak{O}$ to $\mathfrak{O}[\pi_{D_\eta}^{-1}]$ as follows. The Weierstraß division theorem for $\mathfrak{O}$ states that there is a decomposition

$$\mathfrak{O} = \mathfrak{O}F \oplus \bigoplus_{i=0}^{\mathrm{ord}^{\mathrm{red}}(F)-1} \mathcal{O}_{D_\eta} X^i$$

Inverting π_{D_η}, or in other words, tensoring with $\mathcal{O}_{D_\eta}[\pi_{D_\eta}^{-1}]$ over $\mathcal{O}_{D_\eta}$ then yields

$$\mathfrak{O}\left[\pi_{D_\eta}^{-1}\right] = \mathfrak{O}\left[\pi_{D_\eta}^{-1}\right] F \oplus \bigoplus_{i=0}^{\mathrm{ord}^{\mathrm{red}}(F)-1} \mathcal{O}_{D_\eta}\left[\pi_{D_\eta}^{-1}\right] X^i$$

With respect to this decomposition, g can be written as

$$g = hF + r$$

Since g and F are in I, we have $r \in I$, but a priori not necessarily in $\mathfrak{O}$.

We claim that there are no π_{D_η}-factors in denominators of coefficients of r. ↯ Suppose there are: then, since there are only finitely many coefficients, there is a unique $a \geq 0$ such that $\pi_{D_\eta}^a r \in \mathcal{O}_{D_\eta}[X]$ and $\pi_{D_\eta}^a r \notin \pi_{D_\eta}\mathcal{O}_{D_\eta}[X]$, i.e. a is minimal such that $\pi_{D_\eta}^a r$ has no π_{D_η}-factors in its denominators. Thus

$$\mathrm{ord}^{\mathrm{red}}\left(\pi_{D_\eta}^a r\right) \leq \deg\left(\pi_{D_\eta}^a r\right) = \deg(r) < \mathrm{ord}^{\mathrm{red}}(F)$$

But since π_{D_η} is a unit in $\mathfrak{O}[\pi_{D_\eta}^{-1}]$, we have $\pi_{D_\eta}^a r \in I$, which contradicts the minimality condition in our choice of f. ↯

Thus $r \in \mathfrak{O}$. Now r is a polynomial of degree at most $\mathrm{ord}^{\mathrm{red}}(F) - 1$, hence $\mathrm{ord}^{\mathrm{red}}(r) < \mathrm{ord}^{\mathrm{red}}(F)$ or $\mathrm{ord}^{\mathrm{red}}(r) = \infty$. The former is impossible by our choice of f. Thus $\mathrm{ord}^{\mathrm{red}}(r) = \infty$, that is, all coefficients must be non-units, so $r \in \bigoplus_{i=0}^{\mathrm{ord}^{\mathrm{red}}(F)-1} \pi_{D_\eta} \mathcal{O}_{D_\eta} X^i$. We claim that $r = 0$. ↯ If not, then we can do the same as in the previous step in reverse: there is a unique $a \geq 0$ such that

$$\pi_{D_\eta}^{-a} r \in \bigoplus_{i=0}^{\mathrm{ord}^{\mathrm{red}}(F)-1} \mathcal{O}_{D_\eta} X^i - \bigoplus_{i=0}^{\mathrm{ord}^{\mathrm{red}}(F)-1} \pi_{D_\eta} \mathcal{O}_{D_\eta} X^i,$$

and thus $\mathrm{ord}^{\mathrm{red}}(\pi_{D_\eta}^{-a} r) < \mathrm{ord}^{\mathrm{red}}(F)$, contradiction. ↯

Hence $r = 0$, so $g = hF$, that is, $g \in \mathfrak{O}[\pi_{D_\eta}^{-1}]F$ as claimed. □

Claim 4.3.4 allows us to utilise Theorem 0.8.3, which tells us that A admits a diagonal reduction via elementary operations. That is, $A = UBV$ where $B \in M_{mn}(\mathfrak{O}[\pi_{D_\eta}^{-1}])$ is a diagonal matrix, and $U, V \in \mathrm{GL}_{mn}(\mathfrak{O}[\pi_{D_\eta}^{-1}])$ are products of elementary, permutation, and scalar matrices.

We want to use Weierstraß preparation again: for this, we need to clear denominators. Let $b \in \mathbb{Z}$ be the minimal integer such that $\pi_{D_\eta}^b B \in M_{mn}(\mathfrak{O})$. Then

$$\det(B) = \det(\pi_{D_\eta}^{-b} \cdot \pi_{D_\eta}^b B) = \pi_{D_\eta}^{-b} \det(\pi_{D_\eta}^b B) = \pi_{D_\eta}^{-b} \beta_B J_B$$

where $\beta_B \in \mathfrak{O}^\times$ is a unit and $J_B \in \mathfrak{O}$ is a monic polynomial. Similarly, let $u, v \in \mathbb{Z}$ be the minimal integers such that $\pi_{D_\eta}^u U, \pi_{D_\eta}^v V \in M_{mn}(\mathfrak{O})$. Then there exist units $\beta_U, \beta_V \in \mathfrak{O}^\times$ and monic polynomials $J_U, J_V \in \mathfrak{O}$ such that

$$\det(U) = \pi_{D_\eta}^{-u} \beta_U J_U, \quad \det(V) = \pi_{D_\eta}^{-v} \beta_V J_V \tag{4.3}$$

We claim that $J_U = J_V = 1$. For this, we go back to the fact that U and V are products of elementary, permutation and scalar matrices. Elementary matrices have Dieudonné determinant 1 by definition. A permutation matrix M satisfies $M^2 = 1$ and hence $\det(M)^2 = 1$; in particular, $\det(M)$ is invertible in $\mathfrak{O}[\pi_{D_\eta}^{-1}]$. So the elementary and permutation factors only contribute β-factors in (4.3). Moreover, since U and V are invertible over $\mathfrak{O}[\pi_{D_\eta}^{-1}]$, their scalar matrix parts may only contribute units in $\mathfrak{O}[\pi_{D_\eta}^{-1}]$, which can be written as a power of π_{D_η} times a unit in $\mathfrak{O}$: that is, these are accounted for in the π_{D_η}- and β-factors in (4.3). We conclude that $J_U = J_V = 1$, as claimed.

Since the Dieudonné determinant factors through the abelian group $K_1(\mathfrak{D})$, we have

$$\det(A) = \det(U)\det(B)\det(V) = (\beta_U \beta_B \beta_V) \cdot J_B \cdot \pi_{D_\eta}^{-u-b-v} = \det(C) \tag{4.4}$$

where $C = \mathrm{diag}((\beta_U \beta_B \beta_V) \cdot J_B \cdot \pi_{D_\eta}^{-u-b-v}, 1, \ldots, 1)$ is an $n \times n$ matrix. In order to prove $C \in M_n(\mathfrak{O}) \cap \mathrm{GL}_n(\mathfrak{D})$, it remains to show that $u + b + v \leq 0$.

In $\mathfrak{z}(\mathfrak{D})$, we have the following equality:

$$\underbrace{\mathrm{nr}_{M_{mn}(\mathfrak{O})/\mathfrak{z}(\mathfrak{O})}(A)}_{\in \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}} = \mathrm{nr}_{\mathfrak{O}/\mathfrak{z}(\mathfrak{O})}(\det(A)) \tag{4.5}$$

$$= \underbrace{\mathrm{nr}_{\mathfrak{O}/\mathfrak{z}(\mathfrak{O})}(\beta_U \beta_B \beta_V)}_{\in \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}^\times} \cdot \underbrace{\mathrm{nr}_{\mathfrak{O}/\mathfrak{z}(\mathfrak{O})}(J_B)}_{\text{monic}} \cdot \mathrm{nr}_{\mathfrak{O}/\mathfrak{z}(\mathfrak{O})}\left(\pi_{D_\eta}^{-(u+b+v)}\right) \tag{4.6}$$

The first equality comes from the relationship between reduced norms and the Dieudonné determinant: for invertible matrices, $\mathrm{nr}_{M_{mn}(\mathfrak{D})/\mathfrak{z}(\mathfrak{D})} = \mathrm{nr}_{\mathfrak{D}/\mathfrak{z}(\mathfrak{D})} \circ \det$, see (0.18). Here we use the assumption that A is invertible. The second equality is (4.4).

On the left hand side of (4.5), $\mathrm{nr}_{M_{mn}(\mathfrak{D})/\mathfrak{z}(\mathfrak{D})}(A)$ is integral, i.e. contained in $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ by Claim 4.3.3. On the right hand side of (4.6), the first term is a unit because $\beta_U\beta_B\beta_V \in \mathfrak{O}^\times$. The second term is a monic polynomial by the upcoming Lemma 4.3.5 because J_B is monic.

Thus neither the first nor the second term is divisible by π_{D_η}, so in order for the right hand side to be integral, the third term must also be integral. The valuation of $\mathrm{nr}_{\mathfrak{D}/\mathfrak{z}(\mathfrak{D})}(\pi_{D_\eta}^{-(u+b+v)})$ is a multiple of $-(u+b+v)$. We conclude that $u+b+v \leq 0$. This completes the proof. □

Lemma 4.3.5. *Let $G(X) \in \mathfrak{D} = \mathrm{Quot}(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]])$ be an element which is a monic polynomial in X. Then its reduced norm $\mathrm{nr}_{\mathfrak{D}/\mathfrak{z}(\mathfrak{D})}(G)$ in $\mathfrak{z}(\mathfrak{D}) = \mathrm{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]])$ is also a monic polynomial in $(1+X)^{w_\chi/v_\chi}$. In particular, it is also a monic polynomial in X.*

Proof. Once again we use the embedding Φ from (3.15). By definition of the reduced norm, we have

$$\mathrm{nr}_{\mathfrak{D}/\mathfrak{z}(\mathfrak{D})}(G) = \det(\Phi(G))$$

We compute the leading term of this determinant. For this, write $G = \sum_{i=0}^{\deg G} \alpha_i \cdot (1+X)^i$ where $\alpha_i \in D_\eta$, $\deg G$ is the degree of G, and $\alpha_{\deg G} = 1$ as G was assumed to be monic.

$$\begin{aligned}
\Phi(G) &= \Phi\left(\sum_{i=0}^{\deg G} \alpha_i(1+X)^i\right) \\
&= \sum_{i=0}^{\deg G} \Phi\left(\alpha_i(1+X)^i\right) \\
&= \sum_{i=0}^{\deg G} \Phi\left(\alpha_i\right)\cdot\Phi\left((1+X)^i\right) \\
&= \sum_{i=0}^{\deg G} \begin{pmatrix} \varphi(\alpha_i) & & & \\ & \tau(\varphi(\alpha_i)) & & \\ & & \ddots & \\ & & & \tau^{\frac{w_\chi}{v_\chi}-1}(\varphi(\alpha_i)) \end{pmatrix} \begin{pmatrix} & \mathbf{1}_{s_\eta} & & \\ & & \ddots & \\ & & & \mathbf{1}_{s_\eta} \\ (1+X)^{\frac{w_\chi}{v_\chi}}\mathbf{1}_{s_\eta} & & & \end{pmatrix}^i \\
&= \sum_{i=0}^{\deg G} \begin{pmatrix} \varphi(\alpha_i) & & & \\ & \tau(\varphi(\alpha_i)) & & \\ & & \ddots & \\ & & & \tau^{\frac{w_\chi}{v_\chi}-1}(\varphi(\alpha_i)) \end{pmatrix} \begin{pmatrix} & & \mathbf{1}_{s_\eta} & & \\ & & & \ddots & \\ & & & & \mathbf{1}_{s_\eta} \\ (1+X)^{\frac{w_\chi}{v_\chi}}\mathbf{1}_{s_\eta} & & & & \\ & \ddots & & & \\ & & (1+X)^{\frac{w_\chi}{v_\chi}}\mathbf{1}_{s_\eta} & & \end{pmatrix} (1+X)^{\frac{w_\chi}{v_\chi}j}
\end{aligned}$$

Here $i = \frac{w_\chi}{v_\chi} j + k$ and $0 \leq k < \frac{w_\chi}{v_\chi}$. In the second block matrix inside the summation, the first $\frac{w_\chi}{v_\chi} - k$ rows have entries $\mathbf{0}_{s_\eta}$ and $\mathbf{1}_{s_\eta}$, and the last k rows have entries $\mathbf{0}_{s_\eta}$ and $(1+X)^{\frac{w_\chi}{v_\chi}} \mathbf{1}_{s_\eta}$

$$= \sum_{i=0}^{\deg G} \begin{pmatrix} & & & \varphi(\alpha_i) & & \\ & & & & \ddots & \\ & & & & & \tau^{i-1}(\varphi(\alpha_i)) \\ \tau^i(\varphi(\alpha_i)) \cdot (1+X)^{\frac{w_\chi}{v_\chi}} & & & & & \\ & \ddots & & & & \\ & & \tau^{\frac{w_\chi}{v_\chi}-1}(\varphi(\alpha_i)) \cdot (1+X)^{\frac{w_\chi}{v_\chi}} & & & \end{pmatrix} (1+X)^{\frac{w_\chi}{v_\chi} j}$$

The determinant of $\Phi(G)$ may be computed using Laplace expansion: we find that it is a polynomial in $(1+X)^{w_\chi/v_\chi}$. Since $\alpha_{\deg G} = 1$, we have $\varphi(\alpha_{\deg G}) = \mathbf{1}_{s_\eta}$. Therefore there is a unique highest degree term with respect to the variable $(1+X)^{w_\chi/v_\chi}$ in the expansion of $\det(\Phi(G))$, namely

$$\left((1+X)^{w_\chi/v_\chi}\right)^{s_\eta \cdot \frac{w_\chi}{v_\chi} j} \cdot \left((1+X)^{w_\chi/v_\chi}\right)^{s_\eta \cdot k} = \left((1+X)^{w_\chi/v_\chi}\right)^{s_\eta \cdot \deg G}$$

where $\deg G = \frac{w_\chi}{v_\chi} j + k$ and $0 \leq k < \frac{w_\chi}{v_\chi}$. Hence the determinant is monic with respect to $(1+X)^{w_\chi/v_\chi}$. Since $(1+X)^{w_\chi/v_\chi}$ is a monic polynomial in X, the determinant is also a monic polynomial in X. □

4.4 Integrality of smoothed equivariant p-adic Artin L-functions

We arrive at our main result concerning integrality of smoothed equivariant p-adic Artin L-functions. Recall that $\mathcal{Q}(\mathcal{G})$ has Wedderburn decomposition

$$\mathcal{Q}(\mathcal{G}) \simeq \prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(D_\chi) \tag{4.7}$$

and that Σ_χ is the preimage of $\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]]$ under the isomorphism in Theorem 3.3.11.

Theorem 4.4.1. *Assume that Condition 2.4.1 holds for all irreducible characters χ of $\mathcal{G}$ with open kernel and every irreducible constituent $\eta \mid \mathrm{res}^{\mathcal{G}}_H \chi$.*

(i) Let $\mathfrak{M}(\mathcal{G})$ be the preimage of

$$\prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(\Sigma_\chi)$$

under the Wedderburn isomorphism (4.7). This is a maximal $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$.

(ii) Assume Conjecture 1.4.4. Then ζ_S^T is in the image of the natural map $\mathfrak{M}(\mathcal{G}) \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))$. In particular, Conjecture 4.1.6 holds with $\mathfrak{M} = \mathfrak{M}(\mathcal{G})$.

Remark 4.4.2. This can be seen as (a special case of) an equivariant version of Greenberg's result that the main conjecture implies the p-adic Artin conjecture (Theorem 4.1.1). ◦

Proof. Nichifor and Palvannan's proof in the direct product case, see [NP19, Theorem 1], can be modified to work in our setup, with their Proposition 2.13 replaced by our Proposition 4.3.1. We now explain this in more detail.

First we prove (i). According to Lemma 3.3.13, for each χ we have that $\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]$ is a maximal $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[(1+X)^{w_\chi/v_\chi}-1]]$-order in the skew field $\mathrm{Quot}(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]])$. Using the assumed validity of Condition 2.4.1, the identification in Theorem 3.3.11 then shows that Σ_χ is a maximal $\Lambda^{\mathcal{O}_{\mathbb{Q}_{p,\chi}}}((\Gamma''_\eta)^{w_\chi/v_\chi})$-order in D_χ. Therefore for all χ, the matrix ring $M_{n_\chi}(\Sigma_\chi)$ is a maximal $\Lambda^{\mathcal{O}_{\mathbb{Q}_{p,\chi}}}((\Gamma''_\eta)^{w_\chi/v_\chi})$-order in $M_{n_\chi}(D_\chi)$ by Proposition 0.6.2. Since $\Lambda^{\mathcal{O}_{\mathbb{Q}_{p,\chi}}}((\Gamma''_\eta)^{w_\chi/v_\chi})$ is a finitely generated $\Lambda(\Gamma_0)$-module, it follows that $M_{n_\chi}(\Sigma_\chi)$ is also a maximal $\Lambda(\Gamma_0)$-order in $M_{n_\chi}(D_\chi)$. Finally, $\mathfrak{M}(\mathcal{G})$ is a maximal order in $\mathcal{Q}(\mathcal{G})$ by Proposition 0.6.3, which establishes (i).

For the proof of (ii), recall from (1.10) that there is a free resolution $\Lambda(\mathcal{G})^m \overset{\alpha}{\hookrightarrow} \Lambda(\mathcal{G})^m \twoheadrightarrow Y_S^T$ where

$$\alpha \in M_m(\Lambda(\mathcal{G})) \cap \mathrm{GL}_m(\mathcal{Q}(\mathcal{G}))$$

In particular, α defines a class $[\alpha] \in K_1(\mathcal{Q}(\mathcal{G}))$. Conjecture 1.4.4, the validity of which has been assumed, provides the following equality in $K_1(\mathcal{Q}(\mathcal{G}))$:

$$\partial\left(\zeta_S^T\right) = \partial([\alpha])$$

Recall the localisation exact sequence of K-theory from (0.13):

$$K_1(\Lambda(\mathcal{G})) \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\partial} K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$$

Using this exact sequence, there exist some $n \geq 1$ and $\beta \in \mathrm{GL}_n(\Lambda(\mathcal{G}))$ such that the class $[\beta] \in K_1(\Lambda(\mathcal{G}))$ satisfies

$$\zeta_S^T = [\alpha] \cdot [\beta]$$

Recall from Section 0.8 that the Dieudonné determinant is defined on the K_1-groups of $\Lambda(\mathcal{G})$ and $\mathcal{Q}(\mathcal{G})$. According to a result of Vaserstein, see (0.17), it is actually an isomorphism onto the abelianisation of the units of $\Lambda(\mathcal{G})$ resp. $\mathcal{Q}(\mathcal{G})$:

$$\det : K_1(\Lambda(\mathcal{G})) \xrightarrow{\sim} \left(\Lambda(\mathcal{G})^\times\right)^{\mathrm{ab}}$$
$$\det : K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\sim} \left(\mathcal{Q}(\mathcal{G})^\times\right)^{\mathrm{ab}}$$

Since $[\beta] \in K_1(\Lambda(\mathcal{G}))$, it follows that $\det\beta$ has a representative in $\Lambda(\mathcal{G})$. This, together with the two isomorphisms, reduces showing integrality of ζ_S^T to showing integrality of $\det\alpha$. That is, it remains to show that $\det(\alpha)$ is in the image of the map

$$\mathfrak{M}(\mathcal{G}) \cap \mathcal{Q}(\mathcal{G})^\times \to \left(\mathcal{Q}(\mathcal{G})^\times\right)^{\mathrm{ab}}$$

This can be done Wedderburn component-wise: that is, we need to show that for every $mn_\chi \times mn_\chi$ matrix

$$\alpha_\chi \in M_m(M_{n_\chi}(\Sigma_\chi)) \cap \mathrm{GL}_m(M_{n_\chi}(D_\chi))$$

there exists an $n_\chi \times n_\chi$ matrix

$$A_\chi \in M_{n_\chi}(\Sigma_\chi) \cap \mathrm{GL}_{n_\chi}(D_\chi)$$

such that their Dieudonné determinants agree:

$$\det\alpha_\chi = \det A_\chi$$

This is the statement of Proposition 4.3.1. The proof is complete. □

Remark 4.4.3. Recall from Proposition 3.3.16 that Σ_χ is the unique maximal $\Lambda^{\mathcal{O}_{\mathbb{Q}_{p,\chi}}}((\Gamma''_\eta)^{w_\chi/v_\chi})$-order in D_χ. However, the maximal order $M_{n_\chi}(\Sigma_\chi)$ in $M_{n_\chi}(D_\chi)$ need not be unique. Indeed, as in the proof of Proposition 3.3.16, we may localise and complete at height 1 prime ideals of $\Lambda^{\mathcal{O}_{\mathbb{Q}_{p,\chi}}}((\Gamma''_\eta)^{w_\chi/v_\chi})$, thus reducing the question to studying maximal orders over complete DVRs. In the resulting matrix rings, maximal orders are only unique up to conjugation by units, see [Rei03, Theorem 17.3(ii)]. ◦

4.5 Direct products of pro-p and finite abelian groups

Let $\mathcal{P}$ be a pro-p group and A a finite abelian group of order coprime to p, and suppose that $\mathcal{G} \simeq A \times \mathcal{P}$. In the context of the splitting $\mathcal{G} \simeq H \rtimes \Gamma$, this means that there is a finite p-group H' such that $\mathcal{P} \simeq H' \rtimes \Gamma$ and $H = A \times H'$.

Remark 4.5.1. In our notation, we shall use the prime symbol to refer to objects associated with $\mathcal{P}$, and reserve the non-primed versions for variants associated with $\mathcal{G}$. In particular, χ will denote a character of $\mathcal{G}$, whereas χ' will be a character of $\mathcal{P}$. The double primes in Γ''_η resp. $\Gamma''_{\eta'}$ and γ''_η resp. $\gamma''_{\eta'}$ are unfortunate, but should not cause confusion. ◦

Remark 4.5.2. Note that this setup does not fall under that of Lemma 2.4.10. Indeed, since A may have arbitrary prime-to-p degree, the extension $\mathbb{Q}_p(\zeta_{\#H})/\mathbb{Q}_p$ may have a subquotient that is unramified of degree divisible by p. ◦

Proposition 4.5.3. *Let $\mathcal{G} \simeq A \times \mathcal{P}$ be as above. Then the skew fields in the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$ are Brauer equivalent to cyclic algebras*

$$\left(\mathcal{Q}^{\mathbb{Q}_p(\eta')(\omega_{\eta'})(\lambda)}\left((\Gamma''_{\eta'})^{w_{\chi'}/v_{\chi'}}\right) \Big/ \mathcal{Q}^{\mathbb{Q}_{p,\chi'}(\lambda)}\left((\Gamma''_{\eta'})^{w_{\chi'}/v_{\chi'}}\right), (\sigma'\tau')^k, \left(\gamma''_{\eta'}\right)^{w_{\chi'}/v_{\chi'}}\right) \tag{4.8}$$

where λ is an irreducible (hence linear) character of A, χ' is an irreducible character of $\mathcal{P}$ with open kernel, η' is an irreducible constituent of $\operatorname{res}^{\mathcal{P}}_{H'}\chi'$, and k is the least positive integer such that $(\sigma'\tau')^k$ fixes the field $\mathbb{Q}_{p,\chi'}(\lambda) \cap \mathbb{Q}_p(\eta')(\omega_{\eta'})$. If $k = 1$, then we have equality (and not just Brauer equivalence).

For the proof, we recall the following two general statements concerning cyclic algebras.

Lemma 4.5.4 ([Rei03, Theorem 30.8])**.** *Let $\mathcal{L}/\mathcal{K}$ be a finite cyclic extension of fields, and let ς be a generator of $\operatorname{Gal}(\mathcal{L}/\mathcal{K})$. Let $a \in \mathcal{K}^\times$. Fix an algebraic closure of $\mathcal{K}$, let $\mathcal{E}$ be a finite field extension of $\mathcal{K}$, and let $\mathcal{E}\mathcal{L}$ denote the compositum of $\mathcal{L}$ and $\mathcal{E}$. Let k be the least positive integer such that ς^k fixes the field $\mathcal{L} \cap \mathcal{E}$. Then the following Brauer classes agree:*

$$[\mathcal{E} \otimes_{\mathcal{K}} (\mathcal{L}/\mathcal{K}, \varsigma, a)] = \left[\left(\mathcal{E}\mathcal{L}/\mathcal{E}, \varsigma^k, a\right)\right] \in \operatorname{Br}(\mathcal{E}\mathcal{L}/\mathcal{E})$$

where we identify the Galois groups $\operatorname{Gal}(\mathcal{L}/\mathcal{L} \cap \mathcal{E}) \simeq \operatorname{Gal}(\mathcal{E}\mathcal{L}/\mathcal{E})$ *with each other.*

The proof of Lemma 4.5.4 goes by identifying the cyclic algebras in question with crossed product algebras, and thus reducing the assertion to the corresponding statement for crossed product algebras, see [Rei03, Theorem 29.13]. In the proof of this Theorem, one can find the following special case:

Lemma 4.5.5 ([Rei03, (29.15)])**.** *If $k = 1$ in the setup of Lemma 4.5.4, then the cyclic algebras are isomorphic:*

$$\mathcal{E} \otimes_{\mathcal{K}} (\mathcal{L}/\mathcal{K}, \varsigma, a) \simeq \left(\mathcal{E}\mathcal{L}/\mathcal{E}, \varsigma^k, a\right)$$

Proof of Proposition 4.5.3. The direct product decomposition $\mathcal{G} \simeq A \times \mathcal{P}$ allows us to decompose the Iwasawa algebra as follows:

$$\Lambda(\mathcal{G}) = \bigoplus_{\lambda \in \operatorname{Irr}(A)} \mathcal{O}_{\mathbb{Q}_p(\lambda)}[\![\mathcal{P}]\!]$$

Let $D^{\mathcal{P}}_{\chi'}$ denote the skew fields in the Wedderburn decomposition of $\mathcal{Q}(\mathcal{P})$, with $\chi' \in \mathrm{Irr}(\mathcal{P})$. Then $\mathcal{Q}(\mathcal{G})$ can be decomposed as

$$\begin{aligned}\mathcal{Q}(\mathcal{G}) &= \bigoplus_{\lambda\in\mathrm{Irr}(A)} \mathbb{Q}_p(\lambda)\otimes_{\mathbb{Q}_p}\mathcal{Q}(\mathcal{P})\\ &= \bigoplus_{\lambda\in\mathrm{Irr}(A)}\ \bigoplus_{\chi'\in\mathrm{Irr}(\mathcal{P})/\sim_{\mathbb{Q}_p}} \mathbb{Q}_p(\lambda)\otimes_{\mathbb{Q}_p} M_{n_{\chi'}}\left(D^{\mathcal{P}}_{\chi'}\right)\\ &= \bigoplus_{\lambda\in\mathrm{Irr}(A)}\ \bigoplus_{\chi'\in\mathrm{Irr}(\mathcal{P})/\sim_{\mathbb{Q}_p}} M_{n_{\chi'}}\left(\mathbb{Q}_p(\lambda)\otimes_{\mathbb{Q}_p} D^{\mathcal{P}}_{\chi'}\right)\end{aligned}$$

The skew fields $D^{\mathcal{P}}_{\chi'}$ are determined by Corollary 3.2.2. Indeed, since $\mathcal{P}$ is a pro-p group, $\mathbb{Q}_p(\eta')$ is contained in some cyclotomic extension $\mathbb{Q}_p(\zeta_{p^m})$ of $\mathbb{Q}_p$, which is totally ramified over $\mathbb{Q}_p$, forcing the extension $\mathbb{Q}_p(\eta')/\mathbb{Q}_{p,\chi'}$ to also be totally ramified; therefore Theorem 2.4.9 makes Corollary 3.2.2 apply in this case. Explicitly, we have an isomorphism

$$D^{\mathcal{P}}_{\chi'} \simeq \left(\mathcal{Q}^{\mathbb{Q}_p(\eta')(\omega_{\eta'})}\left((\Gamma''_{\eta'})^{w_{\chi'}/v_{\chi'}}\right)\Big/\mathcal{Q}^{\mathbb{Q}_{p,\chi'}}\left((\Gamma''_{\eta'})^{w_{\chi'}/v_{\chi'}}\right), \sigma'\tau', \left(\gamma''_{\eta'}\right)^{w_{\chi'}/v_{\chi'}}\right)$$

Then

$$\mathbb{Q}_p(\lambda)\otimes_{\mathbb{Q}_p} D^{\mathcal{P}}_{\chi'} = \mathbb{Q}_{p,\chi'}(\lambda)\otimes_{\mathbb{Q}_{p,\chi'}} D^{\mathcal{P}}_{\chi'}$$

because $\mathbb{Q}_p(\lambda)$ and $\mathbb{Q}_{p,\chi'}$ are disjoint over $\mathbb{Q}_p$. Indeed, $\mathbb{Q}_p(\lambda)$ is contained in the unramified extension $\mathbb{Q}_p(\zeta_{\#A})/\mathbb{Q}_p$ while $\mathbb{Q}_{p,\chi'}/\mathbb{Q}_p$ is totally ramified, as explained above. The algebra $\mathbb{Q}_{p,\chi'}(\lambda)\otimes_{\mathbb{Q}_{p,\chi'}} D^{\mathcal{P}}_{\chi'}$ is Brauer equivalent to the algebra in (4.8) by Lemma 4.5.4.

We have $k = 1$ if and only if $\mathbb{Q}_p(\eta')(\omega_{\eta'})$ and $\mathbb{Q}_{p,\chi'}(\lambda)$ are disjoint over $\mathbb{Q}_{p,\chi'}$: in this case, Lemma 4.5.5 shows the claimed isomorphism. □

Corollary 4.5.6. *Suppose $k = 1$ for all χ' and λ in Proposition 4.5.3. Then Theorem 4.4.1 holds for $\mathcal{G}$ under the assumption of the equivariant main conjecture (Conjecture 1.4.4).*

Proof. Irreducible characters of $\mathcal{G}$ with open kernel are of the form $\chi = \lambda\times\chi'$ by [Isa76, Theorem 4.21] where λ and χ' are as above. The irreducible constituents of the restriction $\mathrm{res}^{\mathcal{G}}_H\chi$ are $\eta = \lambda\times\eta'$. According to Schilling's theorem, see [CR87, (74.15)], each $D_{\eta'}$ is a field, namely $D_{\eta'} = \mathbb{Q}_p(\eta')$. This together with the assumption $k = 1$ and the proof of Proposition 4.5.3 shows that $D_\eta = \mathbb{Q}_p(\eta) = \mathbb{Q}_p(\eta',\lambda)$ and $\mathbb{Q}_{p,\chi} = \mathbb{Q}_{p,\chi'}(\lambda)$. (Aside: while $s_\eta = s_{\eta'} = 1$, the numbers q_η and ω_η need not agree with their primed versions. Indeed, $\mathbb{Q}_p(\lambda)$ is unramified over $\mathbb{Q}_p$, so the residue field extension may be nontrivial.)

We conclude that the descriptions from Sections 3.2 and 3.3 as well as the results of Section 4.3 are still valid in this case, with the appropriate definition of Σ_χ. Therefore in this case, the proof of Theorem 4.4.1 goes through under the assumption of the main conjecture. □

4.6 p-abelian extensions

As we have seen in Lemma 2.4.10, Condition 2.4.1 is valid when $\mathcal{G}$ is pro-p, thus making Theorem 4.4.1 hold in this case. In this section, we study another special case, which in some sense is orthogonal to the pro-p one. In this section, $\mathcal{G}'$ denotes the commutator subgroup of $\mathcal{G}$.

Definition 4.6.1. $\mathcal{G}$ is called p-abelian if its commutator subgroup $\mathcal{G}'$ has prime-to-p order. ◦

This makes sense: since $\mathcal{G}/H$ is abelian, the commutator subgroup is always contained in H, hence it is finite. In particular, $\mathcal{G}$ is p-abelian whenever H has order coprime to p.

Remark 4.6.2. In this setup, Condition 2.4.1 may fail. Indeed, we have seen that this happens for Example 2.4.12, which falls under this setup: in that case, $p = 3$ and $\mathcal{G} \simeq C_7 \rtimes \mathbb{Z}_3$, so $H \simeq C_7$ has order coprime to p. ∘

Theorem 4.6.3 ([JN20, Theorem 11.1 and Corollary 12.17]). *$\mathcal{G}$ is p-abelian if and only if $\Lambda(\mathcal{G}) \simeq \prod_j M_{n_j}(R_j)$ where the R_j are complete commutative local rings. Moreover, if this is the case, then Conjecture 1.4.2 holds.*

Remark 4.6.4. In fact, [JN20, Corollary 12.17] states that the equivariant main conjecture with uniqueness (Conjecture 1.4.2) holds whenever $\mathcal{G}$ has an abelian p-Sylow subgroup, which follows from $p \nmid \#\mathcal{G}'$. Indeed, a p-Sylow subgroup of a profinite group is—by definition—a closed pro-p subgroup whose index is coprime to p, and by Sylow's theorem, these are all conjugate. If $\mathcal{G}$ is p-abelian, then for every p-Sylow subgroup $P \subseteq \mathcal{G}$ we have $\mathcal{G}' \cap P = 1$; since $P' \leq \mathcal{G}' \cap P$, this means that P is abelian.

The converse does not hold. If some p-Sylow subgroup $P \subseteq \mathcal{G}$ is abelian, then all p-Sylow subgroups are abelian, because they are conjugate to P by Sylow's theorem. We still have $P' \leq \mathcal{G}' \cap P$, but this need not be an equality.

As an explicit counterexample, consider $\mathcal{G} \simeq \mathfrak{S}_p \times \Gamma$, where $\mathfrak{S}_p$ is the symmetric group of degree p. The symmetric group has order $p!$, hence all p-Sylows of $\mathfrak{S}_p$ are of order p, so in particular abelian. Every p-Sylow of $\mathcal{G}$ is a direct product of a p-Sylow of $\mathfrak{S}_p$ with Γ, hence abelian. Since $p \neq 2$, the commutator subgroup is the alternating group $\mathcal{G}' \simeq \mathfrak{S}_p' \simeq A_p$ (see [Hup67, Kapitel II, Satz 5.1]), whose order $p!/2$ is divisible by p. ∘

In particular, Theorem 4.6.3 combined with Lemma 1.4.5 shows that Conjecture 1.4.4 holds for p-abelian extensions.

Corollary 4.6.5. *If $\mathcal{G}$ is p-abelian, then $\Phi_S^T \in \mathfrak{z}(\Lambda(\mathcal{G}))$. Moreover, Conjecture 4.1.3 holds with $\mathfrak{M} = \Lambda(\mathcal{G})$.*

Remark 4.6.6. The result is unconditional: we assume neither Conjecture 1.4.4 nor Condition 2.4.1. ∘

Proof. As in the proof of Theorem 4.4.1, using Conjecture 1.4.4, we have

$$\zeta_S^T = [\alpha] \cdot [\beta] \tag{4.9}$$

where $\alpha \in M_m(\Lambda(\mathcal{G})) \cap \mathrm{GL}_m(\mathcal{Q}(\mathcal{G}))$ and $\beta \in \mathrm{GL}_n(\Lambda(\mathcal{G}))$. Replacing α by $\alpha\beta$, and potentially enlarging m in case $n > m$, applying the reduced norm map on $K_1(\mathcal{Q}(\mathcal{G}))$ to (4.9) yields

$$\Phi_S^T = \mathrm{nr}\, \zeta_S^T = \mathrm{nr}[\alpha] = \mathrm{nr}_{M_n(\mathcal{Q}(\mathcal{G}))/\mathfrak{z}(\mathcal{Q}(\mathcal{G}))}\, \alpha \tag{4.10}$$

where the first equality is again due to Conjecture 1.4.4.

Let us now study the reduced norm map on $M_n(\Lambda(\mathcal{G}))$ in general. This is the restriction of the reduced norm map $M_n(\mathcal{Q}(\mathcal{G})) \to \mathfrak{z}(\mathcal{Q}(\mathcal{G}))$. Recall that $\mathcal{Q}(\mathcal{G})$ is semisimple while $\Lambda(\mathcal{G})$ is not, therefore a priori, the reduced norm on $M_n(\Lambda(\mathcal{G}))$ takes values in $\mathfrak{z}(\mathcal{Q}(\mathcal{G}))$. However, using the description of $\Lambda(\mathcal{G})$ from Theorem 4.6.3, we can see that in the p-abelian case, the reduced norm actually maps to $\mathfrak{z}(\Lambda(\mathcal{G}))$. Indeed, the following diagram commutes (without the dashed arrow), and thus the left vertical arrow induces the dashed arrow.

$$\begin{array}{ccccc}
M_m(\Lambda(\mathcal{G})) & & \xrightarrow{\quad\sim\quad} & & \prod_j M_{mn_j}(R_j) \\
\downarrow{\scriptstyle \mathrm{nr}} & \overset{\mathrm{nr}}{\searrow} & & & \downarrow{\scriptstyle \prod_j \det} \\
\mathfrak{z}(\mathcal{Q}(\mathcal{G})) & \hookleftarrow & \mathfrak{z}(\Lambda(\mathcal{G})) & \xrightarrow{\ \sim\ } & \prod_j R_j
\end{array}$$

We conclude that the reduced norm of α on the right hand side of (4.10) is in $\mathfrak{z}(\Lambda(\mathcal{G}))$, and hence the same is true for Φ_S^T. The first assertion is now proven.

Let $z \in \mathfrak{z}(\Lambda(\mathcal{G}))$ be an arbitrary central element, and let $m := 1$ in the diagram above. Let $z_j \in R_j$ be the image of z in R_j under the bottom horizontal map. Let $\tilde{Z}$ be the tuple in $\prod_j M_{n_j}(R_j)$ whose jth component is $\operatorname{diag}(z_j, 1, \ldots, 1)$. (Note that this element is not central unless $n_j = 1$ for all indices j.) Then the image of $\tilde{Z}$ under $\prod_j \det$ is $(z_j)_j$. Let $\tilde{z} \in \Lambda(\mathcal{G})$ be the element corresponding to $\tilde{Z}$ under the top horizontal isomorphism. So in the diagram above, we have

$$\begin{array}{ccc} \tilde{z} & \longmapsto & \tilde{Z} \\ \Big\downarrow{\scriptstyle \mathrm{nr}} & & \Big\downarrow{\scriptstyle \prod_j \det} \\ z & \longmapsto & (z_j)_j \end{array}$$

In particular, we have $\mathrm{nr}(\tilde{z}) = z$. Setting $z := \Phi_S^T$, it follows that Φ_S^T is in the image of

$$\Lambda(\mathcal{G}) \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$$

The second and third assertions follow. □

Remark 4.6.7. The 'if' part of Theorem 4.6.3 shows that when $\mathcal{G}$ is not p-abelian, the reduced norm won't just be a product of determinants, and the argument above cannot be generalised. ∘

Chapter 5

Study of D_χ via higher local fields

In this chapter, we study the skew field $\mathfrak{D}$ using the theory of higher local fields, using results from Section 3.3. This was carried out by Lau in [Lau12a, §3] and [Lau12b] for $\mathcal{G}$ pro-p, and indeed the results in this chapter are mostly generalisations of her results. Assuming the validity of Condition 2.4.1, the results on $\mathfrak{D}$ carry over to D_χ under the isomorphism of Theorem 3.3.11.

In this chapter, overline always means reduction, e.g. taking residue (skew) fields.

5.1 Centre and maximal subfield

As in the proof of Corollary 3.3.8, we write $T := (1+X)^{w_\chi/v_\chi}-1$, so that $\mathfrak{z}(\mathfrak{D}) = \mathrm{Frac}(\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]])$. This is not a higher local field, but as we will now show, it becomes one after localising and then completing at any height 1 prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$, which is the standard method of constructing higher local fields, as explained in Section 0.9. Height 1 primes in $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$ are described by Weierstraß theory: they are principal ideals, generated either by a distinguished irreducible polynomial $P(T) \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[T]$ or by a uniformiser π_χ of $\mathcal{O}_{\mathbb{Q}_{p,\chi}}$.

Lemma 5.1.1. *Completions of localisations of $\mathfrak{z}(\mathfrak{D})$ at height 1 primes admit the following description as 2-dimensional higher local fields, distinguishing between equal and mixed characteristic.*

- ***Equal characteristic case.*** *Let $\mathfrak{p} = (P(T))$ for a distinguished irreducible polynomial $P(T) \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[T]$. Then the completed localisation at $\mathfrak{p}$ of $\mathfrak{z}(\mathfrak{D})$ resp. its residue field is the following 2- resp. 1-dimensional local field:*

$$\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}} = \mathrm{Frac}\left(\widehat{\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_\mathfrak{p}}\right), \quad \overline{\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}}} = \mathbb{Q}_{p,\chi}[T]/(P(T))$$

If ξ is a root of $P(T)$ in some extension of $\mathbb{Q}_{p,\chi}$, then these can be identified with the standard fields

$$\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}} \simeq \mathbb{Q}_{p,\chi}(\xi)((y)), \quad \overline{\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}}} \simeq \mathbb{Q}_{p,\chi}(\xi)$$

- ***Mixed characteristic case.*** *If $\mathfrak{p} = (\pi_\chi)$ then*

$$\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}} = \mathrm{Frac}\left(\widehat{\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_\mathfrak{p}}\right) \simeq \mathbb{Q}_{p,\chi}\{\{T\}\}, \quad \overline{\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}}} = \overline{\mathbb{Q}_{p,\chi}}((T))$$

where $\overline{\mathbb{Q}_{p,\chi}}$ denotes the residue field of $\mathbb{Q}_{p,\chi}$.

Lemma 5.1.1 generalises Lemma 2 and Corollary 2 of [Lau12a].

Proof. We begin with the equal characteristic case. The first equation is by definition. We now justify the second one. A generic element of the localisation $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}$ is of the form

$$\frac{G_1(T)}{G_2(T)} \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}$$

where $G_1(T), G_2(T) \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$ and $P(T) \nmid G_2(T)$. Since Weierstraß preparation works in $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$ and we are in the equal characteristic case, we may assume that $G_2(T) \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[T]$ by moving powers of the uniformiser as well as the unit part into the numerator and thus relaxing the condition on G_1 to $G_1(T) \in \mathbb{Q}_{p,\chi} \otimes_{\mathcal{O}_{\mathbb{Q}_{p,\chi}}} \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$. This allows us to apply Euclidean division to $G_2(T)$ in the polynomial ring $\mathbb{Q}_{p,\chi}[T]$: we find a polynomial $H(T) \in \mathbb{Q}_{p,\chi}[T]$ such that

$$G_2(T)H(T) \equiv 1 \pmod{P(T)}$$

Then the following congruence holds in $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}$:

$$\frac{G_1(T)}{G_2(T)} = \frac{G_1(T)H(T)}{G_2(T)H(T)} \equiv G_1(T)H(T) \pmod{P(T)}$$

Let $e \geq 0$ be the least non-negative integer such that $G_1(T)H(T)\pi_\chi^e \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$, so that the Weierstraß division theorem is applicable to $G_1(T)H(T)\pi_\chi^e$:

$$G_1(T)H(T)\pi_\chi^e = Q(T)P(T) + R(T)$$

where $Q(T) \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]$, and $R(T) \in \mathcal{O}_{\mathbb{Q}_{p,\chi}}[T]$ with degree $\deg R < \deg P$. The uniformiser π_χ becomes a unit in the localisation $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}$, and we conclude every element of $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}$ is congruent modulo $P(T)$ to one of the form $R(T)$. The polynomials $R(T)$ run through the elements of $\mathbb{Q}_{p,\chi}[T]$ of degree lower than $\deg P$ as $G_1(T)/G_2(T)$ runs through $\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}$. Quotienting out by $\mathfrak{p} = P(T)$, we find that the residue field is as claimed.

If ξ is as in the statement, then the ring $\mathbb{Q}_{p,\chi}[T]/(P(T))$ is isomorphic to $\mathbb{Q}_{p,\chi}(\xi)$, justifying the fourth equation. The third equation then comes from the classification theorem of 2-dimensional higher local fields stated in Theorem 0.9.4. This isomorphism is the identity on $\mathbb{Q}_{p,\chi}$, and sends the uniformiser $P(T)$ to the uniformiser y. The element ξ is sent to an element $x \in \widehat{\mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]]_{\mathfrak{p}}}$ such that $P(x) = 0$: this is shown in [Ser80, II§4, p. 34, Proposition 7], where x is constructed using Hensel's lemma. (Note: x is not unique, but it is uniquely determined by its image in the residue field.)

In the mixed characteristic case, the first equality is by definition. Taking the quotient modulo π_χ, we obtain the second equality. The isomorphism comes from the classification theorem. □

Let $E \subseteq D_\eta$ be a maximal subfield with ring of integers $\mathcal{O}_E$. Then the field

$$\mathfrak{E}(E) := \operatorname{Frac}(\mathcal{O}_E[[T]])$$

is a maximal subfield of $\mathfrak{D}$ by Lemma 3.3.17. We will write

$$\mathcal{O}_{\mathfrak{E}(E)} := \mathcal{O}_E[[T]]$$

for its ring of integers. If $E = \mathbb{Q}_p(\eta)(\omega_\eta)$, then we have $\mathfrak{E}(\mathbb{Q}_p(\eta)(\omega_\eta)) = \mathfrak{E}$ in the notation of Section 3.3. We give an interpretation of the field $\mathfrak{E}(E)$ in the context of higher local fields. Let $\mathfrak{P}$ be a height 1 prime ideal of $\mathcal{O}_E[[T]]$.

Lemma 5.1.2. *We have the following description of completions of localisations of the maximal subfield* $\mathfrak{C}(E) = \operatorname{Frac}(\mathcal{O}_E[[T]])$ *of* $\mathfrak{D}$ *at height* 1 *primes as* 2*-dimensional higher local fields, distinguishing between equal and mixed characteristic.*

- ***Equal characteristic case.*** *Let* $\mathfrak{P} = (P(T))$ *for a distinguished irreducible polynomial* $P(T) \in \mathcal{O}_E[T]$*. Then the completed localisation at* $\mathfrak{P}$ *of* $\mathfrak{C}(E)$ *resp. its residue field is the following* 2*- resp.* 1*-dimensional local field:*

$$\widehat{\mathfrak{C}(E)}_{\mathfrak{P}} = \operatorname{Frac}\left(\widehat{\mathcal{O}_E[[T]]_{\mathfrak{P}}}\right), \quad \overline{\widehat{\mathfrak{C}(E)}_{\mathfrak{P}}} = E[T]/(P(T))$$

If ξ *is a root of* $P(T)$ *in some extension of* E*, then these can be identified with the standard fields*

$$\widehat{\mathfrak{C}(E)}_{\mathfrak{P}} \simeq E(\xi)((y)), \quad \overline{\widehat{\mathfrak{C}(E)}_{\mathfrak{P}}} \simeq E(\xi)$$

- ***Mixed characteristic case.*** *If* $\mathfrak{P} = (\pi_\chi)$*, then*

$$\widehat{\mathfrak{C}(E)}_{\mathfrak{P}} = \operatorname{Frac}\left(\widehat{\mathcal{O}_E[[T]]_{\mathfrak{P}}}\right) \simeq E\{\{T\}\}, \quad \overline{\widehat{\mathfrak{C}(E)}_{\mathfrak{P}}} = \overline{E}((T))$$

where $\overline{E}$ *denotes the residue field of* E*.*

Proof. The proof is completely analogous to that of Lemma 5.1.1. □

Lemmata 5.1.1 and 5.1.2 are stated in terms of prime ideals $\mathfrak{p}$ resp. $\mathfrak{P}$ of the respective ring. The connexion between these are made by the following isomorphism, see [Neu99, Proposition II.8.3]:

$$\mathfrak{C}(E) \otimes_{\mathfrak{z}(\mathfrak{D})} \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}} \simeq \prod_{\mathfrak{P}|\mathfrak{p}} \widehat{\mathfrak{C}(E)}_{\mathfrak{P}} \tag{5.1}$$

The statement and the proof in the reference are given without the hats, but also work for completions. The condition that $\mathfrak{C}(E)/\mathfrak{z}(\mathfrak{D})$ be separable is satisfied as these are fields of characteristic zero.

We may also consider a special case of this. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$, and assume that $\mathfrak{p}$ is inert in $\mathfrak{C}(E)/\mathfrak{z}(\mathfrak{D})$. That is, there is a unique prime ideal $\mathfrak{P}$ of $\mathcal{O}_{\mathfrak{C}(E)}$ above $\mathfrak{p}$, and $\mathfrak{P}$ is unramified above $\mathfrak{p}$. More explicitly, this means that either one of the following holds.

- In equal characteristic: $\mathfrak{p} = (P(T))$ as above, such that $P(T)$ remains irreducible in $\mathcal{O}_E[T]$, so that $\mathfrak{P} = (P(T))$ is a height 1 prime.
- In mixed characteristic: $\mathfrak{p} = (\pi_\chi)$, and the extension $E/\mathbb{Q}_{p,\chi}$ is unramified, so that π_χ is a uniformiser in E.

Remark 5.1.3. If $E = \mathbb{Q}_p(\eta)(\omega_\eta)$, then $E/\mathbb{Q}_p(\eta)$ is unramified, and thus the condition in mixed characteristic is equivalent to $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ being unramified. Note that this is orthogonal to Condition 2.4.1. ◦

If $\mathfrak{p}$ is inert in $\mathfrak{C}(E)/\mathfrak{z}(\mathfrak{D})$, then $\mathfrak{C}(E) \otimes_{\mathfrak{z}(\mathfrak{D})} \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}} \simeq \widehat{\mathfrak{C}(E)}_{\mathfrak{P}}$: indeed, there is only one $\mathfrak{P}$ above $\mathfrak{p}$ by assumption, and the claim follows from (5.1).

5.2 Another way towards describing D_χ

In Proposition 3.2.4 we have seen that D_χ lives in the relative Brauer group

$$\mathrm{Br}\left(\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma'_\chi)/\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)\right)$$

Fixing an isomorphism $\Lambda(\Gamma'_\chi) \simeq \mathbb{Z}_p[[T]]$, the results of Section 5.1 carry over to these fields.

Recall the following elementary property of Brauer groups.

Lemma 5.2.1. *Let $\mathcal{L}/\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ be field extensions. Then the tensor product $\mathcal{L} \otimes_{\mathcal{F}} \mathcal{E}$ is isomorphic to a product of fields $\prod_{i\in I} \mathcal{E}_i$. If $\mathcal{A}$ is a central simple $\mathcal{F}$-algebra split by $\mathcal{L}$, that is, if $[\mathcal{A}] \in \mathrm{Br}(\mathcal{L}/\mathcal{F})$, then for all indices $i \in I$, the $\mathcal{F}$-algebra $\mathcal{A} \otimes_{\mathcal{F}} \mathcal{E}$ is split by $\mathcal{E}_i$, that is, $[\mathcal{A} \otimes_{\mathcal{F}} \mathcal{E}] \in \mathrm{Br}(\mathcal{E}_i/\mathcal{F})$.*

Proof. Indeed, if $\mathcal{A} \otimes_{\mathcal{F}} \mathcal{L} \simeq M_n(\mathcal{L})$, then

$$(\mathcal{A} \otimes_{\mathcal{F}} \mathcal{E}) \otimes_{\mathcal{E}} \mathcal{E}_i \simeq \mathcal{A} \otimes_{\mathcal{F}} \mathcal{E}_i \simeq \mathcal{A} \otimes_{\mathcal{F}} \mathcal{L} \otimes_{\mathcal{L}} \mathcal{E}_i \simeq M_n(\mathcal{L}) \otimes_{\mathcal{L}} \mathcal{E}_i \simeq M_n(\mathcal{E}_i) \qquad \square$$

Applying Lemma 5.2.1 to (5.1), we get classes

$$\left[D_\chi \otimes_{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)} \widehat{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)}_{\mathfrak{p}}\right] \in \mathrm{Br}\left(\widehat{\mathcal{Q}^{\mathbb{Q}_p(\eta)(\omega_\eta)}(\Gamma'_\chi)}_{\mathfrak{P}}\Big/\widehat{\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma'_\chi)}_{\mathfrak{p}}\right)$$

for all $\mathfrak{P} \mid \mathfrak{p}$. The latter group has been studied by Kato among others, see [FK00, §5].

For a higher local field $\mathcal{K}$, Kato defined certain cohomology groups $H^q(\mathcal{K})$, which coincide with usual Galois cohomology in characteristic zero, while in positive characteristic they are defined via de Rham–Witt complexes. Kato described an inflation-like homomorphism $i : H^q(\overline{\mathcal{K}}) \oplus H^{q-1}(\overline{\mathcal{K}}) \to H^q(\mathcal{K})$; in equal characteristic, this is given by usual inflation and a cup product, and is more involved in mixed characteristic, see [FK00, p. 55] and [Kat82, Theorem 3]. This homomorphism i is bijective away from p and injective on p-parts.

The case of interest to us is $q = 2$. The cohomology group $H^2(\mathcal{K})$ agrees with the Brauer group of $\mathcal{K}$, and the above results make it possible to study completed localisations of D_χ by means of the inflation map i. Its source is the direct sum of $H^2(\overline{\mathcal{K}})$, the Brauer group of a 1-dimensional local field, and $H^1(\overline{\mathcal{K}})$, the group of characters; both of these groups are well understood. These local pieces of data could then be placed together for all $\mathfrak{p}$ using a local–global map studied by Saito in [Sai86, Proposition 3.1]. There is, however, considerable difficulty in actually computing i in explicit terms in the mixed characteristic case, which is why this approach of describing D_χ was not pursued further.

5.3 Cohomological dimension

We have the following generalisation of Corollary 3 of [Lau12a].

Lemma 5.3.1. *Let $\mathfrak{p}$ resp. $\mathfrak{P}$ be height 1 prime ideals of $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ resp. $\mathcal{O}_{\mathfrak{E}(E)}$. Then the 2-dimensional higher local fields $\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$ and $\widehat{\mathfrak{E}(E)}_{\mathfrak{P}}$ have cohomological dimension 3.*

Proof. Lau's proof carries over without modification.

Indeed, let us first consider the equal characteristic case. Let ℓ be a rational prime. Then the residue field is a 1-dimensional local field, and as such has ℓ-cohomological dimension 2, see [NSW20, Theorem 7.1.8(i)]. Then since the 2-dimensional higher local fields $\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$ and $\widehat{\mathfrak{E}(E)}_{\mathfrak{P}}$ have characteristic $0 \neq \ell$, [NSW20, Theorem 6.5.15] shows that they both have ℓ-cohomological dimension $2 + 1 = 3$. Since this is true for all rational primes ℓ, the assertion follows.

In the mixed characteristic case, the proof is due to Morita in [Mor08, §3], see the assertion after the statement of Theorem 3.1. There it is shown that if $\mathscr{K}$ is a complete discretely valued field of characteristic 0 with residue field $\mathscr{k}$ of characteristic p, and $(\mathscr{k} : \mathscr{k}^p) = p^n < \infty$, then $\operatorname{cd} \mathscr{K} = n+2$. Let $\mathscr{K}$ be either of the fields in the statement: then the residue field $\mathscr{k}$ is the field $\mathbb{F}(\!(T)\!)$ of Laurent series over some finite field $\mathbb{F}$ of p-power order. We have $\mathbb{F}(\!(T)\!)^p = \mathbb{F}(\!(T^p)\!)$, and thus $n = 1$, hence Morita's result is applicable, and the cohomological dimension is $1+2=3$. □

This leads to the following generalisation of [Lau12a, Theorem 2].

Proposition 5.3.2. $\operatorname{cd}(\mathfrak{z}(\mathfrak{D})) = 3$.

Proof. First we prove the inequality

$$\operatorname{cd}(\mathfrak{z}(\mathfrak{D})) \geq 3 \tag{5.2}$$

Recall from [NSW20, (3.3.5)] that if $\mathcal{G}$ is a profinite group and $\mathscr{H}$ a closed subgroup, then for every rational prime ℓ, there is an inequality $\operatorname{cd}_\ell \mathscr{H} \leq \operatorname{cd}_\ell \mathcal{G}$, and consequently $\operatorname{cd} \mathscr{H} \leq \operatorname{cd} \mathcal{G}$. Set $\mathcal{G}$ to be the absolute Galois group of $\mathfrak{z}(\mathfrak{D})$, let $\mathfrak{p}$ be a height 1 prime ideal of $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$, and let $\mathscr{H}$ be the absolute Galois group of $\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$. Then $\mathscr{H}$ is indeed a closed subgroup of $\mathcal{G}$: this can be seen by viewing $\mathscr{H}$ as the decomposition group of some prime above $\mathfrak{p}$. Then (5.2) follows from Lemma 5.3.1.

In light of (5.2), it remains to show that $\operatorname{cd}(\mathfrak{z}(\mathfrak{D})) \leq 3$. Using a result of Saito from [Sai86, Theorem 5.1], Lau proved

$$\operatorname{cd} \mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{w_\chi}) \leq 3$$

as part of the proof of [Lau12a, Theorem 2]. Fixing an isomorphism

$$\begin{aligned} \Lambda^{\mathcal{O}_{\mathbb{Q}_{p,\chi}}}(\Gamma^{w_\chi}) &\xrightarrow{\sim} \mathcal{O}_{\mathbb{Q}_{p,\chi}}[[T]] = \mathcal{O}_{\mathfrak{z}(\mathfrak{D})} \\ \gamma^{w_\chi} &\to 1+T \end{aligned}$$

yields an isomorphism of the respective fields of fractions:

$$\mathcal{Q}^{\mathbb{Q}_{p,\chi}}(\Gamma^{w_\chi}) \simeq \mathfrak{z}(\mathfrak{D})$$

Therefore $\operatorname{cd} \mathfrak{z}(\mathfrak{D}) \leq 3$, and the result follows. □

Proposition 5.3.2 therefore places us in the setup of Suslin's conjecture on the vanishing of SK_1 mentioned in Remark 0.8.2.

5.4 Completions

The completed localisations of the centre of $\mathfrak{D}$ considered above give rise to completions of $\mathfrak{D}$ under the respective $\mathfrak{p}$-adic valuations:

$$\widehat{\mathfrak{D}}_{\mathfrak{p}} := \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}} \otimes_{\mathfrak{z}(\mathfrak{D})} \mathfrak{D}$$

We have already encountered this algebra in Lemma 3.3.14 and Proposition 3.3.16. It is a central simple algebra over its centre

$$\mathfrak{z}\left(\widehat{\mathfrak{D}}_{\mathfrak{p}}\right) = \mathfrak{z}\left(\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}} \otimes_{\mathfrak{z}(\mathfrak{D})} \mathfrak{D}\right) = \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}} \otimes_{\mathfrak{z}(\mathfrak{D})} \mathfrak{z}(\mathfrak{D}) = \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$$

Moreover, since for any field extension $\mathscr{L}/\mathscr{K}$ and a finite dimensional $\mathscr{K}$-vector space V, $\dim_{\mathscr{L}}(V\otimes_{\mathscr{K}}\mathscr{L}) = \dim_{\mathscr{K}} V$, the dimension of $\widehat{\mathfrak{D}}_{\mathfrak{p}}$ over its centre is

$$\dim_{\mathfrak{z}(\widehat{\mathfrak{D}}_{\mathfrak{p}})} \widehat{\mathfrak{D}}_{\mathfrak{p}} = \dim_{\mathfrak{z}(\mathfrak{D})} \mathfrak{D} = \left(\frac{w_\chi}{v_\chi}\right)^2 s_\eta^2$$

Applying Lemma 5.2.1 to (5.1), we find that for each $\mathfrak{P} \mid \mathfrak{p}$ as above, $\widehat{\mathfrak{E}(E)}_{\mathfrak{P}}$ splits $\widehat{\mathfrak{D}}_{\mathfrak{p}}$. This is a field extension of $\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$ of the following degree:

$$\begin{aligned}\left(\widehat{\mathfrak{E}(E)}_{\mathfrak{P}} : \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}\right) &= \begin{cases} (E(\Xi)((y)) : \mathbb{Q}_{p,\chi}(\xi)((y))) & \text{in equal characteristic} \\ (E\{\{T\}\} : \mathbb{Q}_{p,\chi}\{\{T\}\}) & \text{in mixed characteristic} \end{cases} \\ &= \begin{cases} (E : \mathbb{Q}_{p,\chi})\,\frac{\deg P_1(T)}{\deg P(T)} & \text{in equal characteristic} \\ (E : \mathbb{Q}_{p,\chi}) & \text{in mixed characteristic} \end{cases}\end{aligned}$$

where the first step is due to Lemmata 5.1.1 and 5.1.2. Here $\mathfrak{p} = (P(T))$, and $P_1(T)$ is an irreducible factor of it over E such that $\mathfrak{P} = (P_1(T))$, and ξ resp. Ξ are roots of $P(T)$ resp. $P_1(T)$.

The embedding $\mathfrak{E}(E) \hookrightarrow \mathfrak{D}$ induces an embedding $\widehat{\mathfrak{E}(E)}_{\mathfrak{P}} \hookrightarrow \widehat{\mathfrak{D}}_{\mathfrak{p}}$. This is because $\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$ is a flat $\mathfrak{z}(\mathfrak{D})$-module, which is due to the facts that localisation and completion are flat, see [Sta23, Lemmata 00HT(1) & 06LE].

Since $\widehat{\mathfrak{D}}_{\mathfrak{p}}$ is a central simple $\mathfrak{z}(\widehat{\mathfrak{D}}_{\mathfrak{p}})$-algebra, there is an isomorphism

$$\widehat{\mathfrak{D}}_{\mathfrak{p}} \simeq M_n\left(\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ\right)$$

where $\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ$ is a skew field with centre $\mathfrak{z}(\widehat{\mathfrak{D}}_{\mathfrak{p}})$ and Schur index dividing $\frac{w_\chi}{v_\chi}s_\eta$. The valuation of $\mathfrak{z}(\widehat{\mathfrak{D}}_{\mathfrak{p}})$ admits a unique extension w to $\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ$ by [Rei03, Theorem 12.10]. The valuation ring $\{x \in \widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ : w(x) \geq 0\}$ has a unique maximal two-sided ideal $\{x \in \widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ : w(x) > 0\}$. The quotient ring is a skew field, which we will call the residue skew field of $\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ$ and denote by $\overline{\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ}$.

We have the following generalisation of [Lau12a, Proposition 2].

Lemma 5.4.1. *In the equal characteristic case,*

$$SK_1\left(\widehat{\mathfrak{D}}_{\mathfrak{p}}\right) = 1$$

Proof. Due to Morita equivalence, the K_1-groups of $\widehat{\mathfrak{D}}_{\mathfrak{p}} \simeq M_n(\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ)$ and $\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ$ are equal, wherefore the same holds for their SK_1-groups: in particular, it suffices to prove $SK_1(\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ) = 1$. We follow the strategy of Lau's proof. Consider the field extension

$$\mathfrak{z}\left(\overline{\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ}\right)\Big/\overline{\mathfrak{z}\left(\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ\right)}$$

The base field here has characteristic zero: indeed, $\mathfrak{z}(\widehat{\mathfrak{D}}_{\mathfrak{p}}^\circ) = \mathfrak{z}(\widehat{\mathfrak{D}}_{\mathfrak{p}}) = \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}}$ is, by Lemma 5.1.1, a 2-dimensional higher local field with characteristic zero residue field. In particular, the field extension is separable.

Recall the following result of Draxl from [Dra77, Korollar 7]. Let $\mathscr{k}$ be a complete discretely valued field with residue field $\overline{\mathscr{k}}$, and let $\mathscr{D}$ be a skew field which is finite dimensional over $\mathfrak{z}(\mathscr{D}) = \mathscr{k}$. Let $\overline{\mathscr{D}}$ denote the residue skew field of $\mathscr{D}$. Suppose that the field extension $\mathfrak{z}(\overline{\mathscr{D}})/\overline{\mathscr{k}}$ is separable and that $\overline{k}$ is 'reasonable' in the sense of [Dra77]. Then Draxl showed that $SK_1(\mathscr{D}) = 1$. This result is applicable in our case, with $k := \mathfrak{z}(\widehat{\mathfrak{D}}^\circ_\mathfrak{p})$ and $\mathscr{D} := \widehat{\mathfrak{D}}^\circ_\mathfrak{p}$. Indeed, separablility follows from the characteristic being 0, and as we have seen above, the residue field $\overline{k} = \overline{\mathfrak{z}(\widehat{\mathfrak{D}}^\circ_\mathfrak{p})}$ is a 1-dimensional local field, which are always reasonable by [Dra77, (13)]. $\square$

Corollary 4 of [Lau12a] generalises as follows. Let $\mathfrak{p}_0 \subset \Lambda(\Gamma_0)$ be a height 1 prime ideal such that $\mathfrak{p}_0 \neq (p)$. Let $\widehat{\Lambda(\Gamma_0)}_{\mathfrak{p}_0}$ denote the completed localisation of $\Lambda(\Gamma_0)$ at $\mathfrak{p}_0$. We define the localised completion of $\Lambda(\mathcal{G})$ resp. $\mathcal{Q}(\mathcal{G})$ as follows:

$$\begin{aligned}
\widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} &:= \mathrm{Frac}\left(\widehat{\Lambda(\Gamma_0)}_{\mathfrak{p}_0}\right) \\
\widehat{\Lambda(\mathcal{G})}_{\mathfrak{p}_0} &:= \widehat{\Lambda(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\Lambda(\Gamma_0)} \Lambda(\mathcal{G}) \\
\widehat{\mathcal{Q}(\mathcal{G})}_{\mathfrak{p}_0} &:= \mathrm{Quot}\left(\widehat{\Lambda(\mathcal{G})}_{\mathfrak{p}_0}\right) = \widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} \mathcal{Q}(\mathcal{G})
\end{aligned}$$

Corollary 5.4.2. *Assume that Condition 2.4.1 holds, and let $\mathfrak{p}_0 \subset \Lambda(\Gamma_0)$ be a height 1 prime ideal such that $\mathfrak{p}_0 \neq (p)$. Then*

$$SK_1\left(\widehat{\mathcal{Q}(\mathcal{G})}_{\mathfrak{p}_0}\right) = 1$$

Proof. The reduced norm map is defined Wedderburn componentwise, so it is sufficient to show vanishing of SK_1 for each of the Wedderburn components of $\widehat{\mathcal{Q}(\mathcal{G})}_{\mathfrak{p}_0}$. The Wedderburn decomposition is as follows:

$$\begin{aligned}
\widehat{\mathcal{Q}(\mathcal{G})}_{\mathfrak{p}_0} &= \widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} \mathcal{Q}(\mathcal{G}) \\
&\simeq \widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} \prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}(D_\chi) \\
&\simeq \prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim_{\mathbb{Q}_p}} M_{n_\chi}\left(\widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} D_\chi\right)
\end{aligned}$$

Using Morita equivalence, it is therefore sufficient to show vanishing of SK_1 for each

$$\widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} D_\chi \tag{5.3}$$

where χ runs through the set of irreducible characters of $\mathcal{G}$ with open kernel.

Theorem 3.3.11 identifies D_χ with the $\mathfrak{D}$ studied above (here we use Condition 2.4.1). Since $\mathcal{Q}(\Gamma_0)$ is central in $\mathcal{Q}(\mathcal{G})$, its image in $\mathfrak{D}$ is also central. We identify $\Lambda(\Gamma_0)$ and $\mathcal{Q}(\Gamma_0)$ with their respective images in $\mathfrak{D}$. The ring $\Lambda(\Gamma_0)$ is normal, and the ring extension $\Lambda(\Gamma_0) \subseteq \mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ is integral, hence the going down theorem applies, see [Sta23, Proposition 00H8]. Consequently, every prime ideal $\mathfrak{P}_0$ of $\mathcal{O}_{\mathfrak{z}(\mathfrak{D})}$ above $\mathfrak{p}_0$ is of height 1. We apply [Neu99, Proposition II.8.3] once again:

$$\widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} \mathfrak{z}(\mathfrak{D}) \simeq \prod_{\mathfrak{P}_0 | \mathfrak{p}_0} \widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{P}_0}$$

Returning to (5.3), we have

$$\begin{aligned}
\widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} D_\chi &\simeq \widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} \mathfrak{D} \\
&\simeq \widehat{\mathcal{Q}(\Gamma_0)}_{\mathfrak{p}_0} \otimes_{\mathcal{Q}(\Gamma_0)} \mathfrak{z}(\mathfrak{D}) \otimes_{\mathfrak{z}(\mathfrak{D})} \mathfrak{D}
\end{aligned}$$

$$\begin{aligned}
&\simeq \prod_{\mathfrak{P}_0 | \mathfrak{p}_0} \widehat{\mathfrak{z}(\mathfrak{O})}_{\mathfrak{P}_0} \otimes_{\mathfrak{z}(\mathfrak{O})} \mathfrak{O} \\
&\simeq \prod_{\mathfrak{P}_0 | \mathfrak{p}_0} \widehat{\mathfrak{O}}_{\mathfrak{P}_0}
\end{aligned}$$

Each of these $\widehat{\mathfrak{O}}_{\mathfrak{P}_0}$ have trivial SK_1-group by Lemma 5.4.1, and the assertion follows. □

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List of notation

See Section 0.1 for standard notation used throughout the text.

$\sim_{\mathbb{Q}_p}$	$\eta \sim_{\mathbb{Q}_p} \eta' \;\Leftrightarrow\; \exists \sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p)$ such that ${}^{\sigma}\eta = \eta'$	2
$\sim_{\mathbb{Q}_p}$	$\chi \sim_{\mathbb{Q}_p} \chi' \;\Leftrightarrow\; \exists \sigma \in \mathrm{Gal}(\mathbb{Q}_{p,\chi}/\mathbb{Q}_p)$ such that ${}^{\sigma}(\mathrm{res}^{\mathcal{G}}_H \chi) = \mathrm{res}^{\mathcal{G}}_H \chi'$	4
$\mathbb{1}$	trivial character	1
$\mathbf{1}_n$	$n \times n$ identity matrix	1
A_{i/v_χ}	$\tau(Y_\eta) \cdots \tau^{i/v_\chi}(Y_\eta)$, controls δ^i_γ-action on $M_{n_\eta}(D_\eta)$	37
a_{i/v_χ}	$\delta_\tau(y_\eta) \cdots \delta_\tau^{i/v_\chi}(y_\eta)$, controls γ^i-conjugation action on $\mathcal{Q}(\Gamma_0)[H]\varepsilon(\eta)$	37
χ	irreducible character of $\mathcal{G}$ with open kernel	3
χ_{cyc}	$\mathrm{Gal}(K(\mu_{p^\infty})/K) \to \mathbb{Z}_p^\times$ cyclotomic character	15
D_χ	skew field in the Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$	25
D_η	skew field in the Wedderburn decomposition of $\mathbb{Q}_p[H]$	24
$\widetilde{D}_\eta$	$\mathcal{Q}^{\mathbb{Q}_p(\eta)}(\Gamma_0) \otimes_{\mathbb{Q}_p(\eta)} D_\eta$, skew field in the Wedderburn decomposition of $\mathcal{Q}(\Gamma_0)[H]$	25
$\mathfrak{D}$	$\mathrm{Quot}(\mathcal{O}_{D_\eta}[[X; \tau, \tau - \mathrm{id}]])$, skew field, isomorphic to D_χ under Condition 2.4.1	59
$\widehat{\mathfrak{D}}_{\mathfrak{p}}$	$\widehat{\mathfrak{z}(\mathfrak{D})}_{\mathfrak{p}} \otimes_{\mathfrak{z}(\mathfrak{D})} \mathfrak{D}$, completion of $\mathfrak{D}$	83
$\mathfrak{E}$	$\mathrm{Frac}\left(\mathcal{O}_{\mathbb{Q}_p(\eta)(\omega_\eta)}[[(1+X)^{w_\chi/v_\chi} - 1]]\right)$, maximal subfield of $\mathfrak{D}$	59
$\mathfrak{E}(E)$	$\mathrm{Frac}(\mathcal{O}_E[[(1+X)^{w_\chi/v_\chi} - 1]])$ for a maximal subfield $E \subseteq D_\eta$	80
$\widehat{\mathfrak{E}(E)}_{\mathfrak{P}}$	completed localisation of $\mathfrak{E}(E)$ at a height 1 prime $\mathfrak{P}$	81
e_χ	sum of $e(\eta)$s over $\mathcal{G}/\mathrm{St}(\eta)$-orbits, idempotent in $\mathcal{F}_\chi[H]$	3
$e(\eta)$	idempotent in $\mathcal{F}(\eta)[H]$	3
ε_χ	sum of $\varepsilon(\eta)$s over $\mathrm{Gal}(\mathbb{Q}_{p,\chi}/\mathbb{Q}_p)$-orbits, idempotent in $\mathbb{Q}_p(\eta)[H]$	3
$\varepsilon(\eta)$	idempotent in $\mathcal{F}[H]$	3
η	irreducible constituent of $\mathrm{res}^{\mathcal{G}}_H \chi$	3
$\eta_{(i)}$	irreducible constituent of $\mathrm{res}^{\mathcal{G}}_H \chi$ s.t. $\sum_i \sum_{\sigma \in \mathrm{Gal}(\mathcal{F}(\eta_{(i)})/\mathcal{F}_\chi)} {}^{\sigma}\eta_{(i)} = \mathrm{res}^{\mathcal{G}}_H \chi$	3
${}^{g}\eta$	${}^{g}\eta(h) = \eta(ghg^{-1})$ for all $h \in H$	3
${}^{\sigma}\eta$	${}^{\sigma}\eta(h) = \sigma(\eta(h))$ for all $h \in H$	3
$\mathcal{F}$	finite extension of $\mathbb{Q}_p$	3
$f^{(j)}_\chi$	idempotent in $\varepsilon_\chi \mathcal{Q}(\mathcal{G})$ corresponding to $\mathrm{diag}(0, \ldots, 1 \ldots, 0) \in M_{n_\chi}(D_\chi)$	26
$f^{(j)}_\eta$	idempotent in $\varepsilon(\eta)\mathbb{Q}_p[H]$ corresponding to $\mathrm{diag}(0, \ldots, 1 \ldots, 0) \in M_{n_\eta}(D_\eta)$	26
$\mathcal{G}$	$\mathrm{Gal}(L^+_\infty/K) \simeq H \rtimes \Gamma$, 1-dimensional p-adic Lie group	14

Π	$\operatorname{diag}(1,0,\ldots,0)M_n(\mathscr{D})\operatorname{diag}(1,0,\ldots,0) \xrightarrow{\sim} \mathscr{D}$	38
π	$f_\eta^{(1)}\mathcal{Q}(\Gamma_0)[H]f_\eta^{(1)} \xrightarrow{\sim} \widetilde{D}_\eta$ defined by Π and the Wedderburn isomorphism	39
π_η	uniformiser of $\mathbb{Q}_p(\eta)$	24
π_{D_η}	s_ηth root of π_η	24
$\mathcal{Q}(\mathcal{G})$	$\operatorname{Quot}(\Lambda(\mathcal{G})) = \operatorname{Quot}(\mathbb{Z}_p[\![\mathcal{G}]\!])$, total ring of quotients	3
$\mathcal{Q}^{\mathrm{c}}(\mathcal{G})$	$\mathbb{Q}_p^{\mathrm{c}} \otimes_{\mathbb{Q}_p} \mathcal{Q}(\mathcal{G})$	3
$\mathbb{Q}_{p,\chi}$	$\mathbb{Q}_p(\chi(h) : h \in H)$	3
$\mathbb{Q}_p(\eta)$	$\mathbb{Q}_p(\eta(h) : h \in H)$	3
r_η	r_η/s_η is the Hasse invariant of D_η	34
$R_p(\mathcal{G})$	additive group generated by $\mathbb{Q}_p^{\mathrm{c}}$-valued characters of $\mathcal{G}$ with open kernel	16
S	finite set of places of K containing all places ramifying in L_∞/K	14
$\operatorname{St}(\eta)$	stabiliser of η in $\mathcal{G}$	3
s_χ	Schur index of χ	25
s_η	Schur index of η	25
Σ_χ	unique maximal order in D_χ	62
T	finite set of places of K disjoint from S subject to some conditions	14
τ	unique generator of $\operatorname{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi})$ such that ${}^{\gamma^{v_\chi}}\eta = {}^{\tau}\eta$	32
τ	extension of τ to D_η or $\widetilde{D}_\eta$	34
V_χ	representation affording χ	14
v_χ	$\min\left\{0 < i \le w_\chi : \exists \tau \in \operatorname{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p), {}^{\gamma^i}\eta = {}^{\tau}\eta\right\}$	32
w_χ	index of the stabiliser of η in $\mathcal{G}$	3
w_∞	place of L_∞^+ above v	14
X_S	S-ramified Iwasawa module	19
Y_η	matrix in $\mathrm{GL}_{n_\eta}(D_\eta)$ corresponding to y_η under the Wedderburn isomorphism	36
y_η	unit in $\mathbb{Q}_p[H]\varepsilon(\eta)$ such that $\delta_\tau^{-1}\left({}^{\gamma^{v_\chi}}(-)\right) = y_\eta(-)y_\eta^{-1}$	36
$[Y_S^T]$	class associated with Y_S^T in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$	20
$\widehat{\mathfrak{z}(\mathfrak{D})_\mathfrak{p}}$	completed localisation of $\mathfrak{z}(\mathfrak{D})$ at a height 1 prime $\mathfrak{p}$	79