Adèles and idèles

Recall. Complete normed fields, e.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $K/\mathbb{Q}_p$, $\mathbb{F}_p((T))$, $\mathbb{C}((t))$

For a number field $K/\mathbb{Q}$, a place of K is an eq. class of a non-trivial norm on K.

$$\{\text{places of } K\} \longleftrightarrow \left\{\begin{matrix}\text{max. ideals of}\\ \mathcal{O}_K\end{matrix}\right\} \coprod \left\{\begin{matrix}\text{complex embeddings}\\ K \hookrightarrow \mathbb{C}\end{matrix}\right\} \Big/ \text{complex conjugation}$$

$$|\cdot|_{\mathfrak{p}} : K \longrightarrow \mathbb{R}_{\geq 0} \longleftarrow\!\!\shortmid \ \mathfrak{p}$$
$$x \longmapsto (N\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$$

$$|\cdot|_{\sigma} : K \longrightarrow \mathbb{R}_{\geq 0} \longleftarrow\!\!\shortmid \ \sigma$$
$$x \longmapsto |\sigma(x)|_{\mathbb{C}}$$

We have the product formula: $\prod_v |x|_v = 1$

where $|\cdot|_v$ is the normalised norm corresponding to v.

Motivation. Try to consider all completions of a number field K at the same time.

$K \hookrightarrow$ "$\prod_v K_v$" where K_v is the completion of K wrt. v.

$\prod_v K_v$ as a product space does not have nice topological properties, so we need to consider sth. else: that will be the adèle

$$\begin{matrix} K & \hookrightarrow & \prod_v K_v \\ & \searrow & \cup\!\!\shortmid \\ & & \mathbb{A}_K \end{matrix}$$

Review of topological groups

Def. A topological group is a group G equipped with a topology s.t.

$$G \times G \longrightarrow G \qquad\quad G \longrightarrow G$$
$$(x,y) \longmapsto x\cdot y \quad \text{and} \quad x \longmapsto x^{-1}$$

or equivalently, $\varphi : G \times G \longrightarrow G$

$$(x,y) \longmapsto xy^{-1}$$

is continuous.

Ex. 1) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $K/\mathbb{Q}_p$

2) $GL_n(\mathbb{R})$, $GL_n(K)$

Lemma. G a top. group. TFAE:

1) G is T2
2) Every point of G is closed (i.e. G is T1)
3) $e \in G$ is closed

Pf: 1) $\to$ 2) $\Rightarrow$ 3) triv

3) $\to$ 1).

$$\begin{tikzcd}
G \times G \arrow[r, "\varphi_G"] & G \\
\varphi^{-1}(\{e\}) = G \arrow[u, hook, "\Delta"] \arrow[r] & \{e\} \arrow[u, hook, "\text{closed}"']
\end{tikzcd} \Rightarrow \Delta \text{ is closed} \Leftrightarrow T2.$$

$\square$

Def. A locally cpt top gp G is a Hausdorff top. gp G s.t. every point (or equivalently, $e \in G$) has a compact nbh.

Ex. 1) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $K/\mathbb{Q}_p$ finite are loc. cpt.

2) $\overline{\mathbb{F}_p}((x))$ is <u>not</u> locally cpt.

$\mathcal{O}_K \subseteq K$ is an open compact subset if $K/\mathbb{Q}_p$ is finite.

To see this, use $\mathcal{O}_K = \varprojlim_n \mathcal{O}_K/\mathfrak{m}_K^n$

Algebraic Number Theory, lecture 22

Lemma. Let $(X_i)_{i\in I}$ be a system of cpt. Hausdorff spaces indexed by a partially ordered set I. Then $\varprojlim_{i\in I} X_i$ is compact in $\prod_{i\in I} X_i$.

Pf: 1) Recall Tychonoff's Theorem: $\prod_{i\in I} X_i$ is T2 and compact.

2) $$\varprojlim_{i\in I} X_i = \left\{ (x_k)_{k\in I} \in \prod X_k \;\middle|\; \alpha_{ji}(x_j) = x_i \quad \forall i\le j \right\}$$

$$= \bigcap_{i\le j} \left\{ (x_k)_{k\in I} \;\middle|\; \underbrace{\alpha_{ji}(x_j) = x_i}_{\text{closed condition}} \right\}$$

$\Rightarrow$ $\varprojlim X_i$ is closed in $\prod_{i\in I} X_i$, hence compact. □

Subgroups

Lemma. 1) Let G be a top. gp., $H\subseteq G$ a subgroup. $\Rightarrow$ $\overline{H}$ is a top. gp.

2) Every open subgp. is closed.

Pf: $$\varphi_G : G\times G \longrightarrow G$$
$$(x,y) \longmapsto x\cdot y^{-1}$$

$\varphi_G(H\times H) \subseteq H$, $\quad H\times H \subseteq \varphi_G^{-1}(H)$

$\varphi_G^{-1}(\overline{H}) \supseteq \overline{H\times H} = \overline{H}\times\overline{H}$ $\quad\Rightarrow\quad$ $\overline{H}$ is stable under multiplication and inversion, hence a top. gp.

2) $H \subset G$

$$H = \underbrace{G \setminus \Big(\overset{\text{open}}{\bigcup_{gH\neq H} gH} \Big)}_{\text{closed}}$$

□

Prop. Every locally closed subgp $H\subset G$ is closed.

Pf: $$\underset{\substack{\uparrow \\ \text{open (hence closed)}}}{H} \subseteq \underset{\substack{\uparrow \\ \text{closed}}}{\overline{H}} \subseteq G$$

□

Cor. Every locally compact subgroup of a Hausdorff top. gp. is closed. In particular, a discrete subgroup is automatically closed. □

Pf: Since any compact subset of a T2-space is closed, every locally compact subset of a T2-space is locally closed, and the Prop. can be used. □

Cor. A subgroup H of a locally compact topological group G is closed iff it is locally compact.

Pf: We have seen that loc. cpt. $\Rightarrow$ closed.
If H is closed □

Ex. $\mathbb{Z} \subseteq \mathbb{R}$ discrete, hence closed, subgroup of $\mathbb{R}$

$\mathbb{Z}_p \subseteq \mathbb{Q}_p$ open, hence closed, subgroup of $\mathbb{Q}_p$

$GL_n(\mathbb{Z}_p) \subseteq GL_n(\mathbb{Q}_p)$ open compact

Quotients

G topological group, $H \subseteq G$ subgroup

$q: G \longrightarrow G/H$ quotient map (Note that G/H need not be a group.)

G/H has a natural quotient topology: a subset $U \subseteq G/H$ is open iff $q^{-1}(U)$ is open

Prop. 1) $q: G \longrightarrow G/H$ is an open map, i.e. $q(\text{open}) \subseteq \text{open}$

2) If H is a normal subgroup, then G/H is a top group and q is continuous.

Pf: 1) $V \subseteq G$ open

$$q^{-1}(q(V)) = \bigcup_{h \in H} Vh \text{ open} \Rightarrow q(V) \text{ is open by def. of the quotient topology}$$

2)
$$\begin{array}{ccc}
(x,y) & \longmapsto & xy^{-1} \\
G \times G & \xrightarrow{\varphi_G} & G \\
\Big\downarrow q\times q & & \Big\downarrow q \\
G/H \times G/H & \xrightarrow{\varphi_{G/H}} & G/H \\
\cup\!| & & \\
\Delta_{G/H} = \varphi_{G/H}^{-1}(\{H\}) & &
\end{array}$$

$\forall U \subseteq G/H$ open:

$$\varphi_{G/H}^{-1}(U) = \underbrace{(q\times q)\Big(\underbrace{\varphi_G^{-1} q^{-1}(U)}_{\text{open since } G \text{ is a top. group}}\Big)}_{\text{open since } q\times q \text{ is an open map}}$$

$\Rightarrow \varphi_{G/H}$ is continuous □

Prop. 1) H is a closed subgroup in G iff G/H is T2

2) H is an open subgroup in G iff G/H is discrete

3) If G is locally compact and H is closed, then G/H is locally compact.

Pf: 1) • If G/H is Hausdorff $\Rightarrow \{H\} \subseteq G/H$ is closed

$\Rightarrow q^{-1}(\{H\}) = H \subseteq G$ is closed.

• If $H \subseteq G$ is closed, then consider the diagram on the bottom of p. 138. with the black extension

$$(q \times q)^{-1}(\Delta_{G/H}) = \varphi_G^{-1}\, q^{-1}(\{H\}) = \underset{\substack{\uparrow \\ \text{closed}}}{\varphi_G^{-1}(H)} \subseteq G \times G$$

$$\Rightarrow \Delta_{G/H} \subseteq G/H \times G/H \text{ is closed}$$

$$\Delta_{G/H} = \underset{\substack{\uparrow \\ \text{open surjective}}}{(q \times q)}\Big((q \times q)^{-1}(\Delta_{G/H})\Big)$$

2) H open $\Leftrightarrow$ the cosets gH are open

$\Leftrightarrow$ all points in G/H are discrete

3) $q : G \longrightarrow G/H$ is open, hence it sends a compact nbhood of any point onto a cpt nbhood of any point in G/H $\square$

Ex. $\mathbb{Z} \subseteq \mathbb{Z}_p$ dense

$\mathbb{Z}_p/\mathbb{Z}$ has the trivial topology: (only $\emptyset$ and itself are open)

if $\emptyset \neq S \subseteq \mathbb{Z}_p/\mathbb{Z}$ closed $\Rightarrow q^{-1}(S) \subseteq \mathbb{Z}_p$ is a closed subset containing $\mathbb{Z}$,

hence $q^{-1}(S) = \mathbb{Z}_p$ because $\mathbb{Z}$ is dense.

Prop. Let $f : G \twoheadrightarrow H$ be a continuous surj. map of top. groups.

Then f induces a bijective continuous map $f' : G/\ker(f) \longrightarrow H$

If f is open, then f' is an isomorphism of top. groups, i.e. f' is a homeomorphism.

Pf: f' is clearly bijective

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "q"'] & & H \\ & G/H \arrow[ur, "f'"'] & \end{tikzcd}$$

$U \subseteq H$ open $\Rightarrow q^{-1}(f'^{-1}(U)) = f^{-1}(U)$ open

$\Rightarrow f'^{-1}(U)$ open $\Rightarrow f'$ is continuous

Assume f is open.

Take $V \subseteq G/_{\ker f}$ open $\Leftrightarrow q^{-1}(V)$ is open

$$f'(V) = f' \, q \, q^{-1}(V) = f \, q^{-1}(V)$$

open

$\Rightarrow f'$ is a homeomorphism $\square$

(This proof is just tautological.)

Ex. $G = \mathbb{R}$, $H = \mathbb{Z}$ subgroup, $L := \lambda . \mathbb{Z}$ for some $\lambda \in \mathbb{R} \setminus \mathbb{Q}$

$$\underbrace{(L+H)/_H}_{\cong \mathbb{Z}} \underset{\text{dense}}{\subseteq} G/_H = \mathbb{R}/_\mathbb{Z} \cong S^1$$

$$L \hookrightarrow G \twoheadrightarrow G/_H$$

$$f: \quad L \longrightarrow L+H/_H \underset{\text{dense}}{\subseteq}$$

$\ker(f) = L \cap H = \{0\}$

$L \subseteq \mathbb{R}$ with the discrete induced topology

f is bijective, continuous, but not an iso of top groups

Adèles

Let $(G_v)_{v \in V}$ be a family of loc. cpt. top. groups.

Assume that there is a finite subset $V_\infty \subseteq V$ s.t. $\forall v \in V_f := V \setminus V_\infty$:

G_v has an open cpt. subgroup H_v

Def. The restricted product of $(G_v)_{v \in V}$ w.r.t. $(H_v)_{v \in V_f}$ is

$$\prod_{v \in V}{}' G_v := \left\{ (x_v)_v \in \prod_v G_v \;\middle|\; x_v \in H_v \text{ for almost all } v \in V \setminus V_\infty \right\} =$$

$$= \bigcup_{\substack{V_\infty \subseteq S \subseteq V \\ \text{finite}}} G_S$$

where $G_S := \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$.

Algebraic Number Theory, lecture 22

Lemma. If each G_v is loc. cpt., then $\prod'_{v\in V} G_v$ is loc. cpt.

Pf: Let $S \supseteq V_\infty$ be a finite set.

Choose $C_v \subseteq G_v$ compact nbh of $e \in G_v$ for $v \in S$

Then $\prod_{v\in S} C_v \times \prod_{v\notin S} H_v$ is a cpt nbh of $e \in \prod'_v G_v$ □

Let K be a number field.

$V := \{\text{places of } K\} \supseteq V_\infty := \{\text{arch. places of } K\}$

$G_v := K_v \quad \forall v \in V$

$H_v := \mathcal{O}_{K_v} \quad \forall v \in V_f = \{\text{non-arch. places of } K\}$

Def. Adèle ring $\mathbb{A}_K$:= restricted product of $(K_v)_{v\in V}$ w.r.t. $(\mathcal{O}_{K_v})_{v\in V_f}$

$\mathbb{A}_K$ is a loc. cpt. top. ring,

$\forall v \in V$: $\quad K_v \hookrightarrow \mathbb{A}_K$ closed

$\quad x \longmapsto (0,\dots,0,x,0,\dots)$

$\uparrow$ v-place

$\exists$ diagonal embedding: $\quad K \hookrightarrow \mathbb{A}_K$

$\quad x \longmapsto (x,\dots,x,\dots)$

Thm. 1) $K \hookrightarrow \mathbb{A}_K$ is a discrete, hence closed subgroup

2) The quotient $\mathbb{A}_K/K$ is a compact Hausdorff topological group.

Pf: 1) $U = \prod_{v\in V_\infty} U_v \times \prod_{v\in V_f} \mathcal{O}_{K_v} \subseteq \mathbb{A}_K$ open

$U_v = \{x \in K_v \mid |x|_v < 1\}$

normalised norm on K_v

$U \cap K = \{x \in K \mid |x|_v < 1 \;\; \forall v \in V_\infty,\; x \in \mathcal{O}_{K_v},\; v \in V_f\}$

$U \cap K^\times = \emptyset$ by the product formula

$U \cap K = \{0\}$. $\quad \Rightarrow K$ is discrete in $\mathbb{A}_K$.

2) Lemma: $\mathcal{O}_K \hookrightarrow \prod_{v \in V_f} \mathcal{O}_{K_v}$ is dense.

Pf: CRT. $\boxed{L}$

Prop. Let $K_\infty = \prod_{v \in V_\infty} K_v$. Then $\mathbb{A}_K = K + K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v}$.

Pf: $\forall x \in \mathbb{A}_K$ $\exists a \in \mathcal{O}_K$ s.t. $a \neq 0$ and $a x_v \in \mathcal{O}_{K_v}$ $\forall v \in V_f$

Let $S \subseteq V_f$ be finite s.t. $a \in \mathcal{O}_{K_v}^\times$ $\forall v \notin S$

By the Lemma, $\forall \varepsilon > 0$ $\exists b \in \mathcal{O}_K$ s.t.

$$|b - a x_v|_v < \varepsilon \qquad \forall v \in S$$

Note that $|b - a x_v| \leq 1$ $\forall v \in V_f$, $\quad |a|_v = 1$ $\forall v \notin S$

Take ε suff. small s.t.

$$\left| \frac{b}{a} - x_v \right|_v \leq 1 \qquad \forall v \in V_f$$

Thus $x = \frac{b}{a} + \left(x - \frac{b}{a}\right) \in K + K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v}$ $\boxed{P}$

Cor. The inclusion $K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v} \hookrightarrow \mathbb{A}_K$ induces an isomorphism of top. gps

$$\left(K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v} \right) \Big/ \mathcal{O}_K \xrightarrow{\sim} \mathbb{A}_K / K$$

Pf: Prop. $\Rightarrow$ $f : K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v} \twoheadrightarrow \mathbb{A}_K / K$ is surjective and open.

$\ker(f) = K \cap K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v} = \mathcal{O}_K$ $\boxed{C}$

End of pf of Thm. 2): Fix a basis $\alpha_1, \dots, \alpha_n$ of $\mathcal{O}_K$ over $\mathbb{Z}$.

Then $D := \sum_{i=1}^n \left[-\frac{1}{2}, \frac{1}{2}\right) \alpha_i \subseteq K_\infty$ is a fundamental domain for $K_\infty / \mathcal{O}_K$.

$\Rightarrow \left(D \times \prod_{v \in V_f} \mathcal{O}_{K_v} \right)$ is a fund. domain for $\left(K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v} \right) \Big/ \mathcal{O}_K$

$\Rightarrow \underbrace{\left(\bar{D} \times \prod_{v \in V_f} \mathcal{O}_{K_v} \right)}_{\text{cpt}}$ is sent surjectively to $\left(K_\infty \times \prod_{v \in V_f} \mathcal{O}_{K_v} \right) \Big/ \mathcal{O}_K \;\Big\downarrow\wr\; \mathbb{A}_K / K$

$\rightarrow \mathbb{A}_K / \mathcal{O}_K$ is cpt. $\square$

This proof explicitly gives a fundamental domain as well.

For $K=\mathbb{Q}$, $\left[-\frac{1}{2}, \frac{1}{2}\right) \times \prod_p \mathbb{Z}_p$ is a fund. domain for $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$

11.1.2018

Review. $K/\mathbb{Q}$ number field

$$\mathbb{A}_K = \prod_v{}' K_v = \left\{ x=(x_v)_v \in \prod_v K_v \;\middle|\; x_v \in \mathcal{O}_{K_v} \text{ for almost all } v \right\}$$

$$= \bigcup_{V_\infty \subset S} \left(\prod_{v\in S} K_v \prod_{v\notin S} \mathcal{O}_{K_v} \right)$$

$K \hookrightarrow \mathbb{A}_K$ $\qquad$ K is a discrete subgroup

$x \longmapsto (x,\dots,x,\dots)$ $\qquad$ $\mathbb{A}_K/K$ is compact

$K_\infty \times \prod_v \mathcal{O}_{K_v} \hookrightarrow \mathbb{A}_K$ induces an iso $\left(K_\infty \times \prod_v \mathcal{O}_{K_v}\right)/\mathcal{O}_K \xrightarrow{\sim} \mathbb{A}_K/K$

Cor. Let $\alpha_1,\dots,\alpha_n$ be a $\mathbb{Z}$-basis.

Then $D = \underbrace{\left(\sum_{i=1}^n [0,1)\ \iota_\infty(\alpha_i)\right)}_{\subseteq K_\infty} \times \prod_v \mathcal{O}_{K_v}$,

where $\iota_\infty : K \hookrightarrow K_\infty \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^n$

is a fund. domain for $\mathbb{A}_K/K$. $\square$

Haar measures

X locally compact Hausdorff space

$C_c(X,\mathbb{R}) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous with compact support}\}$

Def. A positive Radon measure is a linear functional

$$\Lambda : C_c(X,\mathbb{R}) \longrightarrow \mathbb{R} : \quad f \longmapsto \int_X f\, d\mu$$

s.t. $\Lambda(f) \geq 0 \quad \forall f \geq 0$.

Rmk. 1) $C_c(X,\mathbb{R}) = \bigcup_K C_K(X,\mathbb{R})$ where $C_K(X,\mathbb{R}) = \left\{ \begin{array}{c}\text{continuous function on } X \text{ with} \\ \text{support in } K\end{array}\right\}$

$C_K(X,\mathbb{R}) \subseteq C(K,\mathbb{R})$ which is a Banach space with $\|f\| = \sup_{x\in X} |f(x)|$

$\Rightarrow C_K(X,\mathbb{R})$ has an induced topology

$C_c(X,\mathbb{R})$ is equipped with the finest topology s.t. the inclusions

$$C_K(X,\mathbb{R}) \hookrightarrow C_c(X,\mathbb{R})$$

are continuous.

Then a pos. Radon measure on X is a continuous map $C_c(X,\mathbb{R}) \to \mathbb{R}$.

2) This notion of positive Radon measure is equivalent to the notion of a positive measure μ on some σ-algebra containing all Borel subsets of X s.t. $\mu(K) < +\infty \quad \forall K \subseteq X$ compact.

Let G be a locally compact topological group.

$f \in C_c(G,\mathbb{R})$, $\quad g \in G$

Def. $(L_g f)(x) := f(g^{-1}x)$ left inverse.

For $\Lambda: C_c(G,\mathbb{R}) \to \mathbb{R}$ define $(L_g \Lambda)(f) := \Lambda(L_{g^{-1}} f)$

Def. A left Haar measure on G is a nonzero positive Radon measure on G s.t. $(L_g \Lambda) = \Lambda \quad \forall g \in G$.

Thm. (Haar) For every loc. cpt. top. gp. G there is a left Haar measure, unique up to positive scalars.

PO, the proof can be found in any book on Harmonic Analysis. □

Lemma. μ a left Haar measure on G. Then

1) $\mu(U) > 0 \quad \forall U \subseteq G$ open

2) $\forall f \in C_c(G,\mathbb{R})$ with $f \geq 0$ and $f \neq 0$ (not identically zero) we have

$$\int_G f \, d\mu > 0.$$

Pf: 1) $\mu \neq 0 \Rightarrow \exists K \subseteq G$ compact subset with $\mu(K) > 0$

Cover K by left translates of U. There is a finite subcovering by compactness.

$$K \subseteq \bigcup_{i=1}^n g_i U \Rightarrow \underbrace{\mu(K)}_{0<} \leq \sum_{i=1}^n \mu(g_i U) \underset{\text{left invariance}}{=} n\,\mu(U) \Rightarrow \mu(U) > 0.$$

2) $\exists \varepsilon > 0 : \quad \mathcal{U}_\varepsilon := f^{-1}(\varepsilon, \infty) \subseteq G$ open

Take ε so that $\mathcal{U}_\varepsilon$ is non-empty.

$\Rightarrow \int f \, d\mu \geq \varepsilon \, \mu(\mathcal{U}_\varepsilon) > 0$

$\square$

Examples. 1) G discrete, μ counting measure $\longrightarrow$ is a Haar measure

(For $E \subseteq G$ finite, $\mu(E) = \# E$).

2) $G = \mathbb{R}$, the Lebesgue measure dx is a Haar measure

3) $G = \mathbb{R}^\times$, $\frac{dx}{|x|}$

4) $G = \mathbb{Q}_p \supseteq \mathbb{Z}_p$, normalise the Haar measure by $\mu(\mathbb{Z}_p) = 1$

$$\mathbb{Z}_p = \coprod_{i=0}^{p^n - 1} (i + p^n \mathbb{Z}_p)$$

$$\mu(\mathbb{Z}_p) = \sum_{i=0}^{p^n-1} \mu(i + p^n \mathbb{Z}_p) = \sum_{i=0}^{p^n-1} \mu(p^n \mathbb{Z}_p) = p^n \mu(p^n \mathbb{Z}_p)$$

$$\Rightarrow \mu(p^n \mathbb{Z}_p) = \frac{1}{p^n} \quad \forall n \in \mathbb{Z}.$$

5) $K/\mathbb{Q}_p$ finite extension, $\mu(\mathcal{O}_K) = 1$ normalisation

$\pi \in \mathcal{O}_K$ uniformiser, $q := \#\left(\mathcal{O}_K/(\pi)\right)$

$$\#\left(\mathcal{O}_K/\pi^n \mathcal{O}_K\right) = q^n \quad \Rightarrow \quad \mu(\pi^n \mathcal{O}_K) = q^{-n} \quad \forall n \in \mathbb{Z}$$

6) $G = \mathbb{Q}_p^\times$, $d\mu$ = restriction of the normalised Haar measure on $\mathbb{Q}_p$.

$\longrightarrow d\mu$ is <u>not</u> a Haar measure on $\mathbb{Q}_p^\times$.

$\frac{d\mu}{|x|_p} \underline{\underline{\text{is}}}$ a Haar measure, just as in the real case.

$$\int_{\mathbb{Z}_p^\times} \frac{d\mu}{|x|_p} = \int_{\mathbb{Z}_p^\times} d\mu = \mu(\mathbb{Z}_p) - \mu(p\mathbb{Z}_p) = 1 - \frac{1}{p}$$

$$\int_{p^n \mathbb{Z}_p^\times} \frac{d\mu}{|x|_p} = \int_{p^n \mathbb{Z}_p^\times} d\mu \cdot p^{+n} = p^{+n} \cdot \mu(p^n \mathbb{Z}_p^\times) = p^{n} \cdot p^{-n} \cdot \mu(\mathbb{Z}_p) = \mu(\mathbb{Z}_p^\times) = 1 - \frac{1}{p}$$

7) $G = \mathbb{C}$ $\quad d\mu = dx\,dy$ is a Haar measure

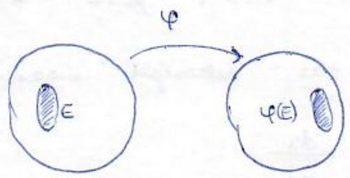
$G = \mathbb{C}^\times$ $\quad \dfrac{d\mu}{|z|^2}$ is a Haar measure

Since we can approximate a function with linear combinations of characteristic functions, what really matters is what the measures of sets are. □

Modulus

G loc cpt top gp., μ left Haar measure

$\varphi : G \longrightarrow G$ automorphism



$\forall E \subseteq G$ compact : $(\varphi^{-1}\mu)(E) := \mu(\varphi(E))$

This defines a Haar measure on G.

By the uniqueness of Haar measures, $\exists c > 0 :$ $\varphi^{-1}\mu = c\mu$

Def. $\mathrm{mod}(\varphi) = c$ is the modulus of φ.

Examples. 1) $G = \mathbb{R}$ $\quad \forall a \in \mathbb{R}^\times :$ $\quad d(ax) = |a|_{\mathbb{R}}\, dx$

2) $G = \mathbb{C}$ $\quad \forall a \in \mathbb{C}^\times :$ $\quad d(a\mu) = |a|_{\mathbb{C}}\, d\mu$

3) $G = K/\mathbb{Q}_p$ finite. $\quad \forall a \in K^\times :$ $\quad d(a\mu) = q^{-v(a)}\, d\mu$

$$\begin{aligned} v : K^\times &\longrightarrow \mathbb{Z} \\ \pi &\longmapsto 1 \end{aligned}$$ normalised additive valuation

$q := \# \mathcal{O}_K/(\pi)$ $\qquad q^{-v(a)} = |a|_K$ where $|\cdot|_K$ is the normalised absolute value on K

4) $G = \mathbb{R}^n$, $\quad \varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear transformation

$\mathrm{mod}(\varphi) = |\det(\varphi)|$.

Prop. If G is either compact or discrete, then $\mathrm{mod}\,\varphi = 1$ for any automorphism $\varphi : G \longrightarrow G$.

Pf: $\underset{\sim}{G \text{ cpt.}}$ $\quad \varphi(G) = G$

$\mu(\varphi(G)) = \mu(G) \Rightarrow (\mathrm{mod}\,\varphi)\cdot\mu(G) = \mu(G) \Rightarrow \mathrm{mod}\,\varphi = 1$

$\underset{\sim}{G \text{ discrete}}$ $\quad e \in G.$ $\quad \mu(\varphi(e)) = \mu(e) = 1$ is the counting measure. □

Algebraic Number Theory, lecture 23

Products

X, Y: loc cpt Hausdorff spaces

μ, ν: pos Radon measures on X resp Y

Thm. (Fubini) $\forall f \in C_c(X \times Y, \mathbb{R})$:

we have $y \longmapsto \int_X f(x,y)\, d\mu \in C_c(Y, \mathbb{R})$

$$x \longmapsto \int_Y f(x,y)\, d\nu \in C_c(X, \mathbb{R})$$

and $$\int_X d\mu \left(\int_Y f(x,y)\, d\nu \right) = \int_Y d\nu \left(\int_X f(x,y)\, d\mu \right)$$ □

Therefore we have a well-defined product measure $\mu \times \nu$ on $X \times Y$.

Rmk. If X, Y are loc cpt groups and μ, ν are left Haar measures then $\mu \times \nu$ is a Haar measure on $X \times Y$.

Infinite products

$(X_i)_{i \in I}$ a family of cpt Hausdorff spaces, $(\mu_i)_{i \in I}$ measures on $(X_i)_{i \in I}$

$$X := \prod_{i \in I} X_i$$

$\forall$ finite subset $J \subseteq I$: $\quad X_J := \prod_{j \in J} X_j$

Prop. Assume $\prod_{i \in I} \mu_i(X_i)$ converges. Then there is a unique positive Radon measure s.t. $\forall J \subseteq I$ finite: $$\int_X f_J \circ pr_J \, d\mu = \prod_{i \notin J} \mu_i(X_i) \int_{X_J} f_J \, d\mu_J$$

$\forall f_J \in C_c(X_J, \mathbb{R})$, $\quad pr_J : X \longrightarrow X_J$ projection.

Thm. (Stone–Weierstrass) Let X be a cpt Hausdorff space.

Then any subalgebra $\mathcal{A}$ of $C(X,\mathbb{R})$ that separates points is dense in $C(X,\mathbb{R})$.

(Separating points: $\forall x,y \in X,\ x \neq y\ \exists f \in \mathcal{A} : f(x) \neq f(y)$.)

□

Pf of Prop: $F := \{ f \in C(X,\mathbb{R}) \mid \exists \mathcal{J} \subseteq I \text{ finite},\ f = f_{\mathcal{J}} \circ pr_{\mathcal{J}} \text{ for some } f_{\mathcal{J}} \in C(X_{\mathcal{J}}, \mathbb{R})\}$

- F is closed under addition and multiplication (easy)

$\Rightarrow F \subseteq C(X,\mathbb{R})$ subalgebra ✓

- F separates points: let $x \neq y \in X$. $\exists i \in I : x_i \neq y_i$

Choose $f_i \in C(X_i, \mathbb{R})$ s.t. $f_i(x_i) \neq f_i(y_i)$

$\Rightarrow (f_i \circ pr_i)(x) \neq (f_i \circ pr_i)(y)$ ✓

By SW $\Rightarrow F \subseteq C(X,\mathbb{R})$ is dense

$$\Lambda : F \longrightarrow \mathbb{R} \qquad f_{\mathcal{J}} \circ pr_{\mathcal{J}} \longmapsto \prod_{i \notin \mathcal{J}} \mu_i(X_i) \cdot \int_{X_j} f_j \, d\mu_j .$$

extends uniquely to a cont functional on $C(X,\mathbb{R})$.

□

Construction

Let $(G_v)_{v \in V}$ be a family of loc cpt groups, $V_\infty \subseteq V$ finite subset.

$H_v \subseteq G_v$ open cpt. subgroups $\forall v \in V_f := V \setminus V_\infty$.

$$G = \prod_v{}' G_v = \bigcup_{\substack{V_\infty \subseteq S \subseteq V \\ \text{finite}}} G_S \quad \text{where} \quad G_S = \prod_{s \in S} G_s \prod_{v \notin S} H_v .$$

Assume $(\mu_v)_{v \in V}$ is a family of $\overset{\text{left}}{}$ Haar measures on $(G_v)_{v \in V}$ s.t.

$\prod_{v \in V_f} \mu_v(H_v)$ converges.

$\Rightarrow \exists$ a unique left Haar measure μ_S on G_S given by the Proposition.

such that $\mu_{S'}|_{G_S} = \mu_S \ \forall S \subseteq S'$ (Note that $\forall S \subseteq S'$: $G_S \hookrightarrow G_{S'}$ is open.)

$\Rightarrow \mu_S$ glue together to a left Haar measure on G.

Algebraic Number Theory, lecture 23 – 24

Application:

$K/\mathbb{Q}$ number field, $V = \{\text{places of } K\}$

$\forall v \in V$: K_v $d\mu_v$ normalised Haar measure

- If v is real: $d\mu_v$ is the Lebesgue measure on $\mathbb{R}$
- If v is complex: $d\mu_v$ is the twice of the Lebesgue measure
- If v is non-archimedean: $\mu_v(\mathcal{O}_{K_v}) = 1$

$\longrightarrow$ we get a Haar measure $d\mu = \prod_v{}' \, d\mu_v$ on $\mathbb{A}_K$.

Since $K \subseteq \mathbb{A}_K$ is discrete, $d\mu$ on $\mathbb{A}_K$ induces a Haar measure on $\mathbb{A}_K/K$.

Namely if $D \subseteq \mathbb{A}_K$ is a measurable subset which is a fundamental domain for $\mathbb{A}_K/K$, $\forall\, \mathcal{U} \subseteq \mathbb{A}_K/K$ open $\mu(\mathcal{U}) := \mu(\pi^{-1}(\mathcal{U}) \cap D)$ where

$\pi : \mathbb{A}_K \longrightarrow \mathbb{A}_K/K$

<u>Prop.</u> Under the normalised Haar measure μ on $\mathbb{A}_K$, we have

$$\mu(\mathbb{A}_K/K) = \sqrt{|\mathrm{disc}_K|}.$$

Pf: Recall that if $\alpha_1, \dots, \alpha_n$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$, then

$$D = \underbrace{\left(\sum_{i=1}^{n} \iota_\infty(\alpha_i) \times [0,1)\right)}_{D_\infty} \times \prod_{v \in V_f} \mathcal{O}_{K_v} \quad \text{is a fun. domain. for } \mathbb{A}_K/K.$$

$$\mu(D) = \mu(D_\infty) \cdot \prod_{v \in V_f} \mu(\mathcal{O}_{K_v}) = \mu(D_\infty) = \mu\left(K_\infty / \iota_\infty(\mathcal{O}_K)\right)$$

$$\begin{tikzcd} \mathcal{O}_K \arrow[r, hook, "\iota_\infty"] \arrow[drr, "\lambda"'] & K_\infty \cong & \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \arrow[d, "\cong"', "\text{Leb.} \quad 2\times\text{Leb.}"] \\ & & \mathbb{R}^{r_1+2r_2} \end{tikzcd}$$

$$\mu\left(K_\infty / \iota_\infty(\mathcal{O}_K)\right) = 2^{r_2} \,\mathrm{Vol}\left(\mathbb{R}^n / \lambda(\mathcal{O}_K)\right) = \sqrt{|\mathrm{disc}_K|} \qquad \square$$

$K \hookrightarrow A_K$ is an analogue of $\mathcal{O}_K \overset{\iota_\infty}{\hookrightarrow} K_\infty$.

Thm. (Minkowski) For constants $(C_v)_{v\in V}$, $C_v \in |K_v^\times|_v$ and $C_v = 1$ for almost all v, satisfying
$$\prod_v C_v > \left(\frac{2}{\pi}\right)^{r_2} \sqrt{|\mathrm{disc}_K|}$$
then is an $a \in K^\times$ s.t. $|a|_v < C_v \ \forall v \in V$ where $|\cdot|_v$ is the normalised absolute value on K_v.

Pf: Let
$$B_v := \begin{cases} \{x \in K_v \mid |x|_v \le C_v\} & v \text{ is non-archimedean} \\ \{x \in K_v \mid |x|_v \le \frac{C_v}{2}\} & v \text{ is real} \\ \{x \in K_v \mid |x|_{\mathbb{C}} \le \frac{\sqrt{C_v}}{2}\} & v \text{ is complex} \end{cases}$$

and $B := \prod_{v\in V} B_v \subseteq A_K$; this is a compact subset.

$$\mu(B) = \prod_v \mu_v(B_v) \qquad \mu_v(B_v) = \begin{cases} C_v & v \text{ is n-arch or real} \\ 2\pi\left(\frac{\sqrt{C_v}}{2}\right)^2 = \frac{\pi}{2} C_v & \text{if } v \text{ is complex} \end{cases}$$
$$= \left(\prod_v C_v\right)\cdot\left(\frac{\pi}{2}\right)^{r_2} > \sqrt{|\mathrm{disc}_K|} = \mu\left(A_K/K\right)$$

$\Rightarrow \exists x, y \in B: \ x \neq y, \quad a := x - y \in K \quad a \neq 0$

$|a|_v \le C_v \ \forall v \in V.$ □

Cor. Let $v_0 \in V$, $(c_v)_{v\in V\setminus\{v_0\}}$ be positive constants s.t. $c_v = 1$ for almost all v. Then $\exists a \in K^\times$ s.t. $|a|_v \le c_v \ \forall v \in V\setminus\{v_0\}$. 15. 1. 2018

Pf: Choose c_{v_0} to be sufficiently large and apply Minkowski's thm. □

Thm: (Strong Approximation)

Let $v_0 \in V$. Then the diagonal embedding $K \hookrightarrow A_K^{(v_0)} = \prod'_{v\neq v_0} K_v$ is dense.

In other words, $K + K_{v_0}$ is dense in A_K. (This is a surprising result!)

Pf: Equivalent: $\forall x \in A_K$ and $(\varepsilon_v)_{v\in V\setminus\{v_0\}}$ positive real numbers s.t. $\varepsilon_v = 1$ for almost all v. $\Rightarrow \exists a \in K$ s.t. $|a - x_v|_v < \varepsilon_v \ \forall v \in V\setminus\{v_0\}$

By the construction of the fundamental domain of A_K/K there exists some constants $(c_v)_{v\in V}$, $c_v = 1$ for almost all v s.t.

$\forall y \in A_K$ $\exists b \in K$ and $y' \in D$ s.t. $y = \underset{\substack{\cap\\ K}}{b} + \underset{\substack{\cap\\ D}}{y'}$, $|y'_v|_v \leq c_v$

There exists $\alpha \in K^\times$ s.t. $|\alpha|_v \leq \frac{\varepsilon_v}{c_v}$ $\forall v \neq v_0$ by the previous corollary.

Put $y := \alpha^{-1}x$. $\Rightarrow$ $\alpha^{-1}x = b + y'$

$$x = \underbrace{\alpha b}_{\in K} + \alpha y' \qquad |\alpha y'_v|_v \leq \frac{\varepsilon_v}{c_v}\cdot c_v = \varepsilon_v \quad \forall v \neq v_0$$

□

Rmk.
The weak approximation was $K \hookrightarrow \prod_{v\in S} K_v$, S finite

Cor. A_K/K is connected.

Pf: Let v_0 be an archimedean place. K_{v_0} is connected since $K_{v_0} \cong \mathbb{R}$ or $K_{v_0} \cong \mathbb{C}$.

The image of $K_{v_0} \hookrightarrow A_K/K$ is dense.

$\Rightarrow A_K/K$ is connected.

□

Idèles (ideal element in French)

$K/\mathbb{Q}$ number field, V and V_∞ as before

Def. The idèle group $\mathbb{I}_K$ of K is the restricted product of $(K_v^\times)_{v\in V}$ w.r.t. $(\mathcal{O}_{K_v}^\times)_{v\in V_f = V\setminus V_\infty}$:

$$\mathbb{I}_K = \bigcup_{\substack{V_\infty \subseteq S\\ \text{finite}}} \left(\prod_{v\in S} K_v^\times \prod_{v\notin S} \mathcal{O}_{K_v}^\times \right) = \left\{ x \in \prod_{v\in V} K_v^\times \;\middle|\; \begin{array}{l} x_v \in \mathcal{O}_{K_v}^\times \text{ for almost} \\ \text{all } v \in V_f \end{array} \right\}$$

$\forall x \in \mathbb{I}_K$: a typical open nbhood in $\mathbb{I}_K$ is $\prod_{v\in V} U_v$ where U_v is an open nbhd of x_v in $K_v^\times$ s.t. $U_v = \mathcal{O}_{K_v}$ for almost all v.

Rmk. • $\mathbb{I}_K$ consists exactly of the invertible elts of $\mathbb{A}_K$, i.e. $\mathbb{I}_K = \mathbb{A}_K^\times$.

• The topology on $\mathbb{I}_K$ is finer than that induced by $\mathbb{A}_K$:

For an open $U = \prod_{v \in V} U_v$ in $\mathbb{I}_K$ as above there is no open subset

$W = \prod_v W_v \subseteq \mathbb{A}_K$ s.t. $W_v = \mathcal{O}_{K_v}$ for almost all v.

Def. For $x \in \mathbb{A}_K$, define the norm of x as

$|x| := \prod_v |x_v|_v$. where v is the normalised absolute value on K_v.

Note that this product is convergent because for almost all v,

$x_v \in \mathcal{O}_{K_v} \Leftrightarrow |x_v|_v \leq 1$.

Lemma. An $x \in \mathbb{A}_K$ belongs to $\mathbb{I}_K$ iff $|x| > 0$ (i.e. $|x| \neq 0$).

Pf: "$\Rightarrow$": easy: all but finitely many terms are 1, and the rest are non-zero.

"$\Leftarrow$": Assume $x \in \mathbb{A}_K$ is not in $\mathbb{I}_K$. Then there exist infinitely many finite places $v \in V_f$ s.t. $x_v \in \pi_v \mathcal{O}_{K_v}$, where $\pi_v \in \mathcal{O}_{K_v}$ is a uniformiser.

$\Rightarrow |x_v|_v \leq |\pi_v|_v = \frac{1}{q_v}$ where $q_v := \left|\mathcal{O}_{K_v}/(\pi_v)\right| \geq 2$

$\Rightarrow |x| = \prod_v |x_v|_v = 0$ $\square$

$\Rightarrow |\cdot| : \mathbb{I}_K \longrightarrow \mathbb{R}_{>0}$ map

Prop. $|\cdot|$ is an open, continuous, surjective homomorphism and it admits a continuous section.

Pf: $|\cdot|$ is clearly surjective and continuous.

1) Let $\mathcal{U} \subseteq \mathbb{I}_K$ be an open subset. NTS: $|\mathcal{U}| \subseteq \mathbb{R}_{>0}$ is open.

We may assume $\mathcal{U} = \prod_v \mathcal{U}_v$ where $U_v = \mathcal{O}_{K_v}^\times$ for almost all v.

We may further assume that $|\mathcal{U}_v|_v$ consists of only one element. $\forall v \in V_f$

(because $|\cdot|_v : K_v^\times \longrightarrow \mathbb{R}_{>0}$ is locally constant).

But $|\cdot|_\infty := \prod_{v \in V_\infty} |\cdot|_v : K_\infty^\times := \prod_{v \in V_\infty} K_v^\times \longrightarrow \mathbb{R}_{>0}$ is open

$\Rightarrow \left| \prod_{v \in V_\infty} \mathcal{U}_v \right|_\infty$ is open in $\mathbb{R}_{>0}$ $\Rightarrow |\mathcal{U}|$ is open in $\mathbb{R}_{>0}$ ✓

2) $s: \mathbb{R}_{>0} \longrightarrow \mathbb{I}_K$

$t \longmapsto (\underbrace{\sqrt[n]{t}, \dots, \sqrt[n]{t}}_{\text{arch. places}}, 1, \dots, 1)$ gives a cont. section for $|\cdot|$ □

Let $\mathbb{I}_K^1 := \operatorname{Ker}\left(|\cdot| : \mathbb{I}_K \longrightarrow \mathbb{R}_{>0}\right) = \{x \in \mathbb{I}_K \mid |x| = 1\}$

<u>Cor.</u> $\mathbb{I}_K^1 \subseteq \mathbb{I}_K$ is a closed subgroup and $\mathbb{I}_K / \mathbb{I}_K^1 \xrightarrow{\sim} \mathbb{R}_{>0}$ canonical.

There is a non-canonical iso $\mathbb{I}_K \cong \mathbb{I}_K^1 \times s(\mathbb{R}_{>0})$ where s is a cont. section of $|\cdot|$. □

Consider the diagonal embedding

$$K^\times \hookrightarrow \mathbb{I}_K$$
$$x \longmapsto (x, x, \dots, x, \dots)$$

By the product formula, $\forall x \in K^\times$:

$$\prod_v |x|_v = 1 \qquad \Rightarrow \qquad K^\times \subseteq \mathbb{I}_K^1$$

<u>Thm.</u> $K^\times$ is a discrete subgroup of $\mathbb{I}_K$, and the quotient $\mathbb{I}_K^1 / K^\times$ is compact.

(This is analogous to the situation of adèles.)

Pf: K is discrete in $\mathbb{A}_K$, $\mathbb{I}_K$ has a finer topology $\Rightarrow K^\times$ is discrete in $\mathbb{I}_K$, hence in $\mathbb{I}_K^1$ as well.

To prove compactness, we need the following

<u>Lemma.</u> Let $c > 1$. Then for all but finitely many $v \in V$:

$|K_v^\times|_v \cap (1, c) = \emptyset \quad (\Leftrightarrow |K_v^\times|_v \cap (\tfrac{1}{c}, 1) = \emptyset)$

Pf: $|K_v^\times|_v = q_v^{\mathbb{Z}}$ for $q_v := \left| \mathcal{O}_{K_v} / (\pi_v) \right|$.

There are only finitely many rational primes (hence only finitely many q_v) $< c$. The assertion follows. □

Prop. $\mathbb{I}_K' \subseteq \mathbb{A}_K$ is a closed subset and the topology on $\mathbb{I}_K'$ coincides with the topology induced from $\mathbb{A}_K$.

Pf: Notation: $\forall x \in \mathbb{A}_K$ $\forall \varepsilon > 0$: Let $S \supseteq V_\infty$ be a finite subset of V. such that $|x_v|_v \leq 1$ $\forall v \notin S$.

$$\underbrace{\mathcal{U}_{S,\varepsilon}(x)}_{\subseteq \mathbb{A}_K \text{ open}} := \{ y \in \mathbb{A}_K \mid |y_v - x_v|_v \leq \varepsilon \ \forall v \in S, \quad |y_v|_v \leq 1 \ \forall v \notin S \}$$

To show that $\mathbb{I}_K$ $\mathbb{A}_K$ is closed, we need to prove $\forall x \in \mathbb{A}_K \setminus \mathbb{I}_K'$ $\exists \mathcal{U}_{S,\varepsilon}(x) \cap \mathbb{I}_K' = \emptyset$.

Two cases:

(1) $|x| < 1$. $\Rightarrow$ $\exists S$ as above s.t. $\prod_{v \in S} |x_v|_v < 1$

We can take ε suff. small so that $\forall y \in \mathcal{U}_{S,\varepsilon}(x)$ we have

$$|y| \leq \prod_{v \in V} |y_v|_v < 1$$

$$\Rightarrow \mathcal{U}_{S,\varepsilon}(x) \cap \mathbb{I}_K' = \emptyset$$

(2) $|x| > 1$ $\Rightarrow$ $x \in \mathbb{I}_K$ $\Rightarrow$ $\exists S \supseteq V_\infty$ finite s.t. $|x_v|_v = 1$ $\forall v \notin S$ and $|K_v^\times|_v \cap \left(\frac{1}{2|x|}, 1\right) = \emptyset$ by the lemma, and we are done.

We can take ε to be suff. small so that

$$\forall y \in \mathcal{U}_{S,\varepsilon}(x) : 1 < \prod_{v \in S} |y_v|_v < 2 \prod_{v \in S} |x_v|_v. = 2|x|$$

Then one has either $|y_v|_v = 1$ $\forall v \notin S$

or $\exists v_0 \notin S$, $|y_{v_0}|_{v_0} < 1$. $\Rightarrow$ $|y_{v_0}|_{v_0} \leq \frac{1}{2|x|}$

So one has either $|y| = \prod_{v \in S} |y_v|_v > 1$

$$\text{or} \quad |y| \leq \left(\prod_{v \in S} |y_v|_v\right) |y_{v_0}|_{v_0} < 2|x| \cdot \frac{1}{2|x|} = 1$$

In either case, $|y| \neq 1$, i.e. $y \notin \mathbb{I}_K'$. $\Rightarrow$ $\mathcal{U}_{S,\varepsilon}(x) \cap \mathbb{I}_K' = \emptyset$.

The topology on $\mathbb{I}_K'$ is clearly finer than the one induced by $\mathbb{A}_K$.

To show the converse, it suffices to show that $\forall x \in \mathbb{I}_K'$ and nbhd. W of x in $\mathbb{I}_K$ there exists some $\mathcal{U}_{S,\varepsilon}(x) \subseteq \mathbb{A}_K$ s.t.

$$\mathcal{U}_{S,\varepsilon}(x) \cap \mathbb{I}_K' \subseteq W \cap \mathbb{I}_K'$$

We may assume that

$$W = \{ y \in \mathbb{I}_K \mid |y_v - x_v|_v < \delta \ \forall v \in S, \ |y_v|_v = 1 \ \forall v \notin S \}$$

for some $S \supseteq V_\infty$ and $\delta > 0$.

Take $\varepsilon > 0$ sufficiently small so that $\forall z \in \mathcal{U}_{S,\varepsilon}(x)$ we have

$$\prod_{v \in S} |z_v|_v < 2 \prod_{v \in S} |x_v|_v = 2 \cdot |x| = 2.$$

If $|z| = 1 \Rightarrow |z_v|_v > \frac{1}{2} \ \forall v \notin S$

$$\underbrace{\left(\prod_{v\in S} |z_v|_v\right)}_{<2} \cdot \prod_{v \notin S} \underbrace{|z_v|_v}_{\leq 1} \quad (= 1)$$

since $q_v \geq 2 \ \forall v \in V_f$ and $|K_v^\times|_v = q_v^{\mathbb{Z}}$

$\Rightarrow |z_v|_v = 1 \ \forall v \notin S$.

$\Rightarrow$ if $\varepsilon < \delta$: $\mathcal{U}_{S,\varepsilon}(x) \cap \mathbb{I}_K' \subseteq W \cap \mathbb{I}_K'$ ☐

Proof of the compactness of $\mathbb{I}_K'/K$:

It suffices to find a compact subset $B \subseteq \mathbb{A}_K$ s.t.

$$B \cap \mathbb{I}_K \longrightarrow \mathbb{I}_K'/K^\times \quad \text{is surjective.}$$

(Implicitly we used the fact that the topology of $\mathbb{I}_K'$ is the same as the induced top. from $\mathbb{A}_K$.)

Fix $c > \left(\frac{2}{\pi}\right)^{r_2} \sqrt{|\mathrm{disc}_K|}$.

By Minkowski's theorem, $\forall x \in \mathbb{I}_K$ with $|x| > c$ $\exists a \in K^\times$ s.t. $|a|_v \leq |x_v|_v \ \forall v \in V$.

Now choose such an x and put $\underset{\substack{\uparrow \\ \text{cpt in } \mathbb{A}_K.}}{B} := \{ y \in \mathbb{A}_K \mid |y_v|_v \leq |x_v|_v \ \forall v \in V \}$

$\forall z \in \mathbb{I}_K$: $|z^{-1}x| = |z^{-1}| \cdot |x| = |x|$

$\underset{\text{Minkowski}}{\Rightarrow} \exists a \in K^\times$ s.t. $|a|_v \leq |z_v^{-1} x_v|_v \Rightarrow |a z_v|_v \leq |x_v|_v \ \forall v \in V$

$\Rightarrow a z_v \in B \cap \mathbb{I}_K'$

Hence the above map is surjective. ☐

Application (revisit of finiteness of class number)

$\mathcal{I}_K = \{$ fractional ideals of $K \}$

$=$ free abelian group generated by max. ideals of $\mathcal{O}_K$

Define $\mathbb{I}_K \xrightarrow{\text{div}} \mathcal{I}_K$ divisor map

$$x = (x_v) \longmapsto \prod_{v \in V_f} \wp^{n_v} \qquad n_v := v\text{-adic valuation of } x_v$$

This makes sense, the product on the right has finitely many non-1 terms.

- div is surjective
- $\text{Ker}(\text{div}) = \prod_{v \in V_\infty} K_v^\times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times$
- $\text{div}(K^\times)$ is the subgroup of principal fractional ideals

$\rightsquigarrow \overline{\text{div}} : \mathbb{I}_K / K^\times \longrightarrow \mathcal{I}_K / \mathcal{P}_K = Cl_K$

$$\text{Ker}(\overline{\text{div}}) = \left(\prod_{v \in V_\infty} K_v^\times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times \right) \Big/ \underbrace{K^\times \cap \left(\prod_{v \in V_\infty} K_v^\times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times \right)}_{\mathcal{O}_K^\times}$$

$$\Rightarrow Cl_K = \mathbb{I}_K \Big/ K^\times \cdot \underbrace{\left(\prod_{v \in V_\infty} K_v^\times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times \right)}_{K_\infty \times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times}$$

Thm. Cl_K is finite.

Pf: $K_\infty' := \{ x \in K_\infty^\times \mid \prod_{v \in V_\infty} |x_v|_v = 1 \}$

$$\mathbb{I}_K' \cap \left(K_\infty^\times \times \prod_v \mathcal{O}_{K_v}^\times \right) = K_\infty' \times \prod_v \mathcal{O}_{K_v}^\times$$

$$\mathbb{I}_K' \Big/ K^\times \left(K_\infty' \times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times \right) \xrightarrow{\sim} \mathbb{I}_K \Big/ K^\times \underbrace{\left(K_\infty^\times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times \right)}_{\text{open in } \mathbb{I}_K}$$

Algebraic Number Theory, lecture 24-25

Snake lemma.

$$\begin{tikzcd}
0 \arrow[r] & K^\times\left(K_\infty^{1} \prod \mathcal{O}_{K_v}^\times\right) \arrow[r] \arrow[d, dash] & K_\infty^\times \prod \mathcal{O}_{K_v} \arrow[r] \arrow[d, dash] & \mathbb{R}_{>0} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{I}_K' \arrow[r] \arrow[d, dash] & \mathbb{I}_K \arrow[r, "|\cdot|"] \arrow[d, dash] & \mathbb{R}_{>0} \arrow[r] & 0 \\
0 \arrow[r] & * \arrow[r, "\sim"] & * \arrow[r] & 0 &
\end{tikzcd}$$

Note that $\mathbb{I}_K' / K^\times \cdot \left(\prod \mathcal{O}_{K_v}^\times \times K_\infty^\times\right) = Cl_K$ is both compact and discrete, hence finite.

□

$K/\mathbb{Q}$ number field, $V = \{\text{places of } K\} \supseteq V_\infty$ 18.1.2018

$\mathbb{I}_K = \{ x \in \prod_{v \in V} K_v^\times \mid x_v \in \mathcal{O}_{K_v}^\times \text{ for almost all } v\}$

$\cup|$ closed

$\mathbb{I}_K' = \{ x \in \mathbb{I}_K \mid |x| = \prod_v |x_v|_v = 1 \}$

$\cup|$ discrete

$K^\times$

Ex. $K = \mathbb{Q}$, $\mathbb{I}_\mathbb{Q} \xrightarrow{\sim} \mathbb{Q}^\times \mathbb{R}_{>0} \prod_p \mathbb{Z}_p^\times$

$$x = (x_\infty, \dots, x_p, \dots) \mapsto \underbrace{\left(\operatorname{sgn}(x_\infty) \prod_p p^{v_p(x_p)}\right)}_{=: \gamma \in \mathbb{Q}^\times} \cdot \left(\gamma^{-1} x_\infty, \dots, \gamma^{-1} x_p, \dots\right)$$

Easy to check: $\gamma^{-1} x_\infty \in \mathbb{R}_{>0}$, $\gamma^{-1} x_p \in \mathbb{Z}_p^\times$

$\mathbb{I}_\mathbb{Q}' = \mathbb{Q}^\times \prod_p \mathbb{Z}_p^\times$

$\mathbb{I}_\mathbb{Q}' / \mathbb{Q}^\times = (+1) \cdot \prod_p \mathbb{Z}_p^\times$

Generalisation

Def. A modulus for K is a function $\mathfrak{m}: V \longrightarrow \mathbb{N}$, $v \longmapsto m_v$

where $m_v = 0$ if v is complex,

$m_v \in \{0,1\}$ if v is real,

$m_v = 0$ for all but finitely many $v \in V_f$.

Rmk. A modulus m for K is equivalent to a pair $(I, V_{\mathbb{R}}^+)$

where $I = \prod_{v \in V_f} \mathfrak{p}_v^{m_v}$, $V_{\mathbb{R}}^+ = \{ v \in V_\infty \mid m_v = 1 \}$

For such a modulus m of K, we associate

$\mathcal{I}_K(m) :=$ subgroup of $\mathcal{I}_K$ generated by $\{ \mathfrak{p}_v \mid m_v = 0, v \in V_f \}$

$\cup\!\!|$

$\mathcal{P}_K(m) := \{$ principal fractional ideals $x\mathcal{O}_K \mid x \in K^\times$, $v(x-1) \geq m_v$ $\forall v \in V_f, m_v > 0$ (⇑ $x \in \mathcal{O}_{K_v}^\times$ if $m_v > 0$), & $\sigma_v(x) > 0$ $\forall m \in V_\infty, m = 1 \}$

$$Cl_K(m) := \mathcal{I}_K(m) / \mathcal{P}_K(m)$$ ray class group of K for m

Ex. 1) $m = 0 \Rightarrow Cl_K(m) = Cl_K$

2) $m_v = 0$ $\forall v \in V_f$, $m_v = 1$ for v real

$\Rightarrow Cl_K(m) = \mathcal{I}_K / \mathcal{P}_K^+$ narrow class group / strict class group of K

where $\mathcal{P}_K^+ = \{ x\mathcal{O}_K \mid x \in K^\times, \sigma(x) > 0 \ \forall \sigma: K \hookrightarrow \mathbb{R} \}$

↗ totally positive principal fractional ideals

Thm. $Cl_K(m)$ is finite for any modulus m of K.

Pf: Similar to that about Cl_K.

$$\mathrm{div}: \mathbb{I}_K \longrightarrow \mathcal{I}_K \quad \text{divisor map}$$
$$x \longmapsto \prod_{v \in V_f} \mathfrak{p}_v^{v(x_v)}$$

Let $\underline{\mathbb{I}_{K,m}} = \left\{ x \in \mathbb{I}_K \;\middle|\; \begin{array}{l} v(x_v - 1) \geq m_v \quad \forall v \in V_f, \ m_v \neq 0 \\ \& \quad x_v > 0 \quad \forall v \in V_\infty, \ m_v \neq 0 \end{array} \right\}$

$$\mathbb{I}_{K,m} = \prod_{\substack{v \in V_\infty \\ m_v = 1}} K_{v,+}^\times \prod_{\substack{v \in V_f \\ v \neq 0}} (1 + \mathfrak{p}_v \mathcal{O}_{K_v})^\times \prod_{v \in V}{}' K_v^\times$$

$\Rightarrow \quad \mathbb{I}_{K,m} \subseteq \mathbb{I}_K$ is both open and closed

$\underline{\mathrm{div}_m}: \mathbb{I}_{K,m} \twoheadrightarrow \mathcal{I}_K(m)$ is the restriction of div, surjective

$$\underline{\mathcal{U}_{K,m}} := \ker(\mathrm{div}_m) = \prod_{\substack{v \in V_\infty \\ m_v = 1}} K_{v,+}^\times \prod_{\substack{v \in V_f \\ m_v \neq 0}} (1 + \mathfrak{p}_v \mathcal{O}_{K_v})^\times \prod_{\substack{v \in V_f \\ m_v = 0}} \mathcal{O}_{K_v}^\times \prod_{\substack{v \in V_\infty \\ m = 0}} K_v^\times \subseteq$$

$$\underset{\text{open}}{\subseteq} \mathbb{I}_{K,m}$$

$$\begin{array}{ccc} \mathbb{I}_{K,m} / \mathcal{U}_{K,m} & \cong & \mathcal{I}_K(m) \\ \cup & & \cup \\ (K^\times \cap \mathbb{I}_{K,m})\mathcal{U}_{K,m} / \mathcal{U}_{K,m} & \cong & \mathcal{P}_K(m) \end{array} \Rightarrow \quad Cl_K(m) \cong \mathbb{I}_{K,m} / (K^\times \cap \mathbb{I}_{K,m})\mathcal{U}_{K,m}$$

Note: $\mathbb{I}_{K,m} K^\times = \mathbb{I}_K$ by approximation

$$\Rightarrow \quad \mathbb{I}_K / K^\times \cong \mathbb{I}_{K,m} K^\times / K^\times \cong \mathbb{I}_{K,m} / (K^\times \cap \mathbb{I}_{K,m})$$

$$\Rightarrow \quad Cl_K(m) \cong \mathbb{I}_{K,m} / K^\times \mathcal{U}_{K,m}$$

$|\cdot| : \mathcal{U}_{K,m} \longrightarrow \mathbb{R}_{>0}$ is surjective! $\Rightarrow \quad \mathbb{I}_K = \mathbb{I}_K^1 \cdot \mathcal{U}_{K,m}$

$$\Rightarrow \quad Cl_K(m) \cong \mathbb{I}_K^1 \mathcal{U}_{K,m} / K^\times \mathcal{U}_{K,m} \cong \mathbb{I}_K^1 / K^\times \mathcal{U}_{K,m}^1 \quad \text{where } \mathcal{U}_{K,m}^1 := \mathbb{I}_K^1 \cap \mathcal{U}_{K,m} \overset{\text{open}}{\subseteq} \mathbb{I}_K^1$$

$$\mathcal{U}_{K,m}^1 = \left(\prod_{\substack{v \in V_\infty \\ m_v = 0}} K_v^\times \prod_{\substack{v \in V_\infty \\ m_v = 1}} K_{v,+} \right)^1 \prod_{\substack{v \in V_f \\ m_v \neq 0}} (1 + \mathfrak{p}_v^{m_v} \mathcal{O}_{K_v})^\times \prod_{\substack{v \in V_f \\ m_v = 0}} \mathcal{O}_{K_v}^\times$$

$\Rightarrow \quad Cl_K(m)$ is both discrete and compact, hence finite. $\square$

Cor. There exists an exact sequence of abelian groups

$$0 \longrightarrow K^\times U_K' / K^\times U_{K,m} \longrightarrow Cl_K(m) \longrightarrow Cl_K \longrightarrow 0 \quad \text{where } U_K' = U_{K,0}.$$

constant 0 modulus.

Moreover, $$K^\times U_K' / K^\times U_{K,m} \cong \left(\pi_0(\mathbb{R})^{V_{\mathbb{R}}^+} \times \prod_{\substack{v \in V_f \\ m_v > 0}} \mathcal{O}_{K,v}^\times / (1+\mathfrak{p}_v^{m_v}) \right) \Big/ \mathcal{O}_K^\times$$

where $\mathcal{O}_K^\times \hookrightarrow \prod_{v \in V_{\mathbb{R}}^+} K_v^\times \times \prod_{\substack{v \in V_f \\ m_v > 0}} \mathcal{O}_{K_v}^\times$,

$$\pi_0(\mathbb{R}^\times) \cong \mathbb{R}^\times / \mathbb{R}_{>0} \cong \{\pm 1\},$$

$$V_{\mathbb{R}}^+ = \{ v \in V_\infty \mid m_v = 1 \}.$$

To see this: $K^\times U_K' / K^\times U_{K,m} \cong U_K' \Big/ \Big(U_{K,m}' \cdot \underbrace{(K^\times \cap U_K')}_{\mathcal{O}_K^\times}\Big)$ and the prev.-proof. □

Ex. (1) $K = \mathbb{Q}$, $N \in \mathbb{Z} \setminus \{0\}$, m is the modulus for $\mathbb{Q}$ associated to N,

i.e. $m_\infty = \begin{cases} 0 & N<0 \\ 1 & N>0 \end{cases}$ and $m_p = v_p(N) \geq 0$

Since $Cl_{\mathbb{Q}} = 0$, $$Cl_{\mathbb{Q}}(m) = \begin{cases} \pi_0(\mathbb{R}^\times) \times \prod_{p \mid N} \mathbb{Z}_p^\times / (1+p^{v_p(N)}\mathbb{Z}_p) \Big/ \{\pm 1\} & \text{if } N<0 \\ \left(\prod_{p \mid N} \mathbb{Z}_p^\times / (1+p^{v_p(N)}\mathbb{Z}_p) \right) \Big/ \{\pm 1\} & \text{if } N>0 \end{cases}$$

$$= \begin{cases} (\mathbb{Z}/N\mathbb{Z})^\times & \text{if } N<0 \\ (\mathbb{Z}/N\mathbb{Z})^\times / \{\pm 1\} & \text{if } N>0 \end{cases}$$

(2) $K/\mathbb{Q}$ quadratic real, $m_v = 0 \ \forall v \in V_f$, $m_v = 1 \ \forall v \in V_\infty$

$\longrightarrow$ we get the narrow class group Cl_K^+

$$0 \longrightarrow \underbrace{\pi_0(\mathbb{R}^\times)^2 / (\mathcal{O}_K^\times)}_{\text{coker } f} \longrightarrow Cl_K^+ \longrightarrow Cl_K \longrightarrow 0$$

$$\mathcal{O}_K^\times = \{\pm 1\} \times \varepsilon^{\mathbb{Z}} \xrightarrow{f} \pi_0(\mathbb{R}^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z})^2$$

There are two cases:

$$\operatorname{coker}(f) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } N_{K/\mathbb{Q}}(\varepsilon) = 1 \\ 0 & \text{if } N_{K/\mathbb{Q}}(\varepsilon) = -1 \end{cases}$$

$$\Rightarrow \# Cl_K^+ = \begin{cases} 2\# Cl_K & \text{if } N_{K/\mathbb{Q}}(\varepsilon) = 1 \\ \# Cl_K & \text{if } N_{K/\mathbb{Q}}(\varepsilon) = -1 \end{cases}$$

Dirichlet's Theorem

Let $C_v := \{ x \in K_v^\times \mid |x_v| = 1 \}$ $\forall v \in V$.

$$\Rightarrow \quad C_v = \begin{cases} \{\pm 1\} & v \text{ is real} \\ S^1 & v \text{ is complex} \\ \mathcal{O}_{K_v}^\times & v \text{ is non-archimedean} \end{cases} \qquad \text{compact}$$

$$C := \prod_{v \in V} C_v \underset{\text{closed}}{\subseteq} \mathbb{I}_K^1 \qquad C \text{ is compact by Tychonoff}$$

Lemma. $C \cap K^\times = \mu_K = \{ \text{roots of unity in } K \}$ is a finite cyclic grp.

Pf: $C \cap K^\times$ is both compact and discrete $\Rightarrow$ finite.

$\Rightarrow$ Every element is of finite order $\Rightarrow$ $C \cap K^\times \subseteq \mu_K$.

Conversely one can directly check that $\mu_K \subseteq C \cap K^\times$ $\square$

Def. S a finite subset of V containing V_∞. An $x \in K$ is called an S-integer resp. S-unit if $|x_v| \le 1$ $\forall v \notin S$. resp. $|x_v| = 1$ $\forall v \notin S$.

$$\mathcal{O}_{K,S} := \{ S\text{-integers in } K \} \overset{\text{subring}}{\subseteq} K$$

$$\cup\!\!\!|$$

$$\mathcal{O}_{K,S}^\times = \{ S\text{-units in } K \}$$

Ex. (1) $S = V_\infty$, $\mathcal{O}_{K,S} = \mathcal{O}_K$, $\mathcal{O}_{K,S}^\times = \mathcal{O}_K^\times$

(2) $K = \mathbb{Q}$, $S = \{\infty, p_1, \dots, p_r\}$ $\qquad \mathcal{O}_{K,S} = \mathbb{Z}\left[\frac{1}{p_1 \cdots p_r}\right]$

$$\mathcal{O}_{K,S}^\times = \underbrace{\{\pm 1\}}_{\mu_{\mathbb{Q}}} \times \underbrace{p_1^{\mathbb{Z}} \times \dots \times p_r^{\mathbb{Z}}}_{\text{free of rk } r = \#S - 1}.$$

Thm. (Dirichlet) We have an isomorphism

$$\mathcal{O}_{K,S}^\times \cong \mu_K \times L$$

where L is a free abelian group of rk $\#S-1$.

Rmk. $S = V_\infty$ yields $\#S = r_1 + r_2 \Rightarrow$ Dirichlet's original thm.

Pf: $\mathbb{I}_{K,S} := \{ x \in \mathbb{I}_K \mid x_v \in \mathcal{O}_{K_v}^\times \ \forall v \notin S \} \underset{\text{open}}{\subseteq} \mathbb{I}_K$

$$\mathbb{I}_{K,S} \cap K^\times = \mathcal{O}_{K,S}^\times \underset{\substack{\text{discrete}\\\text{subgp.}}}{\hookrightarrow} \mathbb{I}_{K,S}$$

Consider $\ell_S : \mathbb{I}_{K,S} \longrightarrow \mathbb{R}^S$

$$x \longmapsto (\log |x_v|_v)_{v \in S}$$

$\ker(\ell_S) = \{ x \in \mathbb{I}_{K,S} \mid \log |x_v|_v = 0 \ \ \forall v \in S \} =$

$= \{ x \in \mathbb{I}_{K,S} \mid |x_v|_v = 1 \ \ \forall v \in S \} =$ $\qquad x_v \in \mathcal{O}_{K_v}^\times \Leftrightarrow |x_v|_v = 1$

$= C$

$\mathrm{im}(\ell_S) = \mathbb{R}^{V_\infty} \times \prod_{v \in S \setminus V_\infty} \log(q_v)\mathbb{Z}$ where $q_v = \#\left(\mathcal{O}_{K_v}/(\pi_v)\right)$

$$\Rightarrow \mathbb{R}^S / \mathrm{im}(\ell_S) \cong \prod_{v \in S \setminus V_\infty} \mathbb{R}/\log(q_v) \cong \underbrace{(\mathbb{R}/\mathbb{Z})^{S \setminus V_\infty}}_{\text{compact}}$$

$$0 \longrightarrow C \longrightarrow \mathbb{I}_{K,S} \xrightarrow{\ell_S} \mathrm{im}(\ell_S) \longrightarrow 0 \quad \text{exact}$$

$$\mathrm{im}(\ell_S) \subseteq \mathbb{R}^S$$

If $x \in \mathbb{I}_{K,S}^1 := \mathbb{I}_{K,S} \cap \mathbb{I}_K^1$ then

$$\sum_{v \in S} \log |x_v|_v = \log \left(\prod_{v \in S} |x_v|_v \right) = \log |x| = 0$$

If $H_S := \{ y \in \mathbb{R}^S \mid \sum_{v \in S} y_v = 0 \}$ then $\ell_S(\mathbb{I}_{K,S}^1) \subseteq H_S$

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & \mathbb{I}_{K,S}^1 \arrow[r, "\ell_S", "\text{open}"'] & \ell_S(\mathbb{I}_{K,S}^1) \arrow[r] & 0 \\
0 \arrow[r] & \mu_K \arrow[r] \arrow[u, hook] & (K^\times \cap \mathbb{I}_{K,S}^1) \arrow[r] \arrow[u, hook, "\text{discrete}"'] & \ell_S(\mathcal{O}_{K,S}^\times) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

$$K^\times \cap \mathbb{I}_{K,S}^1 = \mathcal{O}_{K,S}^\times$$

$$0 \longrightarrow C/\mu_K \longrightarrow \mathbb{I}^1_{K,S}/\mathcal{O}^\times_{K,S} \xrightarrow{\ell_S} \ell_S(\mathbb{I}^1_{K,S})/\ell_S(\mathcal{O}^\times_{K,S}) \longrightarrow 0$$

$$\begin{array}{ccc} \mathbb{I}^1_{K,S} & \subseteq & \mathbb{I}^1_K \\ \cup & & \cup \\ \mathcal{O}^\times_{K,S} & \subseteq & K^\times \end{array}$$ open ($\Rightarrow$ closed) subgp.

$\Rightarrow$ $\underset{\text{compact}}{\mathbb{I}^1_{K,S}/\mathcal{O}^\times_{K,S}} \subseteq \underset{\text{compact}}{\mathbb{I}^1_K/K^\times}$ is open ($\Rightarrow$ closed) subgp.

$\Rightarrow$ $\ell_S(\mathbb{I}^1_{K,S})/\ell_S(\mathcal{O}^\times_{K,S})$ is compact

Note: H_S is an $\mathbb{R}$-space of dim $\#S-1$

and $\underset{\text{compact}}{H_S/\underbrace{\ell_S(\mathbb{I}^1_{K,S})}_{=\ \ell_S(\mathbb{I}_{K,S})\cap H_S}} \underset{\text{closed}}{\subseteq} \underset{\text{compact}}{\mathbb{R}^S/\ell_S(\mathbb{I}_{K,S})}$

$\Rightarrow$ $H_S \underset{\text{quotient is compact}}{\supseteq} \ell_S(\mathbb{I}^1_{K,S}) \underset{\text{quotient is cpt}}{\supseteq} \ell_S(\mathcal{O}^\times_{K,S})$ $\Rightarrow$ $H_S/\ell_S(\mathcal{O}^\times_{K,S})$ is compact

By discreteness and openness in the diagram above (in black),

we obtain that $\ell_S(\mathcal{O}^\times_{K,S})$ is a free abelian gp of rank $\#S-1$. $\square$

Rmk.

In general, ℓ_S is not surjective. It is if $S = V_\infty$: in this case, one obtains

$$0 \longrightarrow C/\mu_K \longrightarrow \mathbb{I}^1_{K,V_\infty}/\mathcal{O}^\times_K \xrightarrow{\ell} H_{V_\infty}/\ell(\mathcal{O}^\times_K) \longrightarrow 0$$

Haar measure on $\mathbb{I}_K$

$K_v^\times$: loc cpt top gp.

Take the Haar measure $d\mu_v$ on K_v normalised as follows:

$$d\mu_v = \begin{cases} \dfrac{dx_v}{|x|} & \text{if } v \text{ is a real place} \\ \dfrac{2\,dx_v\,dy_v}{|z_v|_{\mathbb{C}}^2} & \text{if } v \text{ is complex, } z_v = x_v + iy_v \text{ the standard coordinates on } K_v \cong \mathbb{C}. \end{cases}$$

and $\mu_v(\mathcal{O}_{K_v}^\times) = 1$ for $v \in V_f$.

$\rightsquigarrow$ Haar measure $d\mu = \prod_v d\mu_v$ on $\mathbb{I}_K$.

$$0 \longrightarrow \mathbb{I}_K^1 \longrightarrow \underset{d\mu}{\mathbb{I}_K} \xrightarrow[\curvearrowleft\ \exists\ \text{section}]{|\cdot|} \underset{\frac{dx}{x}}{\mathbb{R}_{>0}}$$

$\Rightarrow$ $\mathbb{I}_K \cong \mathbb{I}_K^1 \cdot \mathbb{R}_{>0}$

$d\mu$ induces a Haar measure on $\mathbb{I}_K^1$ so that*

$$\forall D \subseteq \mathbb{I}_K^1,\ \mathcal{U} \subseteq \mathbb{R}_{>0} : \ \mu(D \times \mathcal{U}) = \mu'(D) \cdot \int_{\mathcal{U}} \frac{dx}{x}$$

<u>Thm.</u> $\mathrm{Vol}\left(\mathbb{I}_K / K^\times\right) = \dfrac{2^{r_1}(2\pi)^{r_2} R_K \cdot h}{w}$ where $w = \#\mu_K$, R_K is the regulator of K, $h = \#\mathrm{Cl}_K$

w.r.t. the Haar measure def'd above.

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<u>Rmk.</u> * alternate formulation: if $s : \mathbb{R}_{>0} \longrightarrow \mathbb{I}_K$ is a cont.-section of $|\cdot| : \mathbb{I}_K \longrightarrow \mathbb{R}_{>0}$ then $\forall C \subseteq \mathbb{I}_K^1\ \forall D \subseteq \mathbb{R}_{>0} : \mu(C \times s(D)) = \mu(C)\,\mu(D)$.

Pf: Recall: $\mathbb{I}_{K,V_\infty} := \prod_{v \in V_\infty} K_v^\times \prod_{v \notin V_\infty} \mathcal{O}_{K_v}^\times \subseteq \mathbb{I}_K$ open

$$\mathbb{I}_{K,V_\infty}^1 := \mathbb{I}_{K,V_\infty} \cap \mathbb{I}_K^1 = \Big(\prod_{v \in V_\infty} K_v^\times\Big)^1 \Big(\prod_v \mathcal{O}_{K_v}^\times\Big)$$

We have exact sequences:

$$0 \longrightarrow \mathbb{I}_{K,V_\infty}^1 \big/ K^\times \cap \mathbb{I}_{K,V_\infty}^1 \xrightarrow{\text{open}} \mathbb{I}_K^1 / K^\times \xrightarrow{\text{div}} \mathrm{Cl}_K \longrightarrow 0$$

$$K^\times \cap \mathbb{I}^1_{K,V_\infty} = \{ x \in K^\times \mid x \in \mathcal{O}_{K_v}^\times \quad \forall v \notin V_\infty \} = \mathcal{O}_K^\times$$

$$\mathrm{Vol}\left(\mathbb{I}_K^1 / K^\times\right) = \mathrm{Vol}\left(\mathbb{I}^1_{K,V_\infty} / \mathcal{O}_K^\times \right) \cdot h$$

Recall:
$$\begin{array}{ccc} \ell : \mathbb{I}_{K,V_\infty} & \longrightarrow & \mathbb{R}^{V_\infty} \\ x & \longmapsto & \left(\log |x_v|_v\right)_{v \in V_\infty} \\ \cup\!| & & \cup\!| \\ \mathbb{I}^1_{K,V_\infty} & \longrightarrow & H = \{ y \in \mathbb{R}^{V_\infty} \mid \sum_v y_v = 0 \} \\ \cup\!| & & \cup\!| \text{ lattice with compact quotient} \\ \mathcal{O}_K^\times & \xrightarrow{\ell} & \ell(\mathcal{O}_K^\times) \end{array}$$

$$\ker(\ell) = \left\{ x \in \mathbb{I}_{K,V_\infty} \;\middle|\; |x_v|_v = 1 \quad \forall v \in V_\infty \right\}$$

$$= \left\{ x \in \mathbb{I}_K \;\middle|\; |x_v|_v = 1 \quad \forall v \in V_\infty \right\} = \prod_v C_v =: C$$

We get an induced exact sequence

$$0 \longrightarrow C / \underbrace{\mathcal{O}_K^\times \cap C}_{\mu_K} \longrightarrow \mathbb{I}^1_{K,V_\infty} / \mathcal{O}_K^\times \xrightarrow{\ell} H / \ell(\mathcal{O}_K^\times) \longrightarrow 0$$

$$\Rightarrow \mathrm{Vol}\left(\mathbb{I}^1_{K,V_\infty} / \mathcal{O}_K^\times \right) = \mathrm{Vol}\left(C/\mu_K \right) \cdot \mathrm{Vol}\left(H / \ell(\mathcal{O}_K^\times) \right)$$

$$\mathrm{Vol}\left(C/\mu_K \right) = \mathrm{Vol}(C) / \underset{\text{finite subgp.}}{\# \mu_K} = \frac{1}{w} \mathrm{Vol}(C) = \frac{1}{w} \prod_v \mathrm{Vol}(C_v)$$

For v non-arch.: $C_v = \mathcal{O}_{K_v}^\times$, $\mathrm{Vol}(C_v) = 1$

v real: $C_v = \{ x \in \mathbb{R}^\times \mid |x| = 1 \} = \{ \pm 1 \} \Rightarrow \mathrm{Vol}(C_v) = 2$

v complex: $C_v = \{ x \in \mathbb{C}^\times \mid |x| = 1 \} = S^1$

The Haar measure on $\mathbb{C}^\times \cong K_v^\times$ is given by

$$d\mu = \frac{2\,dx\,dy}{|z|_{\mathbb{C}}^2} = \frac{2\,dr\,d\vartheta}{r^2} = \frac{d(r^2)}{r^2}\,d\vartheta$$

$$0 \to \underset{\frac{d(r^2)}{r^2}d\vartheta}{S^1 \longrightarrow \mathbb{C}^\times} \xrightarrow{|\cdot|_v = |\cdot|_{\mathbb{C}}^2} \underset{\frac{dx}{x}}{\mathbb{R}^\times_{>0}} \longrightarrow 0 \quad \text{exact sequence}$$

The induced Haar measure on S^1 is $d\vartheta$.

$$\Rightarrow \operatorname{Vol}(C_v) = \int_0^{2\pi} d\theta = 2\pi \quad \text{for } v \text{ complex.}$$

Hence $\operatorname{Vol}\left(C/\mu_K\right) = \dfrac{2^{r_1}(2\pi)^{r_2}}{w}$

To finish the proof, it suffices to see $\operatorname{Vol}\left(H/\ell(\mathcal{O}_K^\times)\right) = R_K$.

Recall: $\mathcal{O}_K^\times = \mu_K \cdot \varepsilon_1^{\mathbb{Z}} \cdots \varepsilon_{r_1+r_2-1}^{\mathbb{Z}}$,

$$\eta_i := \ell(\varepsilon_i) \in H \leq \mathbb{R}^{V_\infty} \qquad i = 1, \dots, r_1+r_2-1$$

$$\eta_0 := \frac{1}{r_1+r_2}(1, \dots, 1) \in \mathbb{R}^{V_\infty}$$

Then $\quad R_K = \left| \det(\eta_0, \eta_1, \dots, \eta_{r_1+r_2-1}) \right|$.

We have 2 exact sequences:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\mathbb{I}^1_{K,V_\infty}/\mathcal{O}_K^\times \arrow[r, "\ell"] \arrow[d, hook] & H/\ell(\mathcal{O}_K^\times) \arrow[d] & \supseteq D^H_{\log} \\
\mathbb{I}_{K,V_\infty}/\mathcal{O}_K^\times \arrow[r, "\ell"] \arrow[d, "1\cdot 1"] & \mathbb{R}^{V_\infty}/\ell(\mathcal{O}_K^\times) \ni y \arrow[d] & \supseteq D_{\log} \arrow[d] \\
\mathbb{R}_{>0} \arrow[r, "\log", "\sim"'] \arrow[d] & \mathbb{R} \arrow[d] & \ni \sum_{v \in V_\infty} y_v \supseteq [0,1] \\
0 & 0 &
\end{tikzcd}$$

$$D^H_{\log} := \left\{ \sum_{i=1}^{r_1+r_2-1} t_i \eta_i \;\middle|\; t_i \in [0,1) \right\} \subseteq H \qquad \text{fund. domain for } H/\ell(\mathcal{O}_K^\times)$$

$$D_{\log} := \left\{ \sum_{i=0}^{r_1+r_2-1} t_i \eta_i \;\middle|\; t_i \in [0,1) \right\} \subseteq \mathbb{R}^{V_\infty} \qquad \text{fund. domain for } \mathbb{R}^{V_\infty}/\ell(\mathcal{O}_K^\times)$$

$$\operatorname{Vol}\left(H/\ell(\mathcal{O}_K^\times)\right) \operatorname{Vol}([0,1)) = \operatorname{Vol}(D_{\log})$$

$$\Rightarrow \operatorname{Vol}\left(H/\ell(\mathcal{O}_K^\times)\right) = \left| \det(\eta_0, \dots, \eta_{r_1+r_2-1}) \right| = R_K$$

$\square$

Algebraic Number Theory, lecture 25

Review of Pontryagin Dual

Recall: G finite abelian gp.

$$\widehat{G} := \mathrm{Hom}(G, S^1)$$

There is a canonical iso $G \xrightarrow{\sim} \widehat{\widehat{G}}$.

This is the Pontryagin duality for fin. ab. groups.

Aim: generalise this for loc. cpt. abelian groups.

Let G be a Hausdorff top. ab. group.

Def. A unitary character of G is a continuous homomorphism $\chi : G \longrightarrow S^1$.

$\widehat{G} = \{ \chi : G \to S^1 \text{ unitary character} \}$, $(\chi_1 \chi_2)(y) = \chi_1(y)\chi_2(y)$.

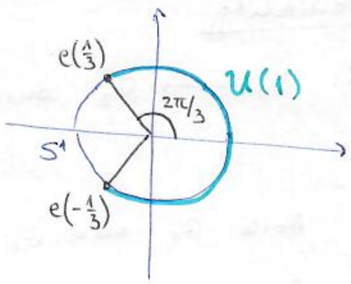
Equip $\widehat{G}$ with the compact-open topology:

A base of the top. on $\widehat{G}$ is given by

$$W(K, \mathcal{U}) = \{ \chi \in \widehat{G} \mid \chi(K) \subseteq \mathcal{U} \}$$

where $K \subseteq G$ cpt, $\mathcal{U} \subseteq S^1$ open.

Rmk. If G is discrete then the compact-open topology on $\widehat{G}$ is induced from $\widehat{G} \subseteq (S^1)^G = \prod_{g \in G} S^1$. In general, the top on $\widehat{G}$ is finer than this induced one.



Notation. $\forall x \in \mathbb{R}$ write $e(x) := \exp(2\pi i x)$

$\forall\, 0 < \varepsilon < 1$: $\mathcal{U}(\varepsilon) := e\left(-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right) \subseteq S^1$

Fact. There is no non-trivial subgp. of S^1 contained in $\mathcal{U}(1)$.

Prop. (1) $\widehat{G}$ is Hausdorff

(2) If G is discrete $\Rightarrow$ $\widehat{G}$ is cpt.

(3) If G is compact $\Rightarrow$ $\widehat{G}$ is discrete.

Pf: (1) Sts. $\{1\}$ is closed in $\widehat{G}$.

$$\{1\} = \bigcap_{g \in G} g^{\perp} \quad \text{where} \quad g^{\perp} := \{\chi \in \widehat{G} \mid \chi(g) = 1\} \subseteq \widehat{G}$$

$$\begin{tikzcd} g^{\perp} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \widehat{G} \times \{g\} \arrow[r] & S^1 \\ (\chi, g) \arrow[r, mapsto] & \chi(g) \end{tikzcd}$$

$\Rightarrow$ $g^{\perp} \subseteq \widehat{G}$ is closed

$\Rightarrow$ $\{1\} \subseteq \widehat{G}$ is closed.

(2) G discrete. $\Rightarrow$ $\widehat{G} \subseteq \prod_{g \in G} S^1$. Here $\widehat{G}$ is closed because

$$\widehat{G} = \bigcap_{(g,h)} V_{g,h} \quad \text{where} \quad V_{g,h} = \{\chi \in \prod_{g \in G} S^1 \mid \chi(gh) = \chi(g)\chi(h)\}.$$

$\prod S^1$ is cpt, hence $\widehat{G}$ is cpt as it is closed.

(3) Consider the nbh of 1 $W(G, \mathcal{U}(1))$ in $\widehat{G}$.

Since $\mathcal{U}(1)$ contains no nontriv subgroups:

$$\forall \chi \in W(G, \mathcal{U}(1)) : \chi(G) = \{1\}. \Rightarrow \chi = 1 \text{ in } \widehat{G}$$

$\Rightarrow W(G, \mathcal{U}(1)) = \{1\}$, hence $\widehat{G}$ is discrete. $\square$

Functoriality

$f: G_1 \longrightarrow G_2$ continuous $\rightsquigarrow$ $\widehat{f}: \widehat{G}_2 \longrightarrow \widehat{G}_1$ continuous

$\chi \longmapsto \chi \circ f$

If both G_1 and G_2 are fin. ab. gps and f is injective then $\widehat{f}: \widehat{G}_2 \twoheadrightarrow \widehat{G}_1$ is surjective.

Prop. If G is an increasing union of discrete finite subgroups G_n, then there is a canonical iso : $\iota: \widehat{G} \xrightarrow{\sim} \varprojlim_n \widehat{G}_n$.
of top gps.

(The inv. lim. has a natural profinite topology.) $\widehat{G}_{n+1} \twoheadrightarrow \widehat{G}_n$

Pf: It is easy to see that $\widehat{G} \longrightarrow \varprojlim_n \widehat{G_n}$ is continuous and bijective.

It suffices to show that $\imath$ is open.

$W(K, \mathcal{U}(\varepsilon))$, $K \subseteq G$ compact

Every K is contained in some G_n.

$$W(G_n, \mathcal{U}(\varepsilon)) \subseteq W(K, \mathcal{U}(\varepsilon))$$

$$\| $$

$$\{\chi \in \widehat{G} \mid \chi(G_n) \subseteq \mathcal{U}(\varepsilon)\} = \{\chi \in \widehat{G} \mid \chi(G_n) = \{1\}\} = \ker(\widehat{G} \longrightarrow \widehat{G}_n)$$

form a fundamental system of nbhds of $1 \in \varprojlim_n \widehat{G}_n$. $\square$

Examples. (1) $G = \mathbb{Z}$, $\widehat{G} \xrightarrow{\sim} S^1$

$$\chi \longmapsto \chi(1)$$

(2) $G = S^1 \Rightarrow \widehat{G} \longleftarrow \mathbb{Z}$

$$(\chi_n : \underbrace{e(\alpha)}_{\in S^1} \mapsto e(n\alpha)) \leftarrow n$$

(3) $G = \mathbb{R} \Rightarrow \widehat{G} \longleftarrow \mathbb{R}$

$$(\chi_\xi : x \mapsto e(x\xi)) \leftarrow \xi$$

All this has been established by classical Fourier analysis.

Exercise. The cpt-open topology on $\widehat{G}$ is compatible with the metric topology on $\mathbb{R}$.

(4) V fin. dim. vct. space $/\mathbb{R}$. $\rightarrow$ $\exists$ an iso

$$V^\vee := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \xrightarrow{\sim} V^\wedge$$

$$\varphi \longmapsto (v \mapsto e(\varphi(v)))$$

In particular, if $V = \mathbb{C}$ then $\mathrm{tr}_{\mathbb{C}/\mathbb{R}}$ induces an isomorphism

$$\mathbb{C}^\vee := \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{R}) \xrightarrow{\sim} \mathbb{C} \text{ so that}$$

$$\mathbb{C} \xrightarrow{\sim} \mathbb{C}^\wedge$$

$$\xi \longmapsto (\psi_\xi : x \longmapsto e(\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x\xi)))$$

(5) $G_n = p^{-n}\mathbb{Z}/\mathbb{Z}$ $\qquad \widehat{G_n} \longleftarrow \mathbb{Z}/p^n\mathbb{Z}$

$$\left(\psi_a : x \mapsto e(ax)\right) \longleftarrow\!\shortmid \; a$$

$$G := \bigcup_{n\geq 1} G_n \implies G \xrightarrow{\sim} \varprojlim_n \widehat{G_n} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

Here the transition map $\widehat{G_{n+1}} \longrightarrow \widehat{G_n}$ is identified with the canonical projection $\mathbb{Z}/p^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$

$$\left(\mathbb{Q}_p/\mathbb{Z}_p\right)^\wedge \xleftarrow{\sim} \mathbb{Z}_p$$

$$\left(\psi_a : x \mapsto e(ax)\right) \longleftarrow\!\shortmid \; a$$

(6) $G = \mathbb{Z}_p \quad \exists \chi \in \widehat{G}, \quad \chi : G \longrightarrow S^1$

$\chi^{-1}(\underbrace{\mathcal{U}(1)}) \subseteq G$

open nbh. of $0 \in \mathbb{Z}_p = G$ $\implies \exists n \in \mathbb{N}$ s.t. $p^n\mathbb{Z}_p \subseteq \chi^{-1}(\mathcal{U}(1))$

$\implies \chi(p^n\mathbb{Z}_p) \subseteq \mathcal{U}(1)$ no nontriv subgps.

$\implies \chi(p^n\mathbb{Z}_p) = \{1\}$ for this $n \in \mathbb{N}$

$$\implies \chi \in \ker\left(\widehat{\mathbb{Z}_p} \longrightarrow (p^n\mathbb{Z}_p)^\wedge\right) \cong \left(\mathbb{Z}_p/p^n\mathbb{Z}_p\right)^\wedge \cong p^{-n}\mathbb{Z}_p/\mathbb{Z}_p \qquad \forall \chi \in \widehat{G}$$

$$\psi_a \longleftarrow a$$

$$\implies \widehat{\mathbb{Z}_p} = \bigcup_n p^{-n}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$$

(7) $G = \mathbb{Q}_p, \quad \chi \in \widehat{G}, \quad \chi : G \longrightarrow S^1$

By the same argument as for $\mathbb{Z}_p$, $\exists n \in \mathbb{N}$: $\chi|_{p^n\mathbb{Z}_p}$ is trivial.

$$\implies \chi \in \underbrace{\left(\mathbb{Q}_p/p^n\mathbb{Z}_p\right)^\wedge}_{\cong_{(5)}\, p^{-n}\mathbb{Z}_p} \subseteq \widehat{\mathbb{Q}_p}$$

$$\implies \widehat{G} = \bigcup_n p^{-n}\mathbb{Z}_p = \mathbb{Q}_p$$

$$\left(\chi_a : x \mapsto e(\lambda_p(ax))\right) \longleftarrow\!\shortmid \; a$$

where $\lambda_p(ax)$ is the fractional part of $ax \in \mathbb{Q}_p$, i.e. for

$$ax = \sum_{n \gg -\infty} a_n p^n, \quad a_n \in \{0, \dots, p-1\} \implies \lambda_p(ax) = \sum_{n<0} a_n p^n.$$

If V is a fin. dim. vct space $/\mathbb{Q}_p$, then

$$V^\vee := \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p) \xrightarrow{\sim} V^\wedge$$
$$\varphi \longmapsto \big(v \mapsto e(\lambda_p(\varphi(v)))\big)$$

In particular, if $K/\mathbb{Q}_p$ is a finite extension, then $\mathrm{Tr}_{K/\mathbb{Q}_p}$ induces an isomorphism $K \xrightarrow{\sim} K^\vee$ so

$$K \xrightarrow{\sim} K^\wedge$$
$$a \longmapsto \big(\psi_a : x \mapsto e(\lambda_p(\mathrm{Tr}_{K/\mathbb{Q}_p}(ax)))\big)$$

Thm. (Pontryagin) Let G be a loc cpt top. abelian gp.
Then $G^\wedge$ is also loc cpt, and the canonical map

$$G \longrightarrow \widehat{\widehat{G}}$$
$$g \longmapsto (\chi \mapsto \chi(g))$$

is an isomorphism of top groups.

Pf: [reference Bourbaki: Spectral Theory, Chap II §1 no° 5.] $\square$

Cor. Let H be a closed subgp of G. Then the short exact sequence

$$0 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} G/H \longrightarrow 0$$

induces an exact sequence of dual groups

$$1 \longrightarrow (G/H)^\wedge \xrightarrow{\hat{p}} \widehat{G} \xrightarrow{\hat{i}} \widehat{H} \longrightarrow 0$$ $\square$

$(G/H)^\wedge$ is also denoted by $H^\perp = \{\chi \in \widehat{G} \mid \chi|_H = 1\}$.

Tate's Thesis ($\longrightarrow$ not included in the oral exam topics)

- Local theory $\longrightarrow$ define local ε-factor
- Global theory $\longrightarrow$ tool: Poisson summation formula

$k = \mathbb{R}, \mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ (i.e. a local field)

Fix $\psi: k \longrightarrow S^1$ as follows:

$$\psi(x) = \begin{cases} e(-x) & k = \mathbb{R} \\ e(-\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(x)) & k = \mathbb{C} \\ e(\lambda_p(\mathrm{Tr}_{k/\mathbb{Q}_p}(x))) & k = \mathbb{Q}_p \end{cases}$$

where $e(x) = \exp(2\pi i x)$.

This ψ is defined so that $\forall x \in \mathbb{Q}$: $\prod_{v \leq \infty} \psi(x) = 1$.

$\psi_v : \mathbb{Q}_v \longrightarrow S^1$ defined as above.

Such a ψ induces an iso $k \longrightarrow k^{\wedge}$

$$\xi \longmapsto (\psi_\xi : x \mapsto \psi(\xi x))$$

If k is p-adic, denote by $\mathcal{O}$ its ring of integers.

Then $\mathcal{O}^\perp = \{x \in k \mid \psi(xy) = 1 \;\; \forall y \in \mathcal{O}\} =$

$$= \{x \in k \mid e(\lambda_p(\mathrm{Tr}_{k/\mathbb{Q}_p}(xy))) = 1 \;\; \forall y \in \mathcal{O}\}$$

$$= \{x \in k \mid \mathrm{Tr}_{k/\mathbb{Q}_p}(xy) \in \mathbb{Z}_p \;\; \forall y \in \mathcal{O}\} = \delta^{-1}$$

where δ is the different of k relative to $\mathbb{Q}_p$.

Let dx be the Haar measure on k s.t.

$$dx = \begin{cases} \text{Lebesgue measure} & \text{if } k = \mathbb{R} \\ 2 \cdot \text{Lebesgue} & \text{if } k = \mathbb{C} \\ \text{normalised by } \int_{\mathcal{O}} dx = (N\delta)^{-1/2} & \text{if } k \text{ is p-adic,} \end{cases}$$

recall that $N\delta = \#(\mathcal{O}/\delta)$.

Rmk. This Haar measure is the unique self-dual Haar measure on k under the iso $k \xrightarrow{\sim} k^\wedge$ induced by ψ.

Fourier Analysis

Recall that a Schwartz function f on $\mathbb{R}^n$ is an $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ s.t.

$$\forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0} : \quad \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| < +\infty \qquad \begin{matrix} \| \\ (x_1, \dots, x_n) \end{matrix} \qquad x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$$

$$S(k) = \begin{cases} \{\text{Schwartz functions on } k\} & k = \mathbb{R}, \mathbb{C} \cong \mathbb{R}^2 \\ C_c^\infty(k, \mathbb{C}) & k / \mathbb{Q}_p \end{cases}$$

$= \{$locally constant and compactly supp. functions on $k\}$

$\forall f \in S(k)$ define its Fourier transform as $\hat{f}(x) = \int_k f(y)\psi(xy)\, dy \in C(k, \mathbb{C})$; here dy denotes the measure from p. 172.

Prop. $S(k)$ is stable under Fourier transform, i.e. $\forall f \in S(k)$: $\hat{f} \in S(k)$

Pf: $k = \mathbb{C}$ or $\mathbb{R}$ $\longrightarrow$ classical Fourier Analysis

If k is p-adic: note that every elt of $C_c^\infty(k, \mathbb{C})$ is a finite linear combination of functions of the form $\mathbb{1}_{a+\mathfrak{p}^l}$ (characteristic function of $a + \mathfrak{p}^l$, $a \in k$, $\mathfrak{p} \subseteq \mathcal{O}$ max ideal, $l \in \mathbb{Z}$)

The Fourier transform is linear $\Rightarrow$ the prop. will follow from:

Lemma. If $f = \mathbb{1}_{a+\mathfrak{p}^l}$ then $\hat{f}(x) = \psi(ax)(N\delta)^{-1/2}(N\mathfrak{p}^l)^{-1}\, \mathbb{1}_{\delta^{-1}\mathfrak{p}^{-l}}$.

Pf: $\hat{f}(x) = \int_k f(y)\psi(xy)\, dy = \int_{a+\mathfrak{p}^l} \psi(xy)\, dy =$

$$= \int_{\mathfrak{p}^l} \psi(x(y+a))\, dy = \psi(ax) \int_{\mathfrak{p}^l} \psi(xy)\, dy$$

$\psi(xy)$ can be viewed as a unitary character ψ_x of $y \in \wp^{\ell}$.

$$\psi_x : \wp^{\ell} \longrightarrow S^1$$

is trivial iff $x\wp^{\ell} \subseteq \delta^{-1}$

$$\psi_x = \begin{cases} 1 & x \in \delta^{-1}\wp^{-\ell} \\ \text{non-trivial} & x \notin \delta^{-1}\wp^{-\ell} \end{cases}$$

Lemma. G cpt abelian gp, dg Haar measure on G, $\chi : G \to S^1$ char on G

$$\Rightarrow \int_G \chi(g)\, dg = \begin{cases} \mathrm{Vol}(G) & \chi = 1 \\ 0 & \chi \neq 1 \end{cases}$$

By this Lemma:

$$\int_{\wp^{\ell}} \psi_x(y)\, dy = \begin{cases} \mathrm{Vol}(\wp^{\ell}) = N(\wp^{\ell})^{-1}\mathrm{Vol}(\mathcal{O}) & \text{if } x \in \delta^{-1}\wp^{-\ell} \\ 0 & \text{if } x \notin \delta^{-1}\wp^{-\ell}, \end{cases}$$

showing the first Lemma.

PF OF 2ND LEMMA: The case $\chi = 1$ is clear.

If $\chi \neq 1$: $\exists h \in G$ s.t. $\chi(h) \neq 1$.

$$\underline{\int_G \chi(g)\, dg} = \int_G \chi(gh)\, d(gh)$$

$$= \int_G \chi(gh)\, dg = \underbrace{\chi(h)}_{\neq 1} \underline{\int_G \chi(g)\, dg} \Rightarrow \int_G \chi(g)\, dg = 0$$

Rmk. If $f = 1_{\mathcal{O}}$ and k is unramified over $\mathbb{Q}_p$ ($\Rightarrow \delta = 0$)

$$\Rightarrow \hat{f} = 1_{\mathcal{O}}.$$

Prop. (Fourier inversion) $\hat{\hat{f}}(x) = f(-x) \quad \forall f \in S(k)$

Rmk. This means that if we replace the chosen Haar measure dx by $dx' := c \cdot dx$ for some $c \in \mathbb{R}_{>0}$, and define the Fourier transform using dx' then we get $\hat{\hat{f}}(x) = c^2 f(-x)$.

This is why we normalised the measure dx.

Algebraic Number Theory, lecture 27

Pf: $k = \mathbb{R}, \mathbb{C} \longrightarrow$ classical Fourier analysis

Assume $k/\mathbb{Q}_p$. Wma $f = 1_{a+\mathfrak{p}^\ell}$.

$$\hat{f}(x) = \psi(ax)^{-\frac{1}{2}} (N\delta)^{-1/2} (N\mathfrak{p}^\ell)^{1} 1_{a+\mathfrak{p}^\ell} \quad \text{by previous discussion.}$$

Lemma. $f \in S(k)$, $a \in k$, $h(x) := \psi(ax) f(x) \Rightarrow \hat{h}(x) = \hat{f}(x+a)$

Pf: $$\hat{h}(x) = \int_k \psi(xy)\psi(ay) f(y)\, dy =$$
$$= \int_k \psi((x+a)y) f(y)\, dy = \hat{f}(x+a)$$

$$\hat{\hat{f}}(x) = (N\delta)^{-1/2} (N\mathfrak{p}^\ell)^{-1} \underline{\left(1_{\delta^{-1}\mathfrak{p}^{-\ell}}\right)^\wedge} (x+a)$$

$$\left(1_{\delta^{-1}\mathfrak{p}^{-\ell}}\right)^\wedge = (N\delta)^{-1/2} \left(N(\delta^{-1}\mathfrak{p}^{\ell})\right)^{-1} 1_{\delta^{-1}(\delta^{-1}\mathfrak{p}^{-\ell})^{-1}}$$
$$= (N\delta)^{-1/2}\ N\delta \cdot N\mathfrak{p}^\ell \cdot 1_{\mathfrak{p}^\ell} = (N\delta)^{+1/2} N\mathfrak{p}^\ell\, 1_{\mathfrak{p}^\ell}$$

$$\Rightarrow \hat{\hat{f}}(x) = \cancel{(N\delta)^{-1/2}}\, \cancel{(N\mathfrak{p}^\ell)^{-1}}\, \cancel{(N\delta)^{+1/2}}\, \cancel{N\mathfrak{p}^\ell} \cdot 1_{\mathfrak{p}^\ell}(x+a) =$$

$$= \begin{cases} 1 & \text{if } a+x \in \mathfrak{p}^\ell \Leftrightarrow x = -a+\mathfrak{p}^\ell \Leftrightarrow -x \in a+\mathfrak{p}^\ell \\ 0 & \text{otherwise} \end{cases}$$

$$= 1_{a+\mathfrak{p}^\ell}(-x)$$

Local zeta integral

Def. A quasicharacter on $k^\times$ is a continuous homomorphism $\chi : k^\times \longrightarrow \mathbb{C}^\times$.

We say that χ is unitary if $\chi(k^\times) \subseteq S^1$.

χ is unramified if χ is trivial on $C = \{x \in k^\times \mid |x| = 1\} \subseteq k^\times$,

Since we have $1 \longrightarrow C \longrightarrow k^\times \xrightarrow{|\cdot|} |k^\times| \longrightarrow 0$,

χ is unramified $\Leftrightarrow \chi = |\cdot|^s$ for some $s \in \mathbb{C}$.

Note that if $k = \mathbb{R}, \mathbb{C} \Rightarrow |k^\times| = \mathbb{R}^\times_{>0}$.

If k is p-adic $\Rightarrow |k^\times| = (N\mathfrak{p})^{\mathbb{Z}}$ so $s \in \mathbb{C}/(2\pi i \log N\mathfrak{p})\mathbb{Z}$, $\mathrm{Re}(s)$ is unique

Notation. $X(k^\times) = \{$ quasicharacters on $k^\times \}$

Classification of quasicharacters

(1) $k = \mathbb{R}$ $\qquad \mathbb{R}^\times = \{\pm 1\}$

Every quasicharacter $\chi \in X(\mathbb{R}^\times)$ writes uniquely as

either $\chi = |\cdot|^s \quad s \in \mathbb{C}$

or $\chi = \mathrm{sgn}\, |\cdot|^s$

(2) $k = \mathbb{C}$ $\qquad \mathbb{C}^\times \cong S^1 \times \mathbb{R}^\times_{>0}$ where $C = S^1$

Since every quasicharacter on S^1 must be unitary (S^1 has the form $\chi_n : S^1 \longrightarrow \mathbb{C}^\times, \; e^{i\theta} \mapsto e^{in\theta}$ for some $n \in \mathbb{Z}$).

Regard χ_n as the quasicharacter on $\mathbb{C}^\times$ by requiring $\chi_n|_{\mathbb{R}^\times_{>0}} = 1$.

Then every quasicharacter on $\mathbb{C}^\times$ writes uniquely as

$\chi = \chi_n \cdot |\cdot|^s$ for some $n \in \mathbb{Z}$, $s \in \mathbb{C}$

(3) k is p-adic. Fix a uniformiser $\pi \in k$ so that $k^\times \cong \mathcal{O}^\times \cdot \pi^{\mathbb{Z}}$

$\forall \chi \in X(k^\times)$ consider the unitary character

$\chi_0 : k^\times \longrightarrow \mathbb{C}^\times$ s.t. $\chi_0|_{\mathcal{O}^\times} = \chi|_{\mathcal{O}^\times}$, $\chi_0(\pi) = 1$.

$\Rightarrow \chi/\chi_0$ is trivial on $\mathcal{O}^\times$, hence of the form $\chi/\chi_0 = |\cdot|^s \Leftrightarrow \chi = \chi_0 \cdot |\cdot|^s$

for some $s \in \mathbb{C}/(2\pi i \log(N\mathfrak{p}))\mathbb{Z}$

Equip $X(k^\times)$ with a structure of a complex manifold so that $\forall \chi_0 \in X(k^\times)$ the function $s \mapsto \chi_0 |\cdot|^s$ is holomorphic in $s \in \mathbb{C}$

(1) $k = \mathbb{R}$: $\quad X(\mathbb{R}^\times) \cong \mathbb{C} \amalg \mathbb{C}$

(2) $k = \mathbb{C}$: $\quad X(\mathbb{C}^\times) \cong \coprod_{n \in \mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{\mathbb{Z}}$

(3) k p-adic: $\quad X(k^\times) = \coprod \mathbb{C}/(2\pi i \log(N\mathfrak{p}))\mathbb{Z}$

Now $\forall \chi \in X(k^\times)$ write $\chi = \chi_0 \cdot |\cdot|^s$ with χ_0 unitary.

Define $\sigma(\chi) := \mathrm{Re}(s) \in \mathbb{R}$.

Haar measure on $k^\times$

$$\underline{d^\times x} := \delta(k) \frac{dx}{|x|} \quad \text{the self-dual Haar measure on } k$$

$$\delta(k) = \begin{cases} 1 & k = \mathbb{R}, \mathbb{C} \\ \frac{N\wp}{N\wp - 1} & k \text{ is p-adic} \end{cases}$$

The constant $\delta(k)$ is chosen so that

$$\int_{\mathcal{O}^\times} d^\times x = (N\delta)^{-1/2}$$

Def. Let $f \in S(k)$, $\chi \in X(k^\times)$. The local zeta integral of f and χ is

$$\underline{\zeta(f, \chi)} := \int_{k^\times} f(x)\,\chi(x)\, d^\times x$$

Ex. k is p-adic, $f := 1_{\wp^n}$, $\chi := |\cdot|^s$, $n \in \mathbb{Z}$

$$\zeta(f, \chi) = \int_{k^\times} 1_{\wp^n}(x) \cdot |x|^s \, d^\times x = \int_{\wp^n \setminus \{0\}} |x|^s \, d^\times x =$$

$$\wp^n \setminus \{0\} = \coprod_{m \geq n} \pi^m \mathcal{O}^\times$$

$$= \sum_{m \geq n} \int_{\pi^m \mathcal{O}^\times} |\pi|^{ms} d^\times x = \sum_{m \geq n} (N\wp)^{-ms} \overbrace{\int_{\pi^m \mathcal{O}^\times} d^\times x}^{= \int_{\mathcal{O}^\times} d^\times x} =$$

$$= (N\delta)^{-1/2} \frac{N\wp^{-n}}{1 - N\wp^{-s}} \int_{\mathcal{O}^\times} d^\times x$$

use $\mathrm{Re}(s) > 0$

Lemma. $\zeta(f, \chi)$ is absolutely convergent if $\sigma(\chi) > 0$.

Pf: $$|\zeta(f, \chi)| \leq \int_{k^\times} |f(x)| \cdot \underbrace{|\chi(x)|}_{|x|^{\sigma(\chi)}} \, d^\times x$$

$|f(x)|$ decreases rapidly as $|x| \to +\infty$ $\Rightarrow$ no problem with the convergence as $|x| \to +\infty$.

$|f(x)|$ is bounded as $|x| \to 0$.

It suffices to show that $\int_{0<|x|\le c} |x|^{\sigma(\chi)} d^\times x$ converges when $\sigma(\chi)>0$

(1) $k=\mathbb{R}$: $\int_{0<|x|\le c} |x_i|^{\sigma(\chi)} d^\times x = 2\int_0^c x^{\sigma(\chi)} \frac{dx}{x} = \frac{2}{\sigma(\chi)} x^{\sigma(\chi)} \Big|_0^c < +\infty$

(2) $k=\mathbb{C}$: similar

(3) k p-adic: $\int_{0<|x|\le 1} |x|^{\sigma(\chi)} d^\times x = \int_{\mathcal{O}\setminus\{0\}} |x|^{\sigma(\chi)} d^\times x =$

$$= (N\delta)^{-1/2} \frac{1}{1-(N\mathfrak{p})^{-\sigma(\chi)}} < +\infty$$

□

Def. For $\chi \in X(k^\times)$ define the local L-factor of χ as follows: 29.1.2018

- $k=\mathbb{R}$: $L(|\cdot|^s) := \Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ where $\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt$

  $L(\operatorname{sgn}|\cdot|^s) := \Gamma_\mathbb{R}(s+1)$

- $k=\mathbb{C}$: $L(\chi_n|\cdot|^s) := \Gamma_\mathbb{C}\left(s+\frac{|n|}{2}\right)$ where $\chi_n(z) = \left(\frac{z}{|z|}\right)^n$, $\Gamma_\mathbb{C}(s) := (2\pi)^{1-s}\Gamma(s)$

- k p-adic: $L(\underbrace{|\cdot|^s}_{\chi}) = \frac{1}{1-N\mathfrak{p}^{-s}} = \frac{1}{1-\chi(\varpi)}$ if χ is unramified

  and $L(\chi)=1$ if χ is ramified.

Rmk. $L(\chi)$ is a meromorphic function in $\chi \in X(k^\times)$ with no poles for $\sigma(\chi)>0$ and no zeros for all χ.

Thm. (Tate). $\forall f \in \mathcal{S}(k)$ $\zeta(f,\chi)$ has an analytic continuation to a meromorphic function in $\chi \in X(k^\times)$ s.t. $\frac{\zeta(f,\chi)}{L(\chi)}$ is holomorphic in $\chi \in X(k^\times)$ and it satisfies:

$$\frac{\zeta(f,\chi_0|\cdot|^s)}{L(\chi)}\varepsilon(\chi) = \frac{\zeta(\hat{f},\overline{\chi_0}|\cdot|^{1-s})}{L(\chi^\vee)}$$

where $\hat{f}$: local Fourier transform of f, $\chi^\vee = \chi^{-1}|\cdot|$

and $\varepsilon(\chi)$ is given: (1) $k=\mathbb{R}$: $\varepsilon(|\cdot|^s)=1$, $\varepsilon(\operatorname{sgn}|\cdot|^s) = -i$

(2) $k=\mathbb{C}$: $\varepsilon(\chi_n|\cdot|^s) = (-i)^{|n|}$

and (3) k is p-adic: $\varepsilon(|\cdot|^s) = (N\delta)^{\frac{1}{2}-s}$ and $\varepsilon(\chi_0|\cdot|^s) = N(\delta f)^{\frac{1}{2}-s}\,\tau_0(\overline{\chi_0})$

where χ_0 is a ramified character on $k^\times$ and trivial on π.

$\wp^c = f \subseteq \mathcal{O}$ is the largest ideal s.t. $\chi_0|_{(1+f)}$ is trivial

$$\tau_0(\overline{\chi_0}) = (Nf)^{-\frac{1}{2}} \sum_{x\in \mathcal{O}^\times/(1+f)} \overline{\chi_0}(x)\,\psi\left(\frac{x}{\pi^m}\right)$$

$\wp^m = \delta f$ so if $\delta = \wp^d \Rightarrow m = c+d$,

$|\tau_0(\overline{\chi_0})|_{\mathbb{C}} = 1$ is the normalised Gauß sum of (χ_0, ψ)

Rmk. (1) $\varepsilon(\chi)\,\varepsilon(\chi^\vee) = \chi(-1)$

(2) $\varepsilon(\overline{\chi}) = \chi(-1)\,\overline{\varepsilon(\chi)}$

For (1): use $\hat{\hat{f}}(x) = f(-x)$.

$$\frac{\zeta(f,\chi)}{L(\chi)}\,\varepsilon(\chi)\,\varepsilon(\chi^\vee) = \frac{\zeta(\hat{f},\chi^\vee)}{L(\chi^\vee)}\,\varepsilon(\chi^\vee) = \frac{\zeta(\hat{\hat{f}},\overbrace{(\chi^\vee)^\vee}^{\chi})}{L(\chi)} \overset{\text{def of }\zeta}{=} \chi(-1)\,\frac{\zeta(f,\chi)}{L(\chi)}$$

For (2): $\hat{\overline{f}}(x) = \overline{\hat{f}}(-x)$

If $\sigma(\chi) = \frac{1}{2} \Rightarrow \overline{\chi} = \chi^\vee = \chi^{-1}|\cdot| = \overline{\chi}_0|\cdot|^{\frac{1}{2}-it}$

$\uparrow$

$\chi = \chi_0|\cdot|^{\frac{1}{2}+it}$

$$\Rightarrow \underbrace{\varepsilon(\chi)\cdot\varepsilon(\overline{\chi})}_{\chi(-1)\,\varepsilon(\chi)\cdot\overline{\varepsilon(\chi)}} = \varepsilon(\chi)\,\varepsilon(\chi^\vee) = \chi(-1) \Rightarrow |\varepsilon(\chi)|_{\mathbb{C}} = 1. \quad \square$$

Lemma. $f, g \in S(k)$, $\chi \in X(k^\times)$, $0 < \sigma(\chi) < 1 \Rightarrow$

$$\Rightarrow \zeta(f,\chi)\,\zeta(\hat{g},\chi^\vee) = \zeta(\hat{f},\chi^\vee)\cdot\zeta(g,\chi)$$

Pf: $\zeta(f,\chi)\,\zeta(\hat{g},\chi^\vee) = \int_{k^\times} f(x)\chi(x)\,d^\times x \int_{k^\times} \hat{g}(y)\,\chi^{-1}(y)\,|y|\,d^\times y$

$= \int_{k^\times}\int_{k^\times} f(x)\chi(x)\,d^\times x\;\hat{g}(xy)\,\chi^{-1}(xy)\,|xy|\,\frac{dy}{|y|}$

Change of variables:
$y \mapsto xy$
$d^\times y \mapsto d^\times(xy) =$
$= d^\times y = \frac{dy}{|y|}$

$= \delta(k)\int_{k^\times}\int_{k^\times} f(x)\,\hat{g}(xy)\,\chi(y)^{-1}\,|x|\,d^\times x\,dy$

$$\hat{g}(xy) = \int_k g(z)\,\psi(xyz)\,dz$$

$$d^\times x = \delta(k)\frac{dx}{|x|}$$

$$\text{LHS} = \int_{k^\times}\Bigg(\underbrace{\int_{k^\times}\int_k f(x)\,g(z)\,\psi(xyz)\,dz\,\cancel{|x|}\,\frac{dx}{\cancel{|x|}}\,\delta(k)}_{\text{symmetric in } f \text{ and } g}\Bigg)\chi(y)^{-1}\,|y|\,d^\times y$$

$= \text{RHS}$ by switching f and g $\square$

Pf OF THM: Choose some special $f \in S(k)$ s.t. $\zeta(f,\chi) = L(\chi)$ and $\zeta(\hat{f},\chi^\vee) = \varepsilon(\chi)\cdot L(\chi^\vee)$.

Then $\forall\, g \in S(k)$ we have $L(\chi)\,\zeta(\hat{g},\chi^\vee) = \varepsilon(\chi)\,L(\chi^\vee)\,\zeta(g,\chi)$

$$\Rightarrow \frac{\zeta(g,\chi)}{L(\chi)}\,\varepsilon(\chi) = \frac{\zeta(\hat{g},\chi^\vee)}{L(\chi^\vee)} \quad \text{functional equation}$$

Once we have this, everything follows.

LHS: $\sigma(\chi) > 0$, RHS: $0 < \sigma(\chi^\vee) = 1 - \sigma(\chi) \iff \sigma(\chi) < 1$

How to choose such an f?

- $k = \mathbb{R}$: $\chi = |\cdot|^s \longrightarrow f := e^{-\pi x^2}$

$$\zeta(f, |\cdot|^s) = \int_{\mathbb{R}^\times} e^{-\pi x^2}\,|x|^s\,\frac{dx}{|x|}$$

$$y := \pi x^2,\quad dy = 2\pi x\,dx,\quad dx = \frac{dy}{2\pi\sqrt{\frac{y}{\pi}}} = \frac{dy}{2\sqrt{\pi y}}$$

$$\zeta(f, |\cdot|^s) = 2\cdot\frac{1}{2\sqrt{\pi}}\int_0^{+\infty} e^{-y}\left(\frac{y}{\pi}\right)^{\frac{s-1}{2}}\frac{dy}{y^{1/2}}$$

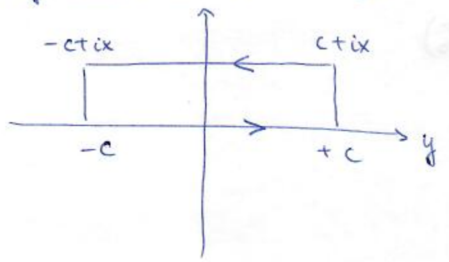
$$= \frac{1}{\sqrt{\pi}}\,\pi^{\frac{1}{2}-\frac{s}{2}}\int_0^{+\infty} e^{-y}\,y^{\frac{s}{2}-1}\,dy$$

$$= \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) = \Gamma_{\mathbb{R}}(s) = L(|\cdot|^s)$$

$$\hat{f}(x) = \int_{\mathbb{R}} e^{-\pi y^2}\,e^{-2\pi i xy}\,dy = \int_{\mathbb{R}} e^{-\pi(y^2 + 2xyi + (ix)^2)}\,e^{-x^2\pi}\,dy =$$

$$= e^{-\pi x^2}\int_{\mathbb{R}} e^{-\pi(y+ix)^2}\,dy$$

Algebraic Number Theory, lecture 28



Standard complex analysis:

$$\int_{\mathbb{R}} e^{-\pi(y+ix)^2}\,dy = \int_{\mathbb{R}} e^{-\pi y^2}\,dy = 1$$

$$\Rightarrow \hat f(x) = f(x) \Rightarrow \zeta(\hat f, \chi^\vee) = \zeta(f,\chi) = L(\chi^\vee).$$

(We won't do this for every case; the basic idea could be observed in this case.)

For $\underline{\chi = \mathrm{sgn}|\cdot|^s} \longrightarrow f(x) := x\, e^{-\pi x^2} \Rightarrow \zeta(f, \mathrm{sgn}|\cdot|^s) = L(\mathrm{sgn}|\cdot|^s)$

$$\Rightarrow \hat f(x) = -ix \Rightarrow \zeta(\hat f, \underbrace{\mathrm{sgn}|\cdot|^{1-s}}_{\chi^\vee}) = -i\,L(\mathrm{sgn}|\cdot|^s)$$

• $k = \mathbb{C}$ $\quad \underline{\chi = \chi_n|\cdot|^s} \longrightarrow f(x) := \begin{cases} \bar z^n e^{-2\pi z\bar z} & n \geq 0 \\ z^{-n} e^{-2\pi z\bar z} & n<0 \end{cases}$

$$\hat f_n = (-i)^{|n|} f_{-n}$$

$$\zeta(f_n, \chi_n|\cdot|^s) = L(\chi_n|\cdot|^s)$$

$$\zeta(\hat f_n, \underbrace{\chi_n^{-1}|\cdot|^{1-s}}_{\chi^\vee}) = (-i)^{|n|}\zeta(f_{-n}, \chi_{-n}|\cdot|^{1-s}) = \underbrace{(-i)^{|n|}}_{\varepsilon(\chi)} L(\chi^\vee)$$

• k is p-adic: $\underline{\chi = |\cdot|^s \text{ unramified}} \longrightarrow f := (N\delta)^{\frac12}\, 1_{\mathcal{O}}$

$$\zeta(f, |\cdot|^s) = \frac{1}{1-N\mathfrak{p}^{-s}} = \frac{1}{1-\chi(\pi)}$$

$$\hat f = (N\delta)^{\frac12}(N\delta)^{-\frac12}\, 1_{\delta^{-1}} = 1_{\delta^{-1}}$$

$$\zeta(\hat f, |\cdot|^{1-s}) = (N\delta)^{-\frac12}\frac{(N\delta)^{1-s}}{1-(N\mathfrak{p})^{1-s}} = \underbrace{(N\delta)^{+\frac12 - s}}_{\varepsilon(\chi)} L(\chi^\vee)$$

When $\underline{\chi = \chi_0|\cdot|^s}$, χ_0 ramified of conductor $\mathfrak{p}^c$

Take $f := \frac{1}{A}\, 1_{1+\mathfrak{p}^c}$, $A := \int_{1+\mathfrak{p}^c} d^\times x$

$$\rightarrow \zeta(f,\chi) = \frac{1}{A}\int_{1+\mathfrak{p}^c} d^\times x = 1$$

Need to check: $\zeta(\hat{f}, \chi^\vee) \overset{?}{=} \varepsilon(\chi) = (N\delta\wp^c)^{\frac{1}{2}-s}\, \tau_0(\overline{\chi_0})$

$$\hat{f} = \underbrace{A^{-1}\cdot (N\delta)^{-\frac{1}{2}} (N\wp)^{-1}}_{\text{constants}} \psi\, 1_{(\delta\wp^c)^{-1}}$$

$$\zeta\left(\psi\, 1_{(\delta\wp^c)^{-1}}, \chi^\vee\right) = \int_{(\delta\wp^c)^{-1}\setminus\{0\}} \psi(x)\, \overline{\chi_0}(x)\, |x|^{1-s}\, d^\times x = *$$

$\chi^\vee = \chi_0 |\cdot|^{1-s}$

$$(\delta\wp^c)^{-1} = \wp^{-m} \qquad m = c+d, \quad \wp^d = \delta$$

$$\wp^{-m}\setminus\{0\} = \coprod_{n=-m}^{+\infty} \pi^n \mathcal{O}^\times$$

$$* = \sum_{n=-m}^{+\infty} (N\wp)^{-n(1-s)} \underbrace{\left(\int_{\mathcal{O}^\times} \psi(\pi^n x)\, \overline{\chi_0}(\pi^n x)\, d^\times x\right)}_{I_n}$$

$\forall x \in \mathcal{O}^\times$: $\psi(\pi^n x) = 1 \iff \pi^n x \in \delta^{-1} \iff n \geq -d$

$n \geq -d$: $\quad I_n = \int_{\mathcal{O}^\times} \overline{\chi_0}(x)\, d^\times x = 0$ because $\overline{\chi_0}|_{\mathcal{O}^\times}$ is non-trivial

Assume $-m \leq n \leq -d$ for $y \in 1 + \wp^{-n}\delta^{-1} = 1 + \wp^{-(n+d)}$

$$0 < -(n+d) \leq c$$

$$\psi(\pi^n x y) = \psi\left(\pi^n x + \pi^n x^{-(n+d)} z\right) \qquad y = 1 + x^{-(n+d)} z, \quad z \in \mathcal{O}$$
$$= \psi(\pi^n x)$$

If S is a set of representations of $\mathcal{O}^\times / \underbrace{(1+\wp^{-n}\delta^{-1})}_{\wp^{-(n+d)}}$

then one has $\displaystyle\sum_{x\in S} \overline{\chi_0}(x)\, \psi(\pi^n x) \left(\int_{1+\wp^{-n}\delta^{-1}} \overline{\chi_0}(y)\, d^\times y\right) = I_n$

For $-d > n > -m$: $\iff 0 < -(n+d) < c$: $\overline{\chi_0}\big|_{1+\wp^{-(n+d)}}$ is non-trivial

$\Rightarrow \displaystyle\int_{1+\wp^{-(n+d)}} \overline{\chi_0} = 0 \quad \Rightarrow \quad I_n \neq 0$ only when $n = -m$

In that case:

$$I_m = \sum_{x \in S} \overline{\chi}_0(x)\,\psi(\pi^{-m} x) \underbrace{\left(\int_{1+\mathfrak{p}^c} d^\times x \right)}_{A} = A \, (N\mathfrak{p}^c)^{\frac{1}{2}} \, \tau_0(\overline{\chi}_0)$$

$$\zeta(\hat{f}, \chi^\vee) = \cancel{A^{-1}} (N\delta)^{-1/2} (N\mathfrak{p}^c)^{-1} \cancel{A} (N\mathfrak{p}^c)^{1/2} \tau_0(\overline{\chi}_0)$$

$$= (N\mathfrak{p}^m)^{-1/2} (N\mathfrak{p})^{m(1-s)} \tau_0(\overline{\chi}_0)$$

$$= (N\mathfrak{p}^m)^{\frac{1}{2}-s} \tau_0(\overline{\chi}_0) = \varepsilon(\chi)$$

$\square$

This finishes the discussion of the local theory.

Global zeta integrals

$K/\mathbb{Q}$ number field

Fix $\psi: \mathbb{A}_K = \prod_v' K_v \longrightarrow \mathbb{C}^\times, \quad \psi = \prod_v \psi_v$

Note $\forall x = (x_v) \in \mathbb{A}_K$: $\psi(x) = \prod_v \psi_v(x_v)$ is a finite product.

$$\Psi: \mathbb{A}_K \longrightarrow \hat{\mathbb{A}_K}$$

$$\xi \longmapsto \left(\psi_\xi : x \longmapsto \psi(x\xi) \right)$$

Prop. Ψ induces an isomorphism of topological groups $\mathbb{A}_K \longrightarrow \hat{\mathbb{A}_K}$

Moreover ψ_ξ is trivial on $K \subseteq \mathbb{A}_K$ if $\xi \in K$.

Pf: The first part follows from the local duality $K_v \xrightarrow{\wedge} \hat{K_v}$;

on top of this, need to check continuity, surjectivity, openness. (PO).

We turn to the 2^{nd} part.

Let $\Gamma \subseteq \mathbb{A}_K$ be a subgp. s.t. $\xi \in \Gamma \Leftrightarrow \psi_\xi$ is trivial.

Clearly $\Gamma \supseteq K$.

Given ψ_ξ s.t. $\psi_\xi|_K$ is trivial, WTS $\xi \in K$.

• $K=\mathbb{Q} \longrightarrow \mathbb{A}_{\mathbb{Q}} = \mathbb{Q} + \left(-\frac{1}{2}, \frac{1}{2}\right] \times \prod_p \mathbb{Z}_p$

Write $\xi = b + c$, $b \in \mathbb{Q}$, $c \in \left(-\frac{1}{2}, \frac{1}{2}\right] \times \prod_p \mathbb{Z}_p$

$$1 = \psi_\xi(1) = \psi(\xi) = \psi(b+c) = \psi(c) = \psi_\infty(c_\infty) \underbrace{\prod_p \psi(c_p)}_{1}$$

$$\Rightarrow \psi_\infty(c_\infty) = 1 \Rightarrow c_\infty \in \mathbb{Z} \cap \left(-\tfrac{1}{2}, \tfrac{1}{2}\right] = \{0\}$$

$$1 = \psi_\xi\left(\frac{1}{p^n}\right) = \psi\left(\frac{c}{p^n}\right) = \left(\prod_{\ell \neq p} \psi_\ell\left(\frac{c_\ell}{p^n}\right)\right) \psi_p\left(\frac{c_p}{p^n}\right) = \psi_p\left(\frac{c_p}{p^n}\right)$$

$$\Rightarrow \frac{c_p}{p^n} \in \mathbb{Z}_p \quad \forall n \Rightarrow c_p = 0.$$

For general K, use the following

Lemma. $x = (x_v)_v \in \mathbb{A}_K$ s.t. $\mathrm{Tr}_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(xy) \in \mathbb{Q} \quad \forall y \in K \Rightarrow x \in K$

PROOF IN TIAN'S NOTES ON TATE'S THESIS. □

$$\Rightarrow 1 = \psi_\xi(y) = \psi\left(\mathrm{Tr}_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(\xi y)\right) \quad \forall y \in K \Rightarrow \mathrm{Tr}_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(\xi y) \in \mathbb{Q} \overset{\text{lemma}}{\Rightarrow} \xi \in K.$$ □

1.2.2018

Review. G loc cpt ab gp.

$\hat{G} = \mathrm{Hom}_{\mathrm{Cont}}(G, S^1)$, dg: Haar measure on G

$L^p(G,\mathbb{C})$:= completion of $C_c(G,\mathbb{C})$ wrt the L^p-norm ($p \geq 1$):

$$\|f\|_p := \left(\int_G |f(g)|^p \, dg\right)^{1/p}$$

$\forall f \in L^1(G,\mathbb{C})$: Fourier transform $\hat{f}(\chi) = \int_G f(g)\chi(g)\, dg \in C(\hat{G}, \mathbb{C})$

Thm. (Plancherel) $\exists$ unique Haar measure $d\hat{g}$ on $\hat{G}$ s.t.

$$\int_{\hat{G}} |\hat{f}(\hat{g})|^2 \, d\hat{g} = \int_G |f(g)|^2 \, dg \quad \forall f \in C_c(G,\mathbb{C})$$

and the map $f \longmapsto \hat{f}$ extends uniquely to an isometry $L^2(G,\mathbb{C}) \xrightarrow{\sim} L^2(\hat{G},\mathbb{C})$.

Thm (Fourier Inversion Theorem) Via a canonical iso $\eta : G \xrightarrow{\sim} \hat{\hat{G}}$,

dg is identified with $d\hat{g}$ and $\hat{\hat{f}}(\eta(x)) = f(-x) \quad \forall f \in L^2(G, \mathbb{C})$.

Ex. G cpt. ab. gp., e.g. $G = \mathbb{R}/\mathbb{Z}$, $\mathbb{A}_K/K, \dots$ and

$\hat{G}$ discrete e.g. $\hat{G} = \mathbb{Z}$, $K, \dots$ then:

the dual Haar measure on $\hat{G}$ is $\frac{1}{\mathrm{vol}(G)} \cdot$ counting measure.

FIT $\Rightarrow \forall f \in C(G, \mathbb{C}) \subseteq L^2(G, \mathbb{C})$: $\quad f(x) = \sum_{\chi \in \hat{G}} \chi(-x) \cdot \hat{f}(\chi) \cdot \frac{1}{\mathrm{vol}(G)}$

Let $K/\mathbb{Q}$ be a number field. Fix $\psi = \prod_v \psi_v : \mathbb{A}_K \longrightarrow S^1$ nontrivial additive character, $\psi(K) = 1$.

$$\begin{aligned} \Psi : \mathbb{A}_K &\xrightarrow{\sim} \hat{\mathbb{A}_K} \\ \xi &\longmapsto \left(\psi_\xi : x \longmapsto \psi(x\xi)\right) \end{aligned}$$

is an iso of lc gps by prev. discussion

Moreover, $\psi_\xi|_K$ is trivial $\Leftrightarrow \xi \in K$

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\cong"', "\text{discrete}"] & \mathbb{A}_K \arrow[r] \arrow[d, "\cong"', "\Psi"] & \mathbb{A}_K/K \arrow[r] \arrow[d, "\cong"', "\text{cpt.}"] & 0 \\ 0 \arrow[r] & (\mathbb{A}_K/K)^\wedge \arrow[r] & \hat{\mathbb{A}_K} \arrow[r] & \hat{K} \arrow[r] & 0 \end{tikzcd}$$

So the dual of a cpt gp is discrete and the dual of a discrete gp is cpt.

Let dx_v be the self-dual Haar measure on K_v, $v \in V$.

$dx := \prod_v dx_v$ Haar measure on $\mathbb{A}_K$

Lemma. Under the self-dual Haar measure:

$\mathrm{vol}(\mathbb{A}_K/K) = \int_D dx = 1$ where $D \subseteq \mathbb{A}_K$ is a fund. domain for $\mathbb{A}_K/K$.

Pf: Let dx_v be the Haar measure on K_v normalised by $\int_{\mathcal{O}_{K_v}} dx'_v = 1$,

$dx_v = (N\delta_v)^{-1/2}\, dx'_v$, where δ_v is the different of $K_v/\mathbb{Q}_p$.

$$\int_D dx = \frac{1}{\prod_v (N\delta_v)^{-1/2}} \int_D dx' = \frac{\sqrt{|d_K|}}{\prod_v (N\delta_v)^{-1/2}} = 1$$

actually a finite product

$$|d_K| = |N_{K/\mathbb{Q}}(\delta_{K/\mathbb{Q}})| = \prod_v N_{K_v/\mathbb{Q}_p}(\delta_v)$$

$$(\delta_{K/\mathbb{Q}})_v^{\wedge} = \delta_v$$

□

Rmk. The dual Haar measure on $(\mathbb{A}_K/K)^{\wedge} \cong K$ is the counting measure.

Def. $S(\mathbb{A}_K) := \{$ finite linear combination of $f = \bigotimes_v f_v \mid f_v \in S(K_v)$,

Schwartz functions on $\mathbb{A}_K$ $\qquad f_v = 1_{\mathcal{O}_{K_v}}$ for almost all $v \}$

Def. $f \in S(\mathbb{A}_K)$: $\quad \hat{f}(x) := \int_{\mathbb{A}_K} \psi(xy) f(y)\, dy$

Lemma. (1) $\forall f \in S(\mathbb{A}_K)$: $\hat{f} \in S(\mathbb{A}_K)$

(2) $\hat{\hat{f}}(x) = f(-x)$.

Pf: Wma $f = \bigotimes_v f_v$ by linearity.

Then $\hat{f} = \bigotimes_v \hat{f}_v$ local Fourier transform.

$\Rightarrow$ we can apply the local theory.

□

Prop. (Poisson Summation Formula) $\forall f \in S(\mathbb{A}_K)$ the series $\sum_{x \in K} f(x)$ converges absolutely and $\sum_{x\in K} f(x) = \sum_{\xi \in K} \hat{f}(\xi)$.

Rmk. Classical analogue: $\forall f \in S(\mathbb{R})$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \text{ where } \hat{f}(x) = \int_{\mathbb{R}} f(y)\cdot e^{-2\pi i x y}\, dy$$

E.g. $f_t(x) = e^{-\pi x^2 t} \quad \Rightarrow \quad \hat{f}_t(x) = \frac{1}{\sqrt{t}} e^{-\pi \frac{x^2}{t}}$

$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{n^2}{t}}$$

Pf of PSF: When $f = \otimes_v f_v$, $f_v = 1_{\mathcal{O}_{K_v}}$ for almost all $v \in V$

$\exists \mathcal{U} \subseteq \mathbb{A}_K^\infty = \prod_{v \in V_f} K_v$ open cpt subgp

$\operatorname{supp}(f) \subseteq \mathcal{U} \times K_\infty$, $\quad K_\infty = \prod_{v \in V_\infty} K_v$

$\mathcal{O}_\mathcal{U} := K \cap (\mathcal{U} \times K_\infty) \underset{\text{lattice}}{\hookrightarrow} K_\infty$

$$\sum_{x \in K} f(x) = \sum_{x \in K \cap (\mathcal{U} \times K_\infty)} f(x) = \sum_{x \in \mathcal{O}_\mathcal{U}} f(x) \qquad f(x) = f^\infty \otimes f_\infty$$

where $f^\infty = \otimes_{v \in V_f} f_v$, $\quad f_\infty = \otimes_{v \in V_\infty} f_v$

$\Rightarrow \operatorname{supp} f^\infty \subseteq \mathcal{U}$, $\quad f_\infty \in \mathcal{S}(K_\infty) \subseteq \mathcal{S}(\mathbb{R}^n)$

$\exists A > 0$: $\quad |f^\infty(x)| \leq A \quad \forall x \in \mathcal{U}$

$$\sum_{x \in \mathcal{O}_\mathcal{U}} |f(x)| \leq A \underbrace{\sum_{x \in \mathcal{O}_\mathcal{U}} |f_\infty(x)|}$$

converges absolutely
by classical analysis
on $\mathbb{R}^n$

To prove the PSF, consider $g(x) := \sum_{y \in K} f(x+y)$ well-def function on $\mathbb{A}_K/K = G$

$\forall \xi \in K \cong (\mathbb{A}_K/K)^\wedge$:
$$\hat{g}(\xi) = \int_{\mathbb{A}_K/K} g(x)\, \psi(x\xi)\, dx$$

$$= \int_{\mathbb{A}_K/K} \left(\sum_{y \in K} f(x+y) \right) \psi(x\xi)\, dx$$

$$= \sum_{y \in K} \int_{\mathbb{A}_K/K} f(x+y)\, \overbrace{\psi(x\xi)}^{= \psi((x+y)\xi)}\, d(x+y)$$

D is a fund. domain

$$= \sum_{y \in K} \int_{D+y} f(x)\, \psi(x\xi)\, dx$$

$$= \int_{\mathbb{A}_K} f(x)\, \psi(x\xi)\, dx = \hat{f}(x)$$

Fourier inversion on $\mathbb{A}_K/K$:

$$g(x) = \frac{1}{\mathrm{vol}(\mathbb{A}_K/K)} \sum_{\xi \in (\mathbb{A}_K/K)^\wedge \cong K} \psi(x\xi)\,\hat{g}(\xi) = \sum_{\xi \in K} \psi(-x\xi)\,\hat{f}(\xi)$$

$$\sum_{y \in K} f(x+y) = \sum_{\xi \in K} \psi(-x\xi)\,\hat{f}(\xi)$$

$$x = 0 \Rightarrow \sum_{y \in K} f(y) = \sum_{\xi \in K} \hat{f}(\xi), \quad \text{as desired.}$$

□

<u>Cor.</u> $f \in S(\mathbb{A}_K)$, $x \in \mathbb{I}_K \Rightarrow \sum_{\xi \in K} f(x\xi) = \frac{1}{|x|} \sum_{\xi \in K} \hat{f}\left(\frac{\xi}{x}\right)$

Pf: Apply PSF to $g(y) := f(xy)$ $(\forall y \in \mathbb{A}_K)$

$$\Rightarrow \hat{g}(z) = \int_{\mathbb{A}_K} g(y)\,\psi(xy)\,dy = \int_{\mathbb{A}_K} f(xy)\,\psi(xy)\,dy =$$

$y' := xy$, $dy' = d(xy) = |x|\,dy$

$$= \int_{\mathbb{A}_K} f(y')\,\psi\left(\frac{y'}{x} z\right) \frac{dy'}{|x|} = \frac{1}{|x|} \hat{f}\left(\frac{z}{x}\right)$$

□

<u>Global zeta integral</u>

local: K_v, $K_v^\times$

global: $\mathbb{A}_K$, $\mathbb{A}_K^\times = \mathbb{I}_K$

$d^\times x_v = \delta(K_v) \cdot \frac{dx_v}{|x_v|_v}$ self-dual Haar measure

$$\delta(K_v) = \begin{cases} 1 & v \text{ is arch} \\ \frac{N\mathfrak{p}_v}{N\mathfrak{p}_v - 1} & v \in V_f \end{cases} \qquad \Rightarrow \int_{\mathcal{O}_{K_v}} d^\times x_v = (N\delta_v)^{-1/2}$$

$d^\times x = \prod_v d^\times x_v$ on $\mathbb{I}_K = \prod_v' K_v^\times$

$$\mathbb{I}_K^1 \subseteq \mathbb{I}_K \longrightarrow \mathbb{I}_K \cong \mathbb{I}_K^1 \times \mathbb{R}_{>0}^\times$$

$\mathbb{I}_K^1 = \{x \in \mathbb{I}_K \mid |x| = 1\}$

$d^\times x$ induces a Haar measure on $\mathbb{I}_K^1$

Lemma. Wrt this Haar measure we get

$$\mathrm{Vol}\left(\mathbb{I}_K^1 / K^\times \right) = \frac{2^{r_1} (2\pi)^{r_2} R_K h}{w \cdot \sqrt{|d_K|}}$$

Pf: If we take the Haar measure $d^\times x_v'$ so that $\int_{\mathcal{O}_{K_v}} d^\times x_v' = 1$

then $$\int_{\mathbb{I}_K^1/K^\times} d^\times x' = \frac{2^{r_1} (2\pi)^{r_2} R_K h}{w}$$

$$d^\times x = \prod_v N(\delta_v)^{-1/2} d^\times x_v' = \frac{1}{\left(\prod_v N(\delta_v)\right)^{1/2}} d^\times x' = \frac{d^\times x'}{\sqrt{|d_K|}}$$ $\square$

Def. A Hecke character on K is a cont homomorphism
$\chi : \mathbb{I}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$.

The character χ is unramified if $\chi = |\cdot|^s$ for some $s \in \mathbb{C}$.

$X_K := \{ \chi \text{ Hecke character on } K \}$ complex structure so that

$\forall \chi \in X_K$: $\quad \begin{aligned} \mathbb{C} &\longrightarrow X_K \\ s &\longmapsto \chi |\cdot|^s \end{aligned}$ is an embedding of complex manifolds.

Choose a splitting $\mathbb{I}_K = \mathbb{I}_K^1 \times \mathbb{R}_{>0}^\times$.

For $\chi \in X_K$ put $\chi_0 \in X_K$ s.t. $\chi_0 |_{\mathbb{I}_K^1/K^\times} = \chi |_{\mathbb{I}_K/K^\times}$ and $\chi |_{\mathbb{R}_{>0}^\times}$ is trivial.

$\rightarrow$ χ_0 is unitary (b/c $\mathbb{I}_K^1/K^\times$ is cpt.), $\chi = \chi_0 |\cdot|^s$

Def. $\sigma(\chi) := \mathrm{Re}\ s$.

One can view χ as a character on $\mathbb{I}_K$ trivial on $K^\times$

Write $\chi = \prod_v \chi_v$ where $\chi_v = \chi |_{K_v^\times}$, $\quad (K_v^\times \hookrightarrow \mathbb{I}_K)$

Def. $\forall f \in S(\mathbb{A}_K)$, $\chi \in X_K$: $\quad \zeta(f, \chi) := \int_{\mathbb{I}_K} f(x) \chi(x)\, d^\times x$ global zeta integral

Lemma. $\zeta(f, \chi)$ converges absolutely when $\sigma(\chi) > 1$.

Pf: Wma $f = \otimes_v f_v$, $f_v = 1_{\mathcal{O}_{K_v}}$ for almost all v.

Write $\chi = \prod_v \chi_v$

$\zeta(f,\chi) = \prod_v \zeta(f_v, \chi_v)$ (this, in some sense, is the Euler product)

<u>Fact:</u> $\chi : \mathbb{I}_K \to \mathbb{C}$ cont. $\Rightarrow$ $\chi_v = |\cdot|_v^s$ for almost all v by continuity

$\Rightarrow$ for almost all v: $f_v = 1_{\mathcal{O}_{K_v}}$, $\chi_v = |\cdot|_v^s$, and thus

$$\zeta(1_{\mathcal{O}_{K_v}}, |\cdot|_v^s) = (N\delta_v)^{-1/2} \frac{1}{1-(N\mathfrak{p}_v)^{-s}}$$

$$\prod_{v\in V_f} (N\delta_v)^{-1/2} \frac{1}{(1-N\mathfrak{p}_v)^{-s}} = \frac{1}{\sqrt{|d_K|}} \cdot \underbrace{\zeta_K(s)}_{\text{converges absolutely at } \mathrm{Re}(s)>1}$$

$\mathrm{Re}(s) = \sigma(\chi)$

Each $\zeta(f_v,\chi_v)$ is meromorphic in χ_v and the local zeta integral converges for $\sigma(\chi_v) > 0$. $\square$

<u>Thm.</u> (Tate) Let $f \in \mathcal{S}(\mathbb{A}_K)$. The zeta function $\zeta(f,\chi)$ has an analytic continuation to a meromorphic function in $\chi \in X_K$ and

$\zeta(f,\chi) = \zeta(\hat{f}, \chi^\vee)$ where $\chi^\vee = |\cdot|\chi^{-1}$

Moreover, $\zeta(f,\chi)$ is meromorphic except for two simple poles at $\chi = 1$ and $\chi = |\cdot|$, with $\underset{\chi=1}{\mathrm{Res}}\, \zeta(f,\chi) = -f(0)\,\mathrm{Vol}(\mathbb{I}_K^1/K^\times)$,

$$\underset{\chi=|\cdot|}{\mathrm{Res}}\, \zeta(f,\chi) = \hat{f}(0)\,\mathrm{Vol}(\mathbb{I}_K^1/K^\times).$$

<u>Rmk.</u> $f = \otimes_v f_v$ with $f_v(x) = \begin{cases} e^{-\pi x_v^2} & v \text{ real} \\ e^{-\pi x_v \overline{x_v}} & v \text{ complex} \\ 1_{\mathcal{O}_{K_v}} & v \in V_f \end{cases}$

$\chi = |\cdot|^s$

Then $\zeta(f,\chi) = \zeta(\hat{f},\chi^\vee) \iff$

$$\iff \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = |d_K|^{\frac{1}{2}-s}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s).$$ $\square$

PF OF TATE'S THM:

$\mathbb{I}_K^{\geq 1} = \{ x \in \mathbb{I}_K \mid |x| \geq 1 \}$

$\mathbb{I}_K^{\leq 1} = \{ x \in \mathbb{I}_K \mid |x| \leq 1 \}$ $\qquad \mathbb{I}_K^1 = \mathbb{I}_K^{\geq 1} \cap \mathbb{I}_K^{\leq 1}$ has 0 Haar measure in $\mathbb{I}_K$.

$$\Rightarrow \zeta(f, \chi) = \underbrace{\int_{\mathbb{I}_K^{\geq 1}} f(x)\chi(x)\, d^\times x}_{} + \int_{\mathbb{I}_K^{\leq 1}} f(x)\chi(x)\, d^\times x$$

$f(x)$ behaves well as $|x| \to \infty$

$\Downarrow$

$\int_{\mathbb{I}_K^{\geq 1}} (\dots)$ defines a holomorphic function in $\chi \in X_K$

$$\int_{\mathbb{I}_K^{\leq 1}} f(x)\chi(x)\, d^\times x = \int_{\mathbb{I}_K^{\leq 1}/K^\times} \left(\sum_{\xi \in K^\times} f(\xi x) \right) \chi(x)\, d^\times x = *$$

(use $\chi(x) = \chi(\xi x)$, $d^\times x = d^\times(\xi x)$)

PSF: $$\sum_{\xi \in K^\times} f(\xi x) = \frac{1}{|x|} \sum_{\xi \in K^\times} \hat{f}\left(\frac{\xi}{x}\right) + \frac{1}{|x|}\hat{f}(0) - f(0)$$

$$* = \underbrace{\int_{\mathbb{I}_K^{\leq 1}/K^\times} \frac{1}{|x|} \left(\sum_{\xi \in K^\times} \hat{f}\left(\frac{\xi}{x}\right)\right) \chi(x)\, d^\times x}_{A} + \underbrace{\int_{\mathbb{I}_K^{\leq 1}/K^\times} \left(\frac{1}{|x|}\hat{f}(0) - f(0)\right) \chi(x)\, d^\times x}_{B}$$

For A: $\quad y := 1/x$. $\quad |x| \leq 1 \iff |y| \geq 1$

$$A = \int_{\mathbb{I}_K^{\geq 1}/K^\times} |y| \left(\sum_{\xi \in K^\times} \hat{f}(\xi y) \right) \chi(y^{-1})\, |y|\, d^\times y =$$

$$= \int_{\mathbb{I}_K^{\geq 1}} \hat{f}(y)\, \chi^\vee(y)\, d^\times y.$$

For B, use $\mathbb{I}_K^{\leq 1}/K^\times = \mathbb{I}_K^1/K^\times \times (0,1]$, $\quad \chi = \chi_0 |\cdot|^s$

$$B = \left(\int_{\mathbb{I}_K^1/K^\times} \chi_0(x)\, d^\times x \right) \cdot \int_{t=0}^{1} \left(\frac{\hat{f}(0)}{t} - f(0) \right) t^{s-1}\, dt =$$

$$= \left(\delta_{\chi_0, 1} \cdot \mathrm{Vol}(\mathbb{I}_K^1/K^\times) \right) \cdot \left(\frac{\hat{f}(0)}{s-1} - \frac{f(0)}{s} \right)$$

$$\Rightarrow \zeta(f,\chi) = \underbrace{\int_{\mathbb{I}_k^{\geq 1}} f(x)\,\chi(x)\,d^\times x + \int_{\mathbb{I}_k^{\leq 1}} \hat{f}(y)\,\chi^\vee(y)\,d^\times y}_{\text{holomorphic in } \chi \in X_k} + \underbrace{\delta_{\chi_0,1}}_{= \begin{cases} 0 & \chi_0 \neq 1 \\ 1 & \chi_0 = 1 \end{cases}} \mathrm{Vol}\left(\mathbb{I}_k^1/k^\times\right)\left(\frac{\hat{f}(0)}{s-1} - \frac{f(0)}{s}\right)$$

Symmetry in $(f,\chi) \longleftrightarrow (\hat{f}, \chi^\vee)$.

$$\Rightarrow \zeta(f,\chi) = \zeta(\hat{f}, \chi^\vee).$$

$\square$